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k-EXTENSIBILITY AND WEAKLY *k*-EXTENSIBILITY FOR SOME SPECIAL TYPES OF GRAPHS

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Abstract

In this paper we discuss the k-extensibility and weakly k-extensibility for some special types of graphs like Bull graph, Butterfly graph, Durer graph, Friendship graph, Crown graph and Banana tree. Also some theorems, lemmas and corollaries are discussed.

1. Introduction

A simple graph is a finite non-empty set of object called vertices together with a set of unordered pairs of distinct vertices called edges. A graph G with

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vertex set $V(G)$ and edge set $E(G)$ is denoted by $G = (V, E)$. The edge $e = \{u, v\}$ and is denoted by $e = uv$ and it is said that *e* joins the vertices *u* and *v*; and *u* and *v* are called adjacent vertices; *u* and *v* said to be incident with *e*. If two vertices are not joined by an edge, then they are said to be non-adjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other. A set of vertices in a graph *G* is independent if no two vertices are adjacent and independent number is denoted by $\beta_0(G)$. An independent set is said to be maximal, if it is not a subset of any larger independent set. Let $G = (V, E)$ be a simple graph. Let *k* be a positive integer. *G* is said to be *k*-extendable if every independent set of cardinality *k* in *G* is contained in a maximum independent set of *G*.

Let *G* be a graph. Let *k* be a positive integer, $1 \leq k \leq |V(G)|$. G is said to be weakly *k*-extendable if every non-maximal independent set of cardinality *k* of *G* is contained in a maximum independent set of *G*.

2. Bull Graph

Definition 2.1. The bull graph is a planar undirected graph with 5 vertices and 5 edges. In the form of a triangle with two distinct pendent edges

Example 2.2. Let *G*:

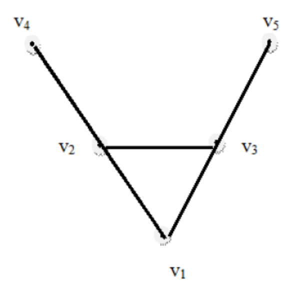

Figure 1

The β_0- set is $\{v_1, v_4, v_5\}$. Here $\beta_0(G) = 3$. $\{v_2\}, \{v_2, v_5\}$ are all independent set of cardinality 1 and 2. Which is not contained in the above β_0- sets of *G*. Therefore bull graph is not 1-extendable and not 2 extendable. It is only $\beta_0(G)$- extendable graph. It is also not weakly 1-extendable, not weakly 2-extendable for all $k, 1 \leq k \leq (\beta_0(G) - 1)$.

3. Butterfly Graph

Definition 3.1. The butterfly graph also called the bowtie graph and the hourglass graph is a planer undirected graph with 5 vertices and 6 edges.

Example 3.2. Let *G*:

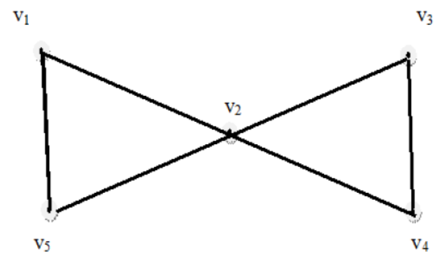

Figure 2

The β_0- sets are $\{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}$. Here $\beta_0(G) = 2$. Since $\{v_2\}$ independent set of cardinality 1 and non-maximal independent set of cardinality 1 which is not contained in any of the above β_0- sets of *G*. Hence *G* is not 1 extendable as well as not weakly 1-extendable.

4. Durer Graph

Definition 4.1. The Durer graph is an undirected graph with 12 vertices and 18 edges.

Example 4.2. Let *G*:

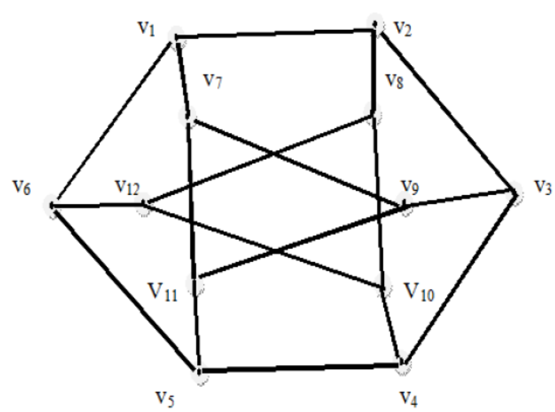

Figure 3

Here $\beta_0(G) = 4$. The β_0- sets are $\{v_1, v_3, v_5, v_8\}, \{v_1, v_3, v_5, v_{10}\}, \{v_1, v_3, v_5, v_{12}\}$ $\{v_2, v_4, v_6, v_7\}, \{v_2, v_4, v_6, v_9\}, \{v_2, v_4, v_6, v_{11}\}, \{v_3, v_5, v_7, v_8\}, \{v_4, v_6, v_7, v_8\}$, $\{v_4, v_6, v_8, v_9\}, \{v_1, v_5, v_8, v_9\}, \{v_1, v_5, v_9, v_{10}\}, \{v_2, v_6, v_9, v_{10}\}, \{v_2, v_6, v_{10}, v_{11}\}$, $\{v_1, v_3, v_{10}, v_{11}\}, \{v_1, v_3, v_{11}, v_{12}\}, \{v_2, v_4, v_{11}, v_{12}\}$. Any independent set of cardinality *k* is contained in any one of the above β_0- sets of *G*. Therefore *G* is *k*-extendable for all $k, 1 \leq k \leq \beta_0(G)$ and it is also weakly *k*-extendable for all $k, 1 \leq k \leq (\beta_0(G) - 1)$.

5. Friendship Graph

Definition 5.1. The friendship graph (or Dutch windmill graph or *n*-fan) F_n is a planer undirected graph with $2n + 1$ vertices and $3n$ edges.

Lemma 5.2. *Let* $G = F_n$ *be Friendship*, F_n *is k-extendable graph for all* $k, 2 \leq k \leq \beta_0(G)$ *except at* $k = 1$.

Proof. Let $G = F_n$ be a Friendship graph. Let *D* be an independent set of *G* cardinality $k, D = \{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{ir}\} \cup \{v_{j1}, v_{j2}, v_{j3}, \ldots, v_{js}\}$, where $r + s = k$ and $i_e \neq j_m$ for all l, m. The maximum independent set is $D \cup \{u_{si} : i \neq i_a$ and $i \neq j_a$ for all $a, b, 1 \leq a \leq r \quad 1 \leq b \leq s\}$ which contains any independent set of cardinality *k*, for all $k, 2 \leq k \leq \beta_0(G)$. Also, *G* contains only one-full degree vertex, which is not contained in *D*. Therefore G is not one-extendable. Hence *G* is *k*-extendable graph for all $k, 2 \leq k \leq \beta_0(G)$, except at $k = 1$.

6. Crown Graph

Definition 6.1. The *n*-Crown graph an integer $n \geq 3$ is the graph with vertex set $\{x_0, x_1, x_2, \ldots, x_{n-1}, y_0, y_1, y_2, \ldots, y_{n-1}\}$ and edge set $\{(x_i, y_i) : 0 \leq i, j \leq n - 1, i \neq j\}$.

Theorem 6.2. *Let G be a crown graph*, $(n \geq 3)\, G$ *is k-extendable graph for all* $k, 1 \leq k \leq \beta_0(G)$ except at $k = 2$.

Proof. Let *G* be a crown graph with vertices $(n \geq 3)$.

The vertex set of G be $V(G) = \{x_0, x_1, x_2, \dots, x_{n-1}, y_0, y_1, y_2, \dots, y_{n-1}\}$ and the edge set of G be $E(G) = \{(x_i, y_i) : 0 \leq i, j \leq n-1, i \neq j\}$. The maximum independent sets of G are $\{x_0, x_1, x_2, \dots, x_{n-1}\}, \{y_0, y_1, y_2, \dots, y_{n-1}\}$. Here $\beta_0(G) = n$. By the definition of Crown graph, there is no edge between (x_i, y_j) where $i = j$. That is $\{x_i, y_i\}$, is the independent set of cardinality two which is not contained in the above maximum independent set of G. Therefore G is not 2-extendable. From the construction of the maximum independent sets of G, it is clear that G is k-extendable for all $k, 1 \leq k \leq \beta_0(G)$ except at $k = 2$. Hence the proof.

Corollary 6.3. *In crown graph* $(n \geq 3)$, *we have n number of maximal independent sets of cardinality two. So we have non-maximal independent set of cardinality one, that clearly contains in any one of the maximum independent set of G. Therefore G is weakly* 1-*extendable*.

7. Banana Tree

Definition 7.1. An $(B_{n,k})$ banana tree graph obtained by connecting one leaf of each of n copies of an k-star graph with a single root vertex that is distinct from all the stars.

Theorem 7.2. *Let* $G = B_{n,k}$, *if* $\beta_0(n, k) = nk - n$, *then G is not extendable for all* $k, 1 \leq k \leq (nk - (2n-1))$ *and it is not weakly extendable for all* $k, 1 \leq k \leq (nk - (2n))$.

Proof. Let $G = B_{n,k}$. To prove this theorem by using induction on n, Let $n = 2$, $G = B_{2,k}$. The vertex set of $B_{2,k}$ be $V_1 = \{v_1, v_2, \dots, v_{2k+1}\}$. The graph of $G = B_{2,k}$ is given below,

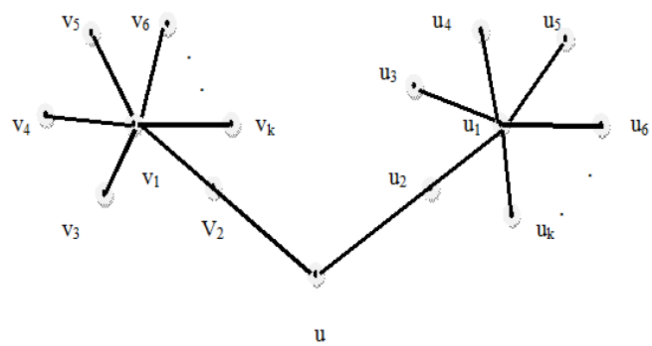

Figure (4)

From the above graph, we seen that the maximum independent set is, $\{v_2, v_3, \ldots, v_k, u_2, u_3, \ldots, u_k\}$, $\beta_0(B_{2,k}) = 2k - 2 \cdot \{u\}, \{u, v_3\}, \{u, v_3, v_4\}$, $\{u, v_3, v_4, v_5\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3\}, \{u, v_3, v_4,$ $v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k\}$ are all independent set of cardinality *k*, for all $k, 1 \leq k \leq (2k - 3)$, which is not contained in the above β_0-sets of $B_{2,k}$. Therefore $B_{2,k}$ is not *k*-extendable for all $k, 1 \leq k \leq (\beta_0(B_{2,k}) - 1)$.

$\{u, v_3, v_4, v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_{k-1}\}$ are all non-maximal independent set of $B_{2,k}$, which is not contained in the above β_0-sets of $B_{2,k}$. Hence $B_{2,k}$ is not weakly *k*-extendable for all $k, 1 \leq k \leq (\beta_0(B_{2,k}) - 2)$. Suppose $n = 3$, $G = B_{n,k}$. The vertex set of $B_{3,k}$ be $V_2 = \{v_1, v_2, \ldots, v_{3k+1}\}$. The graph of $G = B_{3,k}$ is given below,

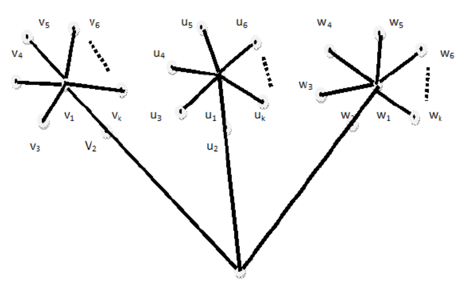

Figure 5

From the above graph, we seen that the maximum independent set is, $\{v_2, v_3, \ldots, v_k, u_2, u_3, \ldots, u_k, w_2, w_3, \ldots, w_k\}$, $\beta_0(B_{3,k}) = 3k - 3$.

$\{u\}, \{u, v_3\}, \{u, v_3, v_4\}, \{u, v_3, v_4, v_5\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k, w_3, w_4, \ldots, w_k\}$ are all independent set of cardinality k, for all $1 \leq k \leq (3k - 5)$, which is not contained in the above β_0-sets of $B_{3,k}$. Therefore $B_{3,k}$ is not k-extendable for all $k, 1 \leq k \leq (\beta_0(B_{3,k}) - 2) \cdot \{u, v_3, v_4, v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k, w_3\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k, w_3, w_4, \ldots, w_{k-1}\}$ are all non-maximal independent set of $B_{3,k}$, which is not contained in the above β_0-sets of $B_{3,k}$. Hence $B_{3,k}$ is not weakly k-extendable for all $k, 1 \leq k \leq (\beta_0(B_{3,k}) - 3)$. Suppose the result is true for n, $B_{n,k}$ is not k-extendable for all $k, 1 \leq k \leq nk - (2n - 1)$. Also $B_{n,k}$ is not weakly k-extendable for all $k, 1 \leq k \leq nk - (2n)$. Since the result is true for n, that means β_0-set of $B_{n,k}$ contains $(nk - n)$-vertices. In $B_{n+1,k}$, the graph $B_{n+1,k}$ is partition into two graphs one is $B_{n,k}$ and another one is a subgraph which is nothing but $K_{1,n-1}$. Let $B_{n+1,k}$. The vertex set of $B_{n+1,k}$ be $V = \{v_1, v_2, \ldots, v_{(nk+1)+k}\}$. The graph of $G = B_{n+1,k}$ is given below,

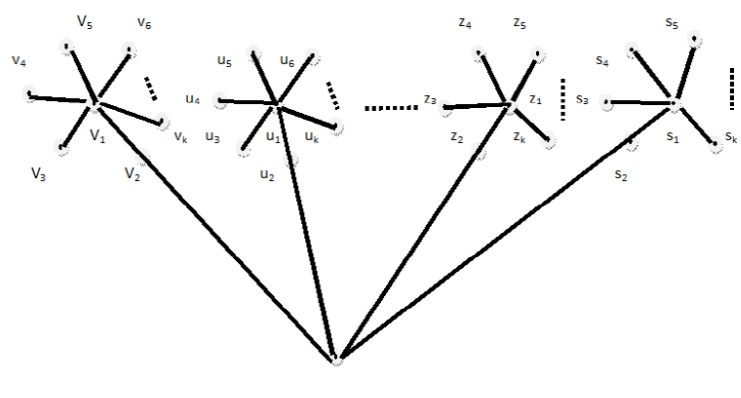

Figure 6

From the above graph, we seen that the maximum independent set is,

$\{v_2, v_3, \ldots, v_k, u_2, u_3, \ldots, u_k, w_2, w_3, \ldots, w_k, \ldots, z_1, z_2, \ldots, z_k, s_1, s_2, \ldots, s_k\}$.

Therefore $\beta_0(B_{n+1,k}) = nk - n + (k-1) \cdot \{u\}, \{u, v_3\}, \{u, v_3, v_4\}, \{u, v_3, v_4, \ldots, v_k\}, \ldots, \{u, v_3, \ldots, v_k, u_3, \ldots, u_k, \ldots, z_1, \ldots, z_k, S_3, \ldots, S_k\}$ are all independent set of $B_{n+1,k}$, which is not contained in the above β_0- sets of $B_{n+1,k}$. Therefore $B_{n+1,k}$ is not k-extendable for all $k, 1 \leq k \leq 1 + (nk - 2n) + (k-2)\{u\}$, $\{u, v_3\}, \{u, v_3, v_4\}, \{u, v_3, v_4, \ldots, v_k\}, \ldots, \{u, v_3, \ldots, v_k, u_3, \ldots, u_k, \ldots, z_1, \ldots, z_k, s_3, \ldots, s_{k-1}\}$ are all non-maximal independent set of $B_{n+1,k}$, which is not contained in the above β_0- sets of $B_{n+1,k}$. Therefore $B_{n+1,k}$ is not weakly k-extendable for all $k, 1 \leq k \leq 1 + (nk - 2n) + (k-3)$. Hence the proof.

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