Advances and Applications in Mathematical Sciences
Volume 20, Issue 9, July 2021, Pages 1977-1984

SCHWARZ BOUNDARY VALUE PROBLEMS FOR POISSON EQUATION ON THE QUARTER PLANE

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Abstract

Explicit representation for the solution of Schwarz boundary conditions for the Poisson equation on the quarter plane is obtained.

1. Introduction

There are principally two different complex second order partial differential operators, the Bitsadze operator $\partial_{\bar{z}}^2$ and the Laplace $\partial_z \partial_{\bar{z}}$. In this article, Schwarz boundary problem is solved explicitly for Laplace operator known as Poisson equation on the quarter plane. The complex variable forms of Cauchy Pompeiu formula and Gauss theorem play a very crucial role in solving boundary value problems in Quarter plane. The area integral written in Cauchy-Pompeiu formula is known as Pompeiu operator and was studied in (see [1, 4, 8, 9, 11, 12, 14]) for bounded domains. Schwarz Boundary Value Problems for different operators are studied on other domains also (see [1, 2, 3, 5, 6, 10, 13, 15]).

For any function $w \in C^1(\mathbb{Q}_1; \mathbb{C}) \cap C(\overline{\mathbb{Q}}_1; \mathbb{C})$ with bounded as a subset of $\mathbb{R}^+$

2010 Mathematics Subject Classification: 30G20, 31A30, 30E25.
Keywords: Schwarz boundary conditions, Cauchy-Pompeiu representation, Gauss theorem.
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Received July 11, 2021; Accepted July 22, 2021

where $m(k, w) = \max\{|w(z)| : |z| = k, 0 \leq \operatorname{Re} z, 0 \leq \operatorname{Im} z\}$, $(1+k)^{\delta} m(k, w)$ and $w_{\bar{z}} \in L_{p,2}(\mathbb{Q}_1; \mathbb{C})$, $2 \leq p$, and $z \in \mathbb{Q}_1$, Gauss theorem on the quarter plane $\mathbb{Q}_1 = \{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ can be written as

$$\frac{1}{2\pi i}\int_0^{+\infty} \frac{w(t)}{t-\bar{z}}\,dt - \frac{1}{2\pi i}\int_0^{+\infty} \frac{w(it)}{t+i\bar{z}}dt - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\varsigma}}(\varsigma)\frac{d\xi d\eta}{\varsigma - \bar{z}} = 0, \tag{1.1}$$

$$\frac{1}{2\pi i}\int_0^{+\infty} \frac{w(t)}{t+\bar{z}}\,dt - \frac{1}{2\pi i}\int_0^{+\infty} \frac{w(it)}{t-i\bar{z}}dt - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\varsigma}}(\varsigma)\frac{d\xi d\eta}{\varsigma + \bar{z}} = 0, \tag{1.2}$$

$$\frac{1}{2\pi i}\int_0^{+\infty} \frac{w(t)}{t+\bar{z}}\,dt - \frac{1}{2\pi i}\int_0^{+\infty} \frac{w(it)}{t-iz}dt - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\varsigma}}(\varsigma)\frac{d\xi d\eta}{\varsigma + \bar{z}} = 0. \tag{1.3}$$

The following Cauchy-Pompeiu formula is a generalization of the Cauchy's integral formula for analytic functions and can be proved using the Gauss theorem.

In case of the Quarter plane, Cauchy-Pompeiu formula can be represented as i.e. any $w \in \mathbb{Q}_1 = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z, 0 \leq \operatorname{Im} z\}$, for which for some $0 \leq \delta$ the function $(1+k)^{\delta} m(k, w)$ is bounded as a subset of $\mathbb{R}^+$ and $w_{\bar{z}} \in L_1(\mathbb{Q}_1; \mathbb{C})$ is representable as

$$w(z) = \frac{1}{2\pi i}\int_0^{+\infty} \frac{w(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_0^{+\infty} \frac{w(it)}{t+iz}\,dt - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\varsigma}}(\varsigma)\frac{d\xi d\eta}{(\varsigma - z)}, \tag{1.4}$$

$$w(z) = \frac{1}{2\pi i}\int_0^{+\infty} \frac{w(t)}{t-\bar{z}}\,dt + \frac{1}{2\pi i}\int_0^{+\infty} \frac{w(it)}{(t-i\bar{z})}\,dt - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\varsigma}(\varsigma)\frac{d\xi d\eta}{(\varsigma - z)}. \tag{1.5}$$

Here the Pompeiu operator T has the following form

$$Tf(z) = -\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)\frac{d\eta d\xi}{\varsigma - z} \tag{1.6}$$

and satisfies the properties $\partial_{\bar{z}} Tf = f$, $\partial_z Tf = \Pi f$, where

$$\Pi f(z) = -\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)\frac{d\eta d\xi}{(\varsigma - z)^2}. \tag{1.7}$$

Here the derivatives are taken in distributional sense.

2. Schwarz Problem for Poisson Equation on the Quarter Plane

The solution of the inhomogeneous Schwarz boundary value problem over $\mathbb{Q}_1$ is given by, (see, [7]),

Theorem 2.1. *The Schwarz boundary value problem on* $\mathbb{Q}_1$. $\partial_{\bar{z}} w(z) = f(z)$ *in* $\mathbb{Q}_1$, *satisfying* $\operatorname{Re} w = \gamma_1$ *on* $y = 0,\ 0 < x < +\infty$, $\operatorname{Im} w = \gamma_2$ *for* $x = 0,\ 0 < y < +\infty$ *is uniquely weakly solvable for* $f \in L_1(\mathbb{Q}_1; \mathbb{C})$, $\gamma_1, \gamma_2 \in C(\mathbb{R}^+; \mathbb{R})$, *such that* γ_1, γ_2 *are bounded on* $\mathbb{R}^+ = [0, \infty)$. *The solution is*

$$w(z) = \frac{2}{\pi i}\int_0^{+\infty} \gamma_1(t)\frac{z}{t^2 - z^2}\,dt - \frac{2}{\pi i}\int_0^{+\infty} \gamma_2(t)\frac{z}{t^2 + z^2}\,dt$$

$$-\frac{2}{\pi}\int_{\mathbb{Q}_1}\left(\frac{zf(z)}{\varsigma^2 - z^2} - \frac{z\overline{f(z)}}{\bar{\varsigma}^2 - z^2}\right)d\xi d\eta. \tag{2.1}$$

Lemma 2.1. *For* $w, w^1_\varsigma(\mathbb{Q}_1; \mathbb{C})C^1(\mathbb{Q}_1; \mathbb{C})$, *with bounded* $(1 + k)^\delta m(k, w_\varsigma)$, $(1 + k)^\delta m(k, w_\varsigma)$ *as a subset of* $\mathbb{R}^+$ *and* $\varsigma = \xi + i\eta$, *we can write the following*

(i) $$I_1 = \frac{1}{\pi}\int_{\mathbb{Q}_1} \partial_{\bar{\varsigma}\varsigma} w(\varsigma)\log|\,\varsigma^2 - z^2\,|^2 d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_0^{+\infty} \partial_\varsigma w(t)\log|\,t^2 - z^2\,|^2 dt$$

$$-\frac{1}{2\pi}\int_0^{+\infty} \partial_\varsigma w(it)\log|\,t^2 + z^2\,|^2 dt$$

$$+\frac{1}{2\pi i}\int_0^{+\infty} w(t)\left(\frac{1}{(t - \bar{z})} + \frac{1}{(t + \bar{z})}\right)dt$$

$$+\frac{1}{2\pi i}\int_0^{+\infty} w(it)\left(\frac{1}{(t - i\bar{z})} + \frac{1}{(t + i\bar{z})}\right)dt + w(z). \tag{2.2}$$

(ii) $$I_2 = \frac{1}{\pi}\int_{\mathbb{Q}_1} \overline{\partial_{\bar{\varsigma}\varsigma} w(\varsigma)}\log|\,\varsigma^{\bar{2}} - z^2\,|^2 d\xi d\eta$$

$$= -\frac{1}{2\pi i}\int_0^{+\infty}\overline{\partial_\varsigma w(t)}\log|t^2 - z^2|^2 dt + \frac{1}{2\pi}\int_0^{+\infty}\overline{\partial_\varsigma w(it)}\log|t^2 + z^2|^2 dt$$

$$+ \frac{1}{2\pi i}\int_{\mathbb{Q}_1}\overline{w(t)}\left(\frac{1}{(t-\bar{z})} + \frac{1}{(t+\bar{z})}\right)dt$$

$$- \frac{1}{2\pi i}\int_{\mathbb{Q}_1}\overline{w(it)}\left(\frac{1}{(t-i\bar{z})} + \frac{1}{(t+i\bar{z})}\right)dt. \quad (2.3)$$

Proof. (i) Starting with the area integral over $\mathbb{Q}_1$ and using Gauss theorem and Cauchy Pompeiu formula on $\mathbb{Q}_1$, we have

$$I_1 = \frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_{\bar{\varsigma}\varsigma}w(\varsigma)\log|\varsigma^2 - z^2|^2 d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_{\bar{\varsigma}}(\partial_\varsigma w(\varsigma)\log|\varsigma^2 - z^2|^2)d\xi d\eta$$

$$- \frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_\varsigma w(\varsigma)\frac{\overline{(\varsigma - z)} + \overline{(\varsigma - z)}}{\varsigma^2 - z^2}d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_\varsigma(\partial_\varsigma w(\varsigma)\log|\varsigma^2 - z^2|^2)d\xi d\eta$$

$$- \frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_\varsigma w(\varsigma)\left(\frac{1}{\overline{(\varsigma - z)}} + \frac{1}{\overline{(\varsigma + z)}}\right)d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_{\bar{\varsigma}}(\partial_\varsigma w(\varsigma)\log|\varsigma^2 - z^2|^2)d\xi d\eta$$

$$- \frac{1}{\pi}\int_{\mathbb{Q}_1}\left(\frac{\partial_\varsigma w(\varsigma)}{\overline{(\varsigma - z)}} + \partial_\varsigma\left(\frac{w(\varsigma)}{\overline{(\varsigma + z)}}\right)\right)d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_0^{+\infty}\partial_\varsigma w(t)\log|t^2 - z^2|^2 dt$$

$$- \frac{1}{2\pi}\int_0^{+\infty}\partial_\varsigma w(it)\log|t^2 + z^2|^2 dt$$

$$+ \frac{1}{2\pi i} \int_{\mathbb{Q}_1} w(t)\left(\frac{1}{(t - \bar{z})} + \frac{1}{(t + \bar{z})}\right)dt$$

$$+ \frac{1}{2\pi i} \int_{\mathbb{Q}_1} w(it)\left(\frac{1}{(t - i\bar{z})} + \frac{1}{(t + i\bar{z})}\right)dt + w(z).$$

(ii) Starting with the area integral and using Gauss theorem and Cauchy Pompeiu formula on $\mathbb{Q}_1$, we have

$$I_2 = \frac{1}{\pi}\int_{\mathbb{Q}_1} \overline{\partial_{\bar{\varsigma}\varsigma}} w(\varsigma) \log|\bar{\varsigma}^2 - z^2|^2 d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{Q}_1} \partial_{\varsigma}(\overline{\partial_{\varsigma} w(\varsigma)} \log|\bar{\varsigma}^2 - z^2|^2) d\xi d\eta$$

$$- \frac{1}{\pi}\int_{\mathbb{Q}_1} \overline{\partial_{\varsigma} w(\varsigma)} \frac{(\varsigma - \bar{z}) + (\varsigma + \bar{z})}{(\varsigma^2 - \bar{z}^2)} d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{Q}_1} \partial_{\varsigma}(\overline{\partial_{\varsigma} w(\varsigma)} \log|\bar{\varsigma}^2 - z^2|^2) d\xi d\eta$$

$$- \frac{1}{\pi}\int_{\mathbb{Q}_1} \overline{\partial_{\varsigma} w(\varsigma)} \left(\frac{1}{(\varsigma - \bar{z})} + \frac{1}{(\varsigma + \bar{z})}\right) d\xi d\eta$$

$$= -\frac{1}{2\pi i}\int_0^{+\infty} \overline{\partial_{\varsigma} w(t)} \log|t^2 - z^2|^2 dt$$

$$+ \frac{1}{2\pi}\int_0^{+\infty} \overline{\partial_{\varsigma} w(it)} \log|t^2 + z^2|^2 dt$$

$$+ \frac{1}{2\pi i}\int_{\mathbb{Q}_1} \overline{w(t)} \left(\frac{1}{(t - \bar{z})} + \frac{1}{(t + \bar{z})}\right) dt$$

$$- \frac{1}{2\pi i}\int_{\mathbb{Q}_1} \overline{w(it)} \left(\frac{1}{(t - i\bar{z})} + \frac{1}{(t + i\bar{z})}\right) dt. \qquad \square$$

Theorem 2.2. *The Schwarz boundary value problem for the Poisson equation in* $\mathbb{Q}_1$

$$\partial_{\bar{z}z} w(z) = f(z) \;\; in \;\; \mathbb{Q}_1,$$

$$\operatorname{Re} w = \psi_0 \ \textit{on} \ 0 < x < +\infty, \ y = 0, \operatorname{Re} w = \psi_1 \ \textit{on} \ 0 < x < +\infty, \ y = 0$$

$$\operatorname{Im} w = \phi_0 \ \textit{on} \ 0 < y < +\infty, \ x = 0, \operatorname{Im} w_z = \phi_1 \ \textit{on} \ 0 < y < +\infty, \ x = 0$$

is uniquely weakly solvable for $f \in L_1(\mathbb{Q}_1; \mathbb{C})$, $\phi_0, \phi_1, \psi_0, \psi_1 \in C(\mathbb{R}^+; \mathbb{R})$ *such that* $t^\delta \phi_0(t)$, $t^\delta \psi_0(t)$, $t^{1+\delta}\phi_1(t)$, $t^{1+\delta}\psi_1(t)$ *are bounded on* $\mathbb{R}^+ = [0, +\infty)$ *for some* $0 < \delta$. *The solution is given by*

$$w(z) = -\frac{2}{\pi i}\int_0^{+\infty} \psi_0(t)\frac{t}{t^2 - \bar{z}^2}\,dt - \frac{2}{\pi}\int_0^{+\infty} \phi_0(t)\frac{t}{t^2 + \bar{z}^2}\,dt$$

$$-\frac{1}{\pi i}\int_{-\infty}^{+\infty} \psi_1(t)\log|t^2 - z^2|^2 dt + \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi_1(t)\log|t^2 + z^2|^2 dt$$

$$+\frac{1}{\pi}\int_{\mathbb{Q}_1} [f(\varsigma)\log|\varsigma^2 - z^2|^2 - \overline{f(\varsigma)}\log|\bar{\varsigma}^2 - z^2|^2]\, d\xi d\eta. \tag{2.4}$$

Proof. Using Lemma 2.1, the last area integral in (2.4) gives

$$\frac{1}{\pi}\int_{\mathbb{Q}_1} [f(\varsigma)\log|\varsigma^2 - z^2|^2 - \overline{f(\varsigma)}\log|\bar{\varsigma}^2 - z^2|^2]\, d\xi d\eta$$

$$= \frac{1}{\pi}\int_{\mathbb{Q}_1} [\partial_{\bar{\varsigma}\varsigma} w(\varsigma)\log|\varsigma^2 - z^2|^2 - \overline{\partial_{\bar{\varsigma}\varsigma} w(\varsigma)}\log|\bar{\varsigma}^2 - z^2|^2]\, d\xi d\eta$$

$$= [I_1 - I_2]$$

$$= \frac{1}{\pi i}\int_0^{+\infty} \operatorname{Re}\partial_\varsigma w(t)\log|t^2 - z^2|^2 dt$$

$$-\frac{1}{\pi}\int_0^{+\infty} \operatorname{Re}\partial_\varsigma w(it)\log|t^2 + z^2|^2 dt$$

$$+\frac{2}{\pi i}\int_0^{+\infty} \operatorname{Re} w(t)\frac{t}{t^2 - \bar{z}^2}\,dt + \frac{2}{\pi}\int_0^{\infty} \operatorname{Im} w(it)\frac{t}{t^2 + \bar{z}^2}\,dt + w(z).$$

Using boundary values, we obtain

$$= \frac{1}{\pi i}\int_{-\infty}^{\infty} \psi_1(t)\log|t^2 - z^2|^2 dt - \frac{1}{\pi}\int_{-\infty}^{\infty} \phi_1(t)\log|t^2 + z^2|^2 dt$$

$$+\frac{2}{\pi i}\int_0^{+\infty}\psi_0(t)\frac{t}{t^2-\bar{z}^2}\,dt+\frac{2}{\pi}\int_0^{+\infty}\phi_0(t)\frac{t}{t^2+\bar{z}^2}+w(z)$$

i.e.

$$w(z)=-\frac{2}{\pi i}\int_0^{+\infty}\psi_0(t)\frac{t}{t^2-\bar{z}^2}\,dt-\frac{2}{\pi}\int_0^{+\infty}\phi_0(t)\frac{t}{t^2+\bar{z}^2}\,dt$$

$$-\frac{1}{\pi i}\int_{-\infty}^{\infty}\psi_1(t)\log|\,t^2-z^2\,|^2dt+\frac{1}{\pi}\int_{-\infty}^{\infty}\phi_1(t)\log|\,t^2-z^2\,|^2dt$$

$$+\frac{1}{\pi}\int_{\mathbb{Q}_1}[f(\varsigma)\log|\,\varsigma^2-z^2\,|^2-\overline{f(\varsigma)}\log|\,\bar{\varsigma}^2-z^2\,|^2]d\xi d\eta$$

which is (2.4). By doing simple calculations, we can verify that (2.4) is solution of the given Schwarz problem. □

3. Conclusion

Explicit solution of one of the forms of Schwarz boundary value problems for Poisson equation is obtained on the Quarter plane. On the basis of Theorem 2.2 and Lemma 2.1, other forms (some are stated below) can also be solved in the similar way,

$$\partial_{\bar{z}z}w(z)=f(z)\ \text{ in }\ \mathbb{Q}_1,$$

$$\operatorname{Im}w=\psi_0\ \text{ on }\ 0<x<+\infty,\ y=0,\ \operatorname{Im}w_z=\psi_1\ \text{ on }\ 0<x<+\infty,\ y=0$$

$$\operatorname{Re}w=\phi_0\ \text{ on }\ 0<y<+\infty,\ x=0,\ \operatorname{Re}w_z=\phi_1\ \text{ on }\ 0<y<+\infty,\ x=0$$

and

$$\partial_{\bar{z}z}w(z)=f(z)\ \text{ in }\ \mathbb{Q}_1,$$

$$\operatorname{Im}w=\psi_0\ \text{ on }\ 0<x<+\infty,\ y=0,\ \operatorname{Re}w_z=\psi_1\ \text{ on }\ 0<x<+\infty,\ y=0$$

$$\operatorname{Re}w=\phi_0\ \text{ on }\ 0<y<+\infty,\ x=0,\ \operatorname{Im}w_z=\phi_1\ \text{ on }\ 0<y<+\infty,\ x=0$$

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