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ST-WIRELENGTH OF EMBEDDINGS OF CERTAIN GRAPHS

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Abstract

Graph Embedding plays a key role in the implementation of Interconnection Networks in a system. In this paper, we have introduced a variant of wirelength in graph embedding. The *ST*-wirelength deals with all the spanning lines of the host graph in an embedding. The *ST*-wirelength of the circulant graph into wheel graph has been obtained.

1. Preliminaries

Finding the wirelength of embedding one graph onto another has been considered by many authors [1, 2, 3, 4, 5]. In this paper, a variation of the wirelength problem with respect to the spanning trees of the host graph has been defined. A spanning tree of a graph G is a subgraph of G which covers all the vertices with minimum number of edges. If $|V(G)| = n,$ then clearly this subgraph is a tree on n vertices and $n-1$ edges. A k-edge connected

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graph on n vertices has atleast $n\left(\frac{k}{2}\right)^{n-1}$ spanning trees [6]. All spanning trees have the same number of vertices and edges. The number of spanning trees of G, denoted by $t(G)$ is the total number of distinct spanning subgraphs of G that are trees. Finding $t(G)$ has applications in experimental design [7]. The number of spanning trees in a network is a parameter of the reliability of the network when the links of the network are subject to failure. Shut down of links for carrying out repairs in the network is also possible till the vertices in the network remain connected minimally, leading to the study of spanning trees. Spanning tree is used in designing of water supply networks, electrical networks, telecommunication network and so on. It is also used to find paths in the map. In recent years the spanning trees of a large scale network has proven to be useful in community mining [8]. In the study of network embeddings, the spanning trees of the host graph have not been considered so far.

We define an embedding parameter called Spanning tree Wirelength denoted by WL_{ST} defined by

$$WL_{ST}(G, H) = \min_{T} WL(G, T)$$

where the minimum is taken over all spanning trees T of H. The Spanning tree wirelength problem is to identify that spanning tree T of H such that $WL(G, T) = WL_{ST}(G, H)$. In the sequence, we require the following definitions:

Definition 1.1. Maximum Subgraph Problem [9]**.** Let G be a graph on n vertices. Given an integer k, $k \leq n$, the maximum subgraph problem is to find an induced subgraph on k vertices with maximum number of edges, among all subgraphs on k vertices.

Definition 1.2 [1]**.** Let G and H be finite graphs with n vertices. An one-one mapping $f : V(G) \to V(H)$ inducing a one-one mapping $P_f : E(G) \to \{P_f(u, v)$ where $P_f(u, v)$ is a path between $f(u)$ and $f(v)$ in H for $(u, v) \in E(G)\}$ is called an embedding of G into H.

Definition 1.3. For $e \in H$, the number of paths $P_f(u, v)$ that contain e is called the congestion on e with respect to f and is denoted by $c_f(e)$. For $S \subseteq E(H)$, we define $c_f(S) = \sum_{e \in S} c_f(e)$.

Definition 1.4 [10]**.** A subgraph K of a graph G is said to be convex if all the shortest paths between any two vertices of K lie in K. An edge cut S of G is said to be a convex cut if $G \setminus S$ splits into two components, each of which is convex.

Definition 1.5. Congestion Lemma [10]**.** Let f be an embedding of a graph G into H. Let S be a convex edge cut of H such that $H \setminus S$ splits into components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Suppose G_1 and G_2 are maximum subgraphs of G and $P_f(u, v)$ with $u \in G_1$ and $v \in G_2$ contains exactly one edge in S, for $(u, v) \in G$, then

$$\begin{aligned} c_f(S) &= \sum_{v \in V(G_1)} \deg_G(v) - 2 \mid E(G_1) \mid \\ &= \sum_{v \in V(G_2)} \deg_G(v) - 2 \mid E(G_2) \mid. \end{aligned} \tag{1.1}$$

Definition 1.6. Circulant graph [11]**.** Let $K \subseteq \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, $n \geq 3$. A graph G with vertex set $V = 0, 1, \ldots, n-1$ and the edge set $E = \{(i, j) : \mid j - 1 \mid \equiv k (\operatorname{mod} n), k \in K\}$ is called a circulant graph and is denoted by $G(n; \pm K)$.

Definition 1.7. Wheel graph [12]**.** Consider a cycle on $n-1$ nodes. Take a new node and join it to each of the $n-1$ nodes on the cycle. The resultant graph is called a wheel graph and is denoted by W_n. The new vertex of degree $n-1$ is called the hub of W_n.

Definition 1.8. Spanning tree [6]**.** A spanning tree is a subgraph of a graph G that has all the vertices covered with minimum possible number of edges. Removing one edge from the spanning tree will make the graph disconnected. Spanning tree is used to find a minimum path to connect all the vertices in a graph.

2. *ST*-Wirelength of embedding circulant graphs into wheels

Let G be the circulant graph $G(n, \pm\{1, 2\})$ on n vertices. Let us consider all the possible spanning trees of W_n and find the wirelength of the embedding G into each of them. In Figure 2, we have considered 4 nonisomorphic spanning trees $T_1 - T_4$ of W_6. The wirelength of embedding $G : (6; \{1, 2\})$ into each of these lines has been computed. See Table 1.

Remark 2.1. It is clear that the $WL(G, T)$ decreases with increase in the number of leaves.

Table 1. $WL(G, T_i)$, $i = 1, 2, \ldots, 4$.

Tree T	$WL(G, T)$
T_1	26
T_2	24
T_3	22
T_4	20

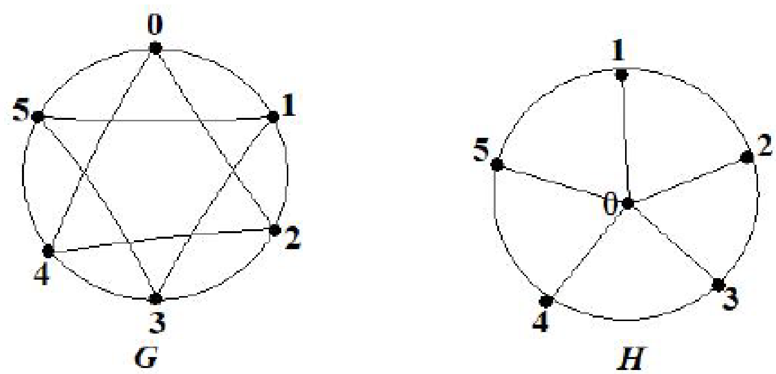

Figure 1. $G : G(6, \{1, 2\})$, $H : W_6$.

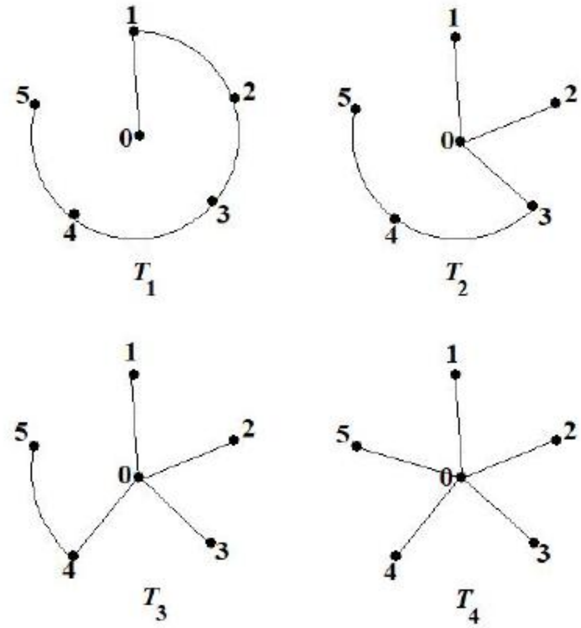

Figure 2. Spanning trees of W_6.

Theorem 2.2. *Let G be* $G(n_i, \pm\{1, 2\})$ *and T be the spanning tree of wheel graph* W_n *with maximum number of leaves.*

$$WL_{ST}(G(n; \pm\{1, 2\}), W_n) = WL(G, T).$$

Proof. Clearly, T is the star graph with one vertex of degree 5 and all other vertices of degree 1. Consider an embedding $f : G \to T$. Since T is the star graph and G is vertex symmetric, the embedding of G into T is unique. Each pendant edge of T is a cut edge whose removal yields components H_1 and H_2 such that $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$ in G are maximum subgraphs of G.

$$\begin{aligned} WL_{ST}(G, W_n) &= WL_f(G(n; \pm\{1, 2\}), T) \\ &= (n-1)[|V(G_i)| - 2|E(G_i)|] \\ &= (n-1)[4(1) - 2(0)] \\ &= 4(n-1). \end{aligned}$$ □

3. Conclusion

The fact that there is exactly one spanning tree with exactly 2, 3, 4 and 5 leaves has yielded the exact ST-wirelength of embedding $G(n; \pm \{1, 2\})$ into W_n. But the difficulty level increases with the increase in the number of spanning trees in the host graph. This paper opens up a challenging line of research in determining the new embedding parameter.

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