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EXACT SOLUTION OF TWO DIMENSIONAL TIME-SPACE FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD

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Abstract

The purpose of this paper is to obtain exact solution of time- space fractional order partial differential equation by using Adomian decomposition method with suitable initial value. These equations arise in biological population model and heat conduction model. The results indicate that Adomian decomposition method is effective, simple and applicable to solve fractional order partial differential equations. To illustrate applicability of present technique, solutions of some differential equations in physical models and their graphical representation are done by MATLAB software.

1. Introduction

In recent years, fractional calculus has been increasingly used for numerous applications in many scientific and technical fields such as medical sciences, biological research, as well as various chemical, biochemical and physical fields. Nonlinear partial differential equation appear in many branches of physics, engineering and applied mathematics. It has turned out that many phenomenon in engineering, physics and other sciences can be

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described very successfully by models using mathematical tool by fractional calculus [16, 17, 23, 24]. For better understanding of phenomenon described by a given nonlinear fractional partial differential equation, the solution of differential equations of fractional order must be involved. Fractional derivatives provide more accurate models of real world problems than integer order derivatives. Because of their many applications in scientific research fields, fractional partial differential equations found to be an effective tool to describe certain physical phenomena, such as diffusion processes, electrical and rheological materials properties and viscoelasticity theories also in earthquake modeling, traffic flow models, diffusion model, heat conduction models, gas dynamics models control and relaxation processes [20, 21, 22].

In recent years, many researcher have paid attention to study the analytical or approximate solutions of nonlinear fractional partial differential equations by using various numerical methods. Among these methods, the Variational Iteration Method, Homotopy Perturbation Method and Differential Transform Method are most popular ones which are used solve to solve differential and integral equations of integral and fractional order. The Adomian Decomposition method (ADM) is universal approach which can be used to solve fractional ordinary differential equations as well as fractional partial differential equations. ADM was originally proposed by Adomian [1, 2]. Wazwaz [30, 31] has applied ADM to solve variety of differential equations. While Shawagfeh [27] has employed Adomian decomposition method for solving nonlinear fractional differential equations, Daftardar-Gejji and Jafri have obtained solution of numerous problems [13, 14] by using Adomian decomposition method. Also Dhaigude and Birajdar [7, 6] extended the discrete ADM for obtaining the numerical solution of system of fractional partial differential equations. Chitalkar-Dhaigude and Bhadgaonkar in [5] have shown that the ADM is more convenient than the Charpit's method to solve first-order nonlinear PDEs. Sontakke and Pandit [28, 29] investigates the iterative solution of linear and nonlinear fractional partial differential equations using fractional ADM. Peng Guo [10] also solve fractional partial differential equation by ADM. Many studies have been conducted to solve time fractional partial differential equations using various methods. We use the Adomian decomposition method to solve a time-space fractional partial differential equations and achieve excellent results. Dhaigude and

Bhadgaonkar [8] are studied time-space fractional models such as Gas dynamics model, Advection model, Wave model, and Klein-Gordon model successfully by virtue of improved ADM developed for nonlinear nonhomogeneous time-space fractional PDEs by using fractional Taylor expansion series to nonhomogeneous functions. Bhadgaonkar and Sontakke [4] obtained exact Solution of time-space fractional partial differential equations by ADM.

There are very few equations which can be solved by applying both time-space fractional order derivative. The main aim of this paper is to implement ADM to solve the nonlinear time-space fractional partial differential equations. The solutions of our model equations are calculated in the form of convergent series with easily computable components. The time-space fractional derivatives are described in the Caputo sense.

The paper is structured in this way: in section (2) few basic results about fractional calculus and related properties are given which are used in this paper, while in section (3) we clarify the steps of the ADM for solving nonlinear time-space fractional order PDEs. The effectiveness and sharpness of the method is shown by obtaining solution of equations in physical models like biological population model and heat conduction model in section (4). Section (5) is conclusions.

2. Preliminaries

In this section, we set up notations, basic definitions and main properties of Riemann-Liouville integral, and the relation between Riemann-Liouville integral and Caputo fractional derivative is also given.

Definition 2.1 [19]**.** A real function $f(x); x > 0$ is said to be in space $\mathcal{C}_\alpha; \alpha \in \Re$ if there exists a real number $p > \alpha$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in \mathcal{C}[0; \infty)$.

Definition 2.2 [26]**.** Let $f \in \mathcal{C}_\alpha$ and $\alpha \geq -1$, then Riemann-Liouville fractional integral operator (RLFIO) of $u(x, t)$ with respect to t of order α is indicated by $J_t^\alpha u(x, t)$ and is explained as

$$J_t^{\alpha} u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{(\alpha - 1)} u(x, \tau) d\tau, \; t > 0, \; \alpha > 0. \tag{2.1}$$

Definition 2.3 [26]**.** Let $m - 1 < \alpha < m, t \in R$ and $t > 0$. The CFDO for the function $f \in H^1([a, b], \mathbb{R}_+)$ with order $\alpha \geq 0$ is explained as

$$^{c}D_t^{\alpha} u(x, t) = \begin{cases} \dfrac{1}{\Gamma(m - \alpha)} \displaystyle\int_0^t (t - \tau)^{m - \alpha - 1} \frac{\partial^m u}{\partial \tau^m} d\tau, & m - 1 < \alpha \leq m \\ \dfrac{\partial^m u}{\partial t^m}, & \alpha = m \in N. \end{cases} \tag{2.2}$$

We have following properties of RLFIO and CFDO

$$^{c}D_t^{\alpha} t^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu - \alpha + 1)} t^{(\mu - \alpha)}, \tag{2.3}$$

$$J_t^{\alpha} t^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} t^{(\mu + \alpha)}, \; \alpha > 0, \; \mu > -1. \tag{2.4}$$

Note that the relation between RLFIO and CFDO is given by:

$$J_t^{\alpha} \, {}^{c}D_t^{\alpha} u(x, t) = u(x, 0) - \sum_{k=0}^{m-1} u^{(k)}(x, 0) \frac{t^k}{k!}, \; m - 1 < \alpha \leq m. \tag{2.5}$$

Definition 2.4. The Mittage-Leffler function [26] for one parameter is defined as follows

$$E_{\alpha}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \; (\alpha \in C, \; \mathrm{Re}(\alpha) > 0),$$

when we apply CFDO on MLF we get,

$$^{c}D_t^{\alpha} E_{\alpha}(at^{\alpha}) = aE_{\alpha}(at^{\alpha}), \tag{2.6}$$

where a is constant.

3. Analysis of Adomian Decomposition Method

In this section, we present the ADM to solve two-dimensional nonlinear time-space fractional partial differential equation. Consider the initial value problem (IVP) for nonlinear time-space fractional partial differential equation of order α

$$L_t^{\alpha} u(x, y, t) + R(u(x, y, t)) + N(u(x, y, t)) = g(x, y, t), \; 0 < \alpha \leq 1 \tag{3.1}$$

$$u(x, y, 0) = h(x, y) \tag{3.2}$$

where $L_t^{\alpha}(u)$ is the fractional differential operator of highest order fractional derivative with respect to t, $u(x, y, t)$ is unknown function which we want to determined, $R(u)$ is linear differential operator, $N(u) = f(u(x, y, t))$ is nonlinear data and $g(x, y, t)$ is nonhomogeneous function.

Now, applying the J_t^{α} on both side of equation (3.1) and using the initial condition (3.2), we get:

$$u(x, y, t) = u(x, y, 0) + J_t^{\alpha}[g(x, y, t) - R(u) - N(u)]. \tag{3.3}$$

The unknown function $u(x, y, t)$ can be decomposed as an infinite series of the form

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) \tag{3.4}$$

The Adomian polynomials A_n for the nonlinear term $N(u)$ can be evaluated by using the following expression

$$N(u(x, y, t)) = \sum_{n=0}^{\infty} A_n \tag{3.5}$$

where

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [N \sum_{i=0}^{n} \lambda^i u_i]_{\lambda=0}, \; n = 0, 1, 2, 3, \ldots \tag{3.6}$$

where A_n are the nonlinear Adomian polynomials. By substituting decomposed series (3.4); (3.5) in (3.3) we have,

$$\sum_{n=0}^{\infty} u(x, y, t) = h(x, y) + J_t^{\alpha}\left[g(x, y, t) - R(\sum_{n=0}^{\infty} u_n(x, y, t)) - \sum_{n=0}^{\infty} A_n(x, t) \right], \tag{3.7}$$

Taking term by term comparison on both side of equation (3.7), we get recursion scheme like.

$$u_0(x, y, t) = h(x, y) + J_t^{\alpha} g(x, y, t),$$

$$u_1(x, y, t) = J_t^{\alpha}[- R(u_0) - A_0],$$

$$u_2(x, y, t) = J_t^{\alpha}[- R(u_1) - A_1],$$

$$u_3(x, y, t) = J_t^{\alpha}[- R(u_2) - A_2],$$

$$\vdots$$

$$u_{(k+1)}(x, y, t) = J_t^{\alpha}[- R(u_k) - A_k],\ k \geq 0,$$

and so forth. Wherever every component can be determined by manipulating the preceding components and we can obtain the solution in a series form by computing the components $u_n(x, y, t),\ n \geq 0$. Eventually, we approximate the solution $u(x, y, t)$ by the reduced series. Then the solution $u(x, y, t)$ of IVP (3.1)-(3.2) is

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x, y, t) \tag{3.8}$$

which gives

$$\lim_{m \to \infty} \phi_{m+1} = u(x, y, t). \tag{3.9}$$

4. Numerical Examples

The benefits and intensity of the ADM can be expressed by applying it to some physical models in time-space fractional partial differential equations.

4.1. Biological population model

General form of time-space fractional biological population model [9, 15] is

$${}^{c}D_t^{\alpha} u(x\ y, t) = {}^{c}D_x^{2\alpha} u^2 + {}^{c}D_y^{2\alpha} u^2 + f(u(x, y, t)),\ 0 < \alpha \leq 1 \tag{4.1}$$

with given initial condition

$$u(x, y, 0) = f(x, y) \tag{4.2}$$

where u denotes the population density and f represents population supply due to births and deaths. Here ${}^cD_x^{2\alpha} = {}^cD_x^{\alpha}\,{}^cD_x^{\alpha}$ and ${}^cD_y^{2\alpha} = {}^cD_y^{\alpha}\,{}^cD_y^{\alpha}$. For $\alpha \to 1$ the equation (4.1) reduces to classical one. Holder estimates of its solution studied in [18]. El-Sayed et al. studied three cases of f for time fractional derivative in [9].

$f(u) = hu,$ where h is constant,(Malthusian Law), [11]

$f(u) = h_1u - h_2u^2,$ where h_1, h_2 are constants, (Verhulst Law),[11]

$f(u) = cu^p, c \geq 0, 0 < p < 1,$ Porous Media [3, 25].

Example 4.1. Consider the nonlinear time-space fractional biological population model

$${}^cD_t^{\alpha}u(x\ y, t) = {}^cD_x^{2\alpha}(u^2) + {}^cD_y^{2\alpha}(u^2) + hu, 0 < \alpha \leq 1 \tag{4.3}$$

with initial condition

$$u(x, y, 0) = \sqrt{x^{\alpha}y^{\alpha}} \tag{4.4}$$

Solution. Applying J_t^{α} on both side of (4.3) we have,

$$J_t^{\alpha}\,{}^cD_t^{\alpha}u(x, y, t) = J_t^{\alpha}[({}^cD_x^{2\alpha} + {}^cD_y^{2\alpha})u^2 + hu]$$

$$u(x, y, t) = u\,(x, y, 0) + J_t^{\alpha}[({}^cD_x^{2\alpha} + {}^cD_y^{2\alpha})u^2 + hu] \tag{4.5}$$

Suppose that the solution $u(x, y, t)$ has the following series form

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) \tag{4.6}$$

Then equation (4.5) has the form

$$\sum_{n=0}^{\infty} u_n(x,\ y,\ t) = \sqrt{x^{\alpha} y^{\alpha}} + J_t^{\alpha}\left[({}^{c}D_x^{2\alpha} + {}^{c}D_y^{2\alpha})\sum_{n=0}^{\infty} A_n + h\sum_{n=0}^{\infty} u_n\right] \tag{4.7}$$

where A_n is the Adomian polynomials to be determined from the nonlinear term u^2. Comparing both side of equation (4.7) we have

$$u_0(x,\ y,\ t) = \sqrt{x^{\alpha} y^{\alpha}}$$

$$u_1(x,\ y,\ t) = J_t^{\alpha}[({}^{c}D_x^{2\alpha} + {}^{c}D_y^{2\alpha})A_0 + hu_0]$$

$$= J_t^{\alpha}[({}^{c}D_x^{2\alpha} + {}^{c}D_y^{2\alpha})(u_0)^2 + hu_0]$$

$$u_1(x,\ y,\ t) = h\sqrt{x^{\alpha} y^{\alpha}}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)$$

$$u_2(x,\ y,\ t) = J_t^{\alpha}[({}^{c}D_x^{2\alpha} + {}^{c}D_y^{2\alpha})A_1 + hu_1]$$

$$= J_t^{\alpha}[({}^{c}D_x^{2\alpha} + {}^{c}D_y^{2\alpha})(2u_0u_1) + hu_1]$$

$$u_2(x,\ y,\ t) = h^2\sqrt{x^{\alpha} y^{\alpha}}\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$$

$$u_3(x,\ y,\ t) = J_t^{\alpha}[({}^{c}D_x^{2\alpha} + {}^{c}D_y^{2\alpha})A_2 + hu_2]$$

$$= J_t^{\alpha}[({}^{c}D_x^{2\alpha} + {}^{c}D_y^{2\alpha})(u_1^2 + 2u_0u_2) + hu_2]$$

$$u_3(x,\ y,\ t) = h^3\sqrt{x^{\alpha} y^{\alpha}}\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right)$$

and so on. Then solution of IVP is

$$u(x,\ y,\ t) = \sum_{n=0}^{\infty} u_n(x,\ y,\ t) = u_0 + u_1 + u_2 + \dots$$

$$= \sqrt{x^{\alpha} y^{\alpha}}\left[1 + \frac{ht^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots\right]$$

$$= \sqrt{x^{\alpha} y^{\alpha}} \sum_{n=0}^{\infty} \frac{(ht^{\alpha})^n}{\Gamma(n\alpha + 1)}$$

$$u(x, y, t) = \sqrt{x^{\alpha} y^{\alpha}} E_{\alpha}(ht^{\alpha}) \tag{4.8}$$

It is exact analytical solution of IVP (4.3)-(4.4).

If $\alpha = 1$ then solution (4.8) reduces to

$$u(x, y, t) = \sqrt{xy}e^{ht} \tag{4.9}$$

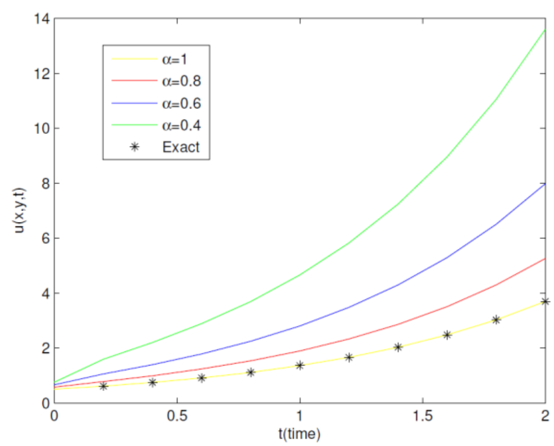

Figure 1. Graphical representation of solution (4.8) of IVP (4.3)-(4.4) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ and exact when $h = 1$ and $x = y = 0.50$.

which is an exact solution to the standard form biological population model.

Remark 4.1. Figure (1) shows the graphical behaviour of ADM solution of equation (4.8) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ and exact solution (4.9) when $x = y = 0.50$. Figure (2)(a); (b) and Figure (3)(c); (d) shows the surface of the 4 terms of the improved ADM solution (4.8) for values of $\alpha = 1, 0.8, 0.6$ and surface of exact solution (4.9). It is clear from Figure (1) and Figure (2) to (3), in the limit while $\alpha \to 1$, (4.8) approaches to the exact solution (4.9). Therefore, the improved ADM is an effective and sharp method which can be handled to detect exact analytical solution of fractional-order biological population equation.

4.2 Heat conduction model

In general heat conduction [12] was defined as the transfer of thermal energy from the more energetic particles of a medium to the adjacent less energetic ones. It was stated that conduction can take place in liquids and gases as well as solids provided that there is no bulk motion involved.

Consider two dimensional time-space fractional heat conduction equation

$$ {}^{c}D_{t}^{\alpha}u(x,\ y,\ t) = {}^{c}D_{x}^{\alpha}u(x,\ y,\ t) + {}^{c}D_{y}^{\alpha}u(x,\ y,\ t) + 2u^{2} - ({}^{c}D_{x}^{\alpha}u(x,\ y,\ t))^{2} - ({}^{c}D_{x}^{\alpha}u(x,\ y,\ t))^{2} \quad (4.10) $$

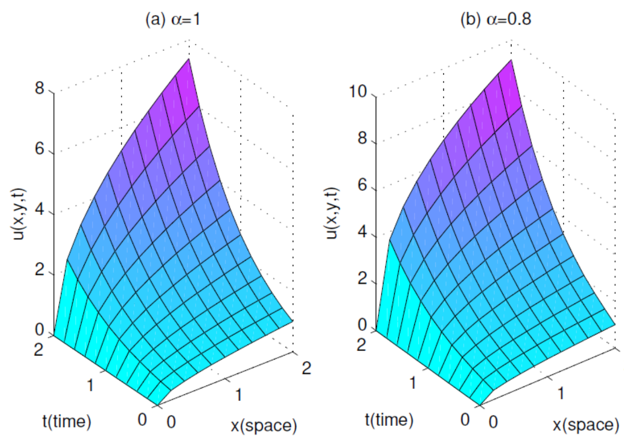

Figure 2. Graphical representation of solution (4.8) of IVP (4.3)-(4.4) when $\alpha = 1,\ 0.8$ with respect to time at $y = 0.50$.

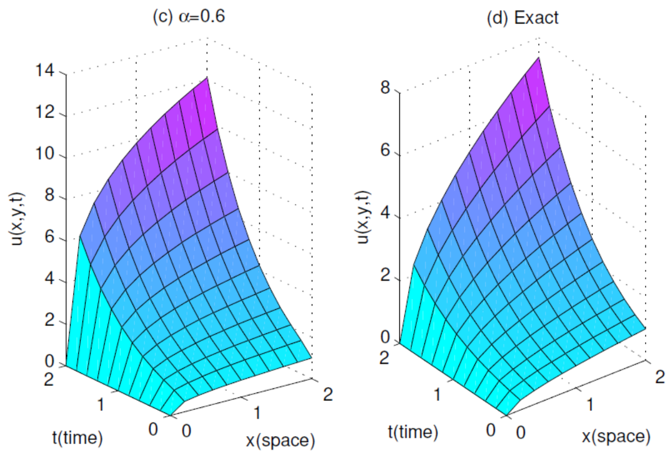

Figure 3. Graphical representation of solution (4.8) of IVP (4.3)-(4.4) when $\alpha = 0.6$ and exact with respect to time at $y = 0.50$.

with initial condition $u(x, y, 0) = f(x, y)$.

Example 4.2. Consider nonlinear time-space fractional order heat conduction equation

$$^{c}D_t^{\alpha}u(x, y, t) = {}^{c}D_x^{2\alpha}u + {}^{c}D_y^{2\alpha}u + 2u^2 - ({}^{c}D_x^{\alpha}u)^2 - ({}^{c}D_x^{\alpha}u)^2, \; 0 < \alpha \leq 1 \quad (4.11)$$

with initial condition

$$u(x, y, 0) = E_{\alpha}(x^{\alpha})E_{\alpha}(y^{\alpha}) \quad (4.12)$$

Solution. Applying J_t^{α} on both side of (4.11) we have,

$$J_t^{\alpha}\, {}^{c}D_t^{\alpha}u(x, y, t) = J_t^{\alpha}[{}^{c}D_x^{2\alpha}u + {}^{c}D_y^{2\alpha} + 2u^2 - ({}^{c}D_x^{2\alpha}u)^2 + {}^{c}D_y^{2\alpha}u)^2]$$

$$u(x, y, t) = u(x, y, 0) + J_t^{\alpha}[{}^{c}D_x^{2\alpha}u + {}^{c}D_y^{2\alpha}u + 2u^2 - ({}^{c}D_x^{\alpha}u)^2 - ({}^{c}D_y^{\alpha}u)^2]$$

$$u(x, y, t) = E_{\alpha}(x^{\alpha})E_{\alpha}(y^{\alpha}) + J_t^{\alpha}[{}^{c}D_x^{2\alpha}u + {}^{c}D_y^{2\alpha}u + 2u^2 - ({}^{c}D_x^{\alpha}u)^2 - ({}^{c}D_y^{\alpha}u)^2] \quad (4.13)$$

Suppose that the solution $u(x, t)$ has the following series form

$$u(x,\ y,\ t) = \sum_{n=0}^{\infty} u_n(x,\ y,\ t) \tag{4.14}$$

Then equation (4.13) has the form

$$\sum_{n=0}^{\infty} u_n(x,\ y,\ t) = E_\alpha(x^\alpha) E_\alpha(y^\alpha)$$

$$+ J_t^\alpha \left[({}^c D_x^{2\alpha} + {}^c D_y^{2\alpha}) \sum_{n=0}^{\infty} u_n + 2 \sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} C_n \right] \tag{4.15}$$

where A_n, B_n and C_n and are the Adomian polynomials to be determined from the nonlinear term u^2, $({}^c D_x^\alpha u)^2$ and $({}^c D_x^\alpha u)^2$. Comparing both side of equation (4.15) we have,

$$u_0(x,\ y,\ t) = E_\alpha(x^\alpha) E_\alpha(y^\alpha) \tag{4.16}$$

$$u_1(x,\ y,\ t) = J_t^\alpha [({}^c D_x^{2\alpha} + {}^c D_y^{2\alpha}) u_0 - 2A_0 - B_0 - C_0]$$

$$= J_t^\alpha [({}^c D_x^{2\alpha} + {}^c D_y^{2\alpha}) u_0 - 2u_0^2 - ({}^c D_x^\alpha u_0)^2 - ({}^c D_x^\alpha u_0)^2]$$

$$u_1(x,\ y,\ t) = 2E_\beta(x^\alpha) E_\alpha(y^\alpha) \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

$$u_2(x,\ y,\ t) = J_t^\alpha [({}^c D_x^{2\alpha} + {}^c D_y^{2\alpha}) u_1 - 2A_1 - B_1 - C_1]$$

$$= J_t^\alpha [({}^c D_x^{2\alpha} + {}^c D_y^{2\alpha}) u_0 - 2 \times 2u_0 u_1 - 2({}^c D_x^\alpha u_0)({}^c D_x^\alpha u_1)$$

$$- 2({}^c D_x^\alpha u_0)({}^c D_x^\alpha u_1)]$$

$$u_2(x,\ y,\ t) = 4E_\beta(x^\alpha) E_\alpha(y^\alpha) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$u_3(x,\ y,\ t) = J_t^\alpha [({}^c D_x^{2\alpha} + {}^c D_y^{2\alpha}) u_2 - 2A_2 - B_2 - C_2]$$

$$= J_t^\alpha [({}^c D_x^{2\alpha} + {}^c D_y^{2\alpha}) u_2 - 2((u_1)^2 + 2u_0 u_1) - ({}^c D_x^\alpha u_1)^2$$

$$- 2({}^c D_x^{\alpha} u_0)({}^c D_x^{\alpha} u_2) - ({}^c D_x^{\alpha} u_0)^2 - 2({}^c D_x^{\alpha} u_0)({}^c D_x^{\alpha} u_2)]$$

$$u_3(x, y, t) = 8E_{\beta}(x^{\alpha}) E_{\alpha}(y^{\alpha}) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

and so on. Then solution of IVP is

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) = u_0 + u_1 + u_2 + \dots$$

$$= E_{\alpha}(x^{\alpha}) E_{\alpha}(y^{\alpha}) \left[1 + \frac{2t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{4t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{8t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots\right]$$

$$u(x, y, t) = E_{\alpha}(x^{\alpha}) E_{\alpha}(y^{\alpha}) E_{\alpha}(2t^{\alpha}) \tag{4.17}$$

It is exact solution of IVP (4.11)-(4.12). If $\alpha = 1$ then solution (4.17) reduces to

$$u(x, y, t) = e^{x+y+2t} \tag{4.18}$$

which is an exact solution to the standard form heat conduction model.

Remark 4.2. Figure (4) shows the graphical behaviour of ADM solution of equation (4.17) for different values of α such as $\alpha = 1, 0.9, 0.8,$ and exact solution (4.18) when $x = y = 0.25.$ Figure (5)(a), (b) and (6)(c), (d) shows the surface of the 4 terms of the improved ADM solution (4.17) for values of $\alpha = 1, 0.8, 0.7$ and surface of exact solution (4.18). It is clear from Figure (4), (5) and (6), in the limit while $\alpha \to 1,$ (4.17) approaches to the exact solution (4.18). Therefore, the improved ADM is an effective and sharp method which can be handled to detect exact analytical solution of fractional-order heat conduction equation.

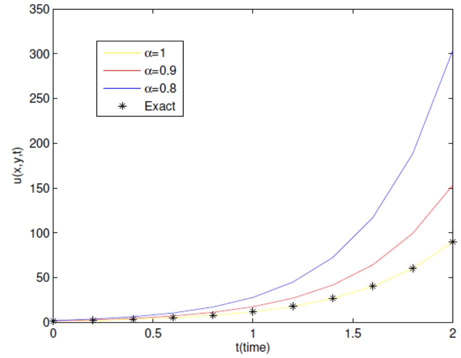

Figure 4. Graphical representation of solution (4.17) of IVP (4.11)-(4.12) for different values of α such as $\alpha = 1, 0.9, 0.8$ and exact when $x = y = 0.25$.

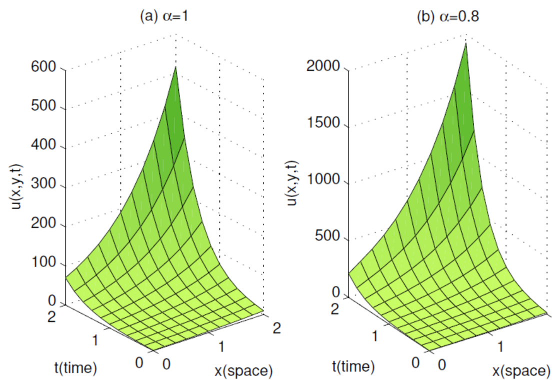

Figure 5. Graphical representation of solution (4.17) of IVP (4.11)-(4.12) when $\alpha = 1, 0.8$ with respect to time at $y = 0.25$.

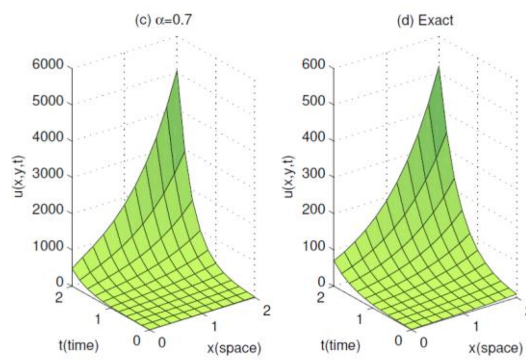

Figure 6. Graphical representation of solution (4.17) of IVP (4.11)-(4.12) when $\alpha = 0.7$ and exact with respect to time at $y = 0.25$.

5. Conclusion

Exact solution of nonlinear fractional order biological population model and heat conduction model with time-space Caputo fractional derivatives has been obtained by successful application of ADM. The solution of these models are in series form may have rapid convergence to a closed-form solution. Two and three dimensional graphical demonstrations ensure the high accuracy of the generated results using ADM. In the fields of applied mathematics generally in the recent emergence of nonlinear fractional differential equations, it is necessary to study various methods to obtain the solutions of such nonlinear fractional partial differential equations. That are why we expect that our work is a pace towards to obtain the exact solution of physical problems devoted to nonlinear fractional partial differential equations. The application of ADM is more convenient in solving every kind of nonlinear time-space fractional partial differential equations.

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