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# Existence and construction of $\alpha$-valuations for quadratic graph $Q(7,4k)$ and its extensions

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**Abstract** A quadratic graph $Q(m,n)$ is a graph with $m$ vertex-disjoint cycles of length $n$. A graceful valuation is a concept in graph theory which has attracted attention from many researchers during the past three decades. In this paper, it is proved that there is an $\alpha$-valuation, which is a stronger form of graceful valuation, for the quadratic graph $Q(7,4k)$ for every $k \geq 1$. Also, some additional results are obtained from the main theorem.

**Keywords** Graph labeling; $\alpha$-valuation; quadratic graph; graceful labeling
**AMS subject classifications** 05C78

## 1 Introduction

Let $G=(V,E)$ be a graph with $m=|V|$ vertices and $n=|E|$ edges. For the rest of this paper the term graph is used to refer to an undirected graph without loops or multiple edges. A graceful valuation (or $\beta$-valuation) of a graph $G=(V,E)$ is a one-to-one mapping $\Psi$ of the vertex set $V(G)$ into the set $\{0,1,2,\dots,n\}$ with this property: If we define, for any edge $e=\{u,v\}\in E(G)$, the value $\Psi^*(e)=|\Psi(u)-\Psi(v)|$, then $\Psi^*$ is a one-to-one mapping of the set $E(G)$ onto the set $\{1,2,\dots,n\}$.

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A graph which has a graceful valuation is called graceful. An $\alpha$-valuation of a graph $G=(V,E)$ is a graceful valuation $\Psi$ of $G$ which satisfies the following additional condition: There exists a number $\gamma$ ($0 \le \gamma \le |E(G)|$) such that, for any edge $e \in E(G)$ with end vertices $u, v \in V(G)$, $\min\{\Psi(u), \Psi(v)\} \le \gamma < \max\{\Psi(u), \Psi(v)\}$. The values of the $\alpha$-valuation $\Psi$ that are at most $\gamma$ are the *small values*, and the remaining values of $\Psi$ are the *large values*.

Reviewing the literature of graph labeling shows that Rosa introduced the problem of graceful labeling and $\alpha$-valuation in [10]. He showed that if a graph $G$ is graceful and all of its vertices are of even degree, then $|E(G)| \equiv 0$ or 3 (mod 4). He observed that if $G$ has an $\alpha$-valuation, then $G$ is bipartite. Thus if in addition, all vertices of $G$ are of even degrees, then $|E(G)| \equiv 0$ (mod 4). He also proved that if $G$ is a cycle, the above mentioned conditions are sufficient. A cycle with $m$ vertices is denoted by $C_m$. For a detailed history of graph labeling problems and an introduction to some basic terms, the reader is referred to Gallian [7] and Eshghi [6], respectively.

One of the earlier results of Abrham and Kotzig (see [8]) is the following.

**Theorem 1** [8] *If $G$ is a 2-regular graph on $n$ vertices and $n$ edges which has a graceful valuation $\Psi$, then there exists exactly one integer $x$ ($0 < x < n$) such that $\Psi(v) \neq x$ for all $v \in V(G)$; this number $x$ is referred to as the missing value of the graceful valuation. Moreover, if $n = 4k$ and $\Psi$ is an $\alpha$-valuation of $G$, then either $x = k$ or $x = 3k$.*

A quadratic graph $Q(r,s)$ is a graph with $r$ components, each of which is an $s$-cycle.

A summary of the main results published in the references regarding to the graceful labeling or $\alpha$-valuation of quadratic graphs are as follows:

1. A $Q(1,s)$-graph (i.e. an $s$-cycle) is graceful if and only if $s \equiv 0$ or 3 (mod 4). It has an $\alpha$-valuation if and only if $s \equiv 0$ (mod 4) [10].

2. A $Q(2,s)$-graph has an $\alpha$-valuation if and only if $s$ is even and $s > 2$ [8].

3. A $Q(r,4)$-graph has an $\alpha$-valuation for $r \neq 3$ [1].

4. A $Q(3,4k)$-graph has an $\alpha$-valuation for each $k > 1$. The graph $Q(3,4)$ does not have an $\alpha$-valuation but it is graceful [8].

5. A $Q(4,4k)$-graph has an $\alpha$-valuation [9].

6. A $Q(5,4k)$-graph has an $\alpha$-valuation [4].

The rest of this paper is organized as follows. In Section 2, it is proved that there is an $\alpha$-labeling for the graph $Q(7,4k)$, and this is the main theorem of this paper. In Section 3, the labelings for a general class of graphs that are derived from $Q(7,4k)$, are found using the standard valuations for $C_{4k}$. The conclusion of the paper and some suggestions for future research are presented in Section 4.

## 2 Main theorem

**Theorem 2** *The quadratic graph $Q(7,4k)$ has an $\alpha$-valuation for all $k \geq 1$.*

*Proof.* We will construct an $\alpha$-valuation of this graph with missing value $21k$ and $\gamma = 14k$. Let us first assume $k \geq 3$.

For $k \geq 3$, the labeling is constructed as follows:

**The first cycle:**

Vertex labels: $\{12k, 16k, 12k+1, 16k-1, \ldots, 13k-2, 15k+2, 13k-1, 15k+1, 13k, 15k-1, 13k+1, 15k-2, \ldots, 14k-2, 14k+1, 14k-1, 14k\}$.

Edge labels: $[1, 2, \ldots, 4k-1, 4k]$.

Figure 1 illustrates this labeling.

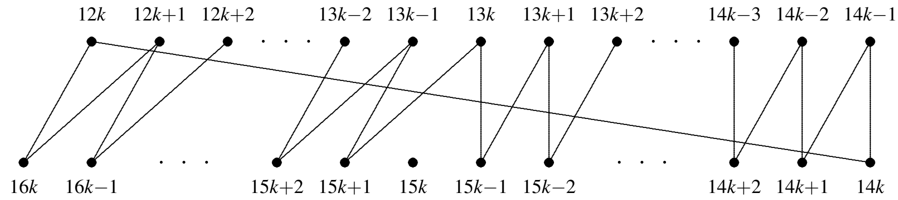


Figure 1. Labeling of the first cycle

**The second cycle:**

Vertex labels: $\{10k-1,\ 18k,\ 10k,\ 18k-1,\ 10k+1,\ \ldots,\ 11k-3,\ 17k+2,\ 11k-2,\ 17k+1, 11k, 17k, 11k+1, \ldots, 12k-2, 16k+2, 12k-1, 16k+1\}$.

Edge labels: $[4k+2, 4k+3, \ldots, 8k, 8k+1]$.

Figure 2 illustrates this labeling.

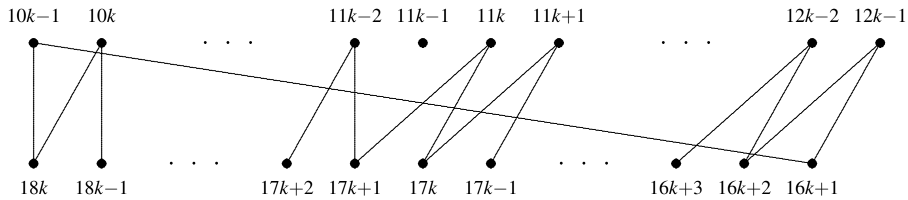


Figure 2. Labeling of the second cycle

**The third cycle:**

Vertex labels: $\{2k+1,\ 26k,\ 2k+2,\ 26k-1,\ \ldots,\ 25k+2,\ 3k,\ 25k+1,\ 3k+1,\ 25k-1, 3k+2, 25k-2, 3k+3, \ldots, 4k-2, 24k+2, 4k-1, 24k+1, 4k, 24k\}$.

Edge labels: $[20k, 20k+1, \ldots, 24k-2, 24k-1]$.

Figure 3 illustrates this labeling.

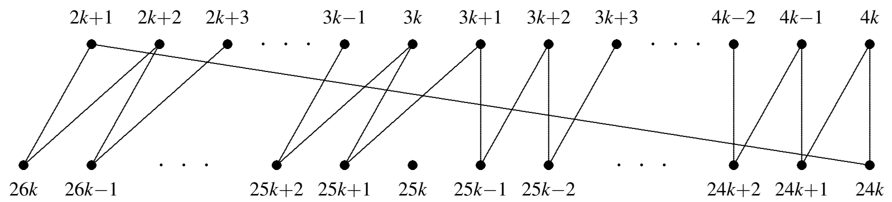


Figure 3. Labeling of the third cycle

**The fourth cycle:**

Vertex labels: $\{0, 28k, 1, 28k-1, \dots, k-1, 27k+1, k+1, 27k, k+2, 27k-1, \dots, 26k+3, 2k-1, 26k+2, 2k, 26k+1\}$.

Edge labels: $[24k+1, 24k+2, \dots, 28k-1, 28k]$.

Figure 4 illustrates this labeling.

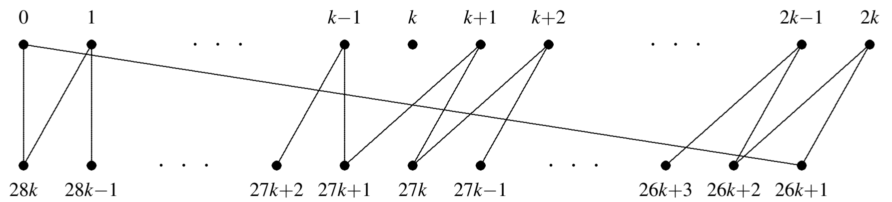


Figure 4. Labeling of the fourth cycle

**The fifth cycle:**

Figure 5 illustrates the labeling for this cycle. As seen in this figure, this cycle is constructed using the following steps:

1. Connect the vertices $(21k-1, 9k-1)$ to generate the edge label $12k$.

2. Beginning from the vertex labeled $9k-1$, successive vertices are labeled by $\{9k-1, 19k, 9k, 19k-1, \dots, 10k-2, 18k+1\}$, which is clearly presented in Figure 5. Therefore, the edge labels $[8k+3, 8k+4, \dots, 10k+1]$ will be obtained.

3. Connect the vertices labeled $(18k+1, 8k-1)$ to create edge label $10k+2$.

4. Remaining labels for vertices are connected as $\{21k-1, 7k, 21k-2, \dots, 8k-2, 20k, 8k-1\}$. Therefore, the edge labels are generated as $[12k+1, 12k+2, \dots, 14k-2, 14k-1]$.

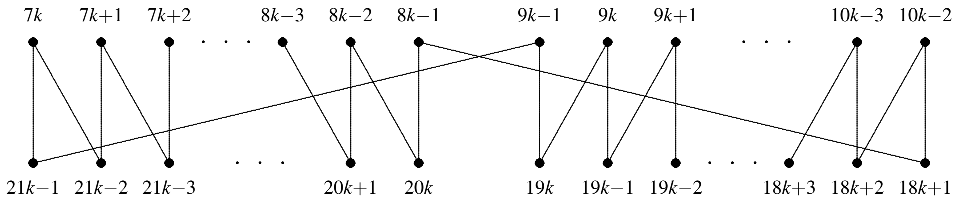


Figure 5. Labeling of the fifth cycle

**The sixth cycle:**

The labeling for this cycle is very similar to the fifth cycle. This cycle is labeled using the following steps:

1. Connect the vertices labeled $(4k+1, 22k)$ to generate the edge label $18k-1$.

2. Beginning from $4k+1$, vertices are successively labeled as $\{4k+1, 24k-1, 4k+2, 24k-2, \ldots, 5k-1, 23k+1, 5k, 23k\}$. Therefore, edge labels are calculated as $[18k, 18k+1, \ldots, 20k-2]$.

3. Connect the vertex labels $(23k, 7k-1)$ to generate the edge label $16k+1$.

4. Remaining labels for the vertices are $\{22k, 6k, 22k-1, 6k+1, \ldots, 21k+1, 7k-1\}$. Edge labels are then calculated as $[14k+2, 14k+3, \ldots, 16k-1, 16k]$.

Figure 6 illustrates labeling for this cycle.

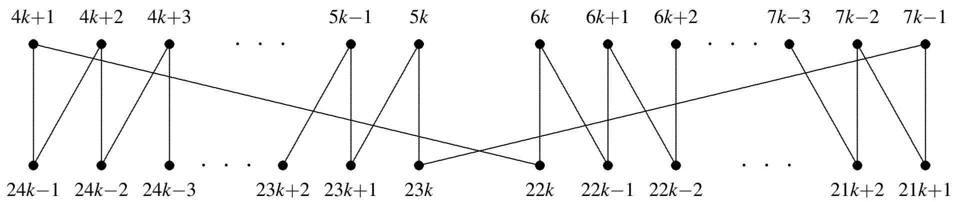


Figure 6. Labeling of the sixth cycle

**The seventh cycle:**

Now we construct the last cycle $C_{4k}$ according to the following stages:

1. Vertex labels $\{5k+1, 23k-1, 5k+2, 23k-2, \ldots, 6k-1, 22k+1\}$ generate the edge labels $[16k+2, 16k+3, \ldots, 18k-2]$.

2. Connect the vertex label $(22k+1, 8k)$ to generate the edge label $14k+1$.

3. Beginning from the vertex label $(8k)$, vertices will be successively labeled as $\{8k$, $20k-1$, $8k+1$, $20k-2$, $\dots$ , $9k-2$, $19k+1\}$. Then, the edge labels $[10k+3$, $10k+4,\dots,12k-1]$ will be generated.

4. Finally, the remaining edge labels $[8k+2, 4k+1, 14k, 24k, 20k-1]$ are generated with the vertex labels: $\{19k+1, 11k-1, 15k, k, 25k, 5k+1\}$.

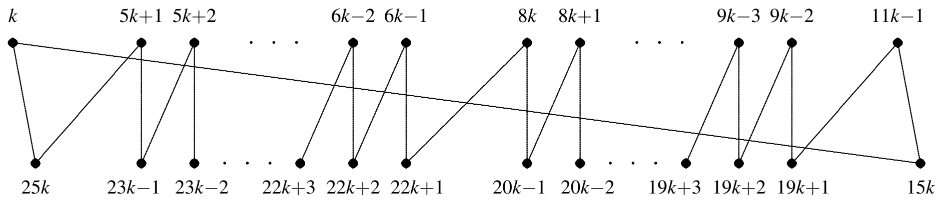


Figure 7. Labeling of the seventh cycle

For $k = 1,2$ the labeling is shown in Table 1.

| $k$ | The construction of an $\alpha$-valuation of $Q(7,4k)$ |
|---|---|
| 1 | $[(28,0,26,1),(25,3,27,4),(24,5,23,7),(16,12,14,13),(17,2,22,8),$ $(19,11,18,6),(15,10,20,9)]$ |
| 2 | $[(24,32,25,31,26,29,27,28),(19,36,20,35,22,34,23,33),$ $(5,52,6,51,7,49,8,48),(0,56,1,55,3,54,4,53),$ $(41,14,40,15,37,18,38,17),(9,47,10,46,13,43,12,44),$ $(21,30,2,50,11,45,16,39)]$ |

Table 1. Construction of $\alpha$-valuation of $Q(7,4k)$ for $k = 1,2$

Therefore, for any positive integer $k$, we showed that there is an $\alpha$-valuation for the quadratic graph $Q(7,4k)$ and the proof is complete. □

## 3 The standard valuation of $C_{4k}$

In this section we define the standard valuation of $C_{4k}$, which was introduced by Eshghi [5]. Then in Theorem 4, the labelings for some graphs which are derived from $Q(7,4k)$ are determined.

To proceed, we recall the following result of Abrham and Kotzig [2].

**Theorem 3** [2] *Let $p,q$ be positive integers, $p \geq 3$, $q \geq 3$. Then the graph $C_p \cup C_q$ has a graceful valuation if and only if $p+q \equiv 0$ or 3 (mod 4). $C_p \cup C_q$ has an $\alpha$-valuation if and only if both $p,q$ are even and $p+q \equiv 0$ (mod 4).*

**Definition 1** The *standard $\alpha$-valuations* of $C_{4k}$ are given by any of the following sequence of values for the consecutive vertices of $C_{4k}$:

(a) $\{4k, 0, 4k-1, 1, 4k-2, 2, \ldots, k-2, 3k+1, k-1, 3k, k+1, 3k-1, k+2, 3k-2, \ldots, 2k+2, 2k-1, 2k+1, 2k\}$ with missing value $x = k$.

(b) $\{0, 4k, 1, 4k-1, 2, 4k-2, \ldots, k-2, 3k+2, k-1, 3k-1, k+1, 3k, k+2, 3k-1, \ldots, 2k-2, 2k+2, 2k, 2k+1\}$ with missing value $x = k$.

(c) $\{4k,\ 0,\ 4k-1,\ 1,\ 4k-2,\ 2,\ \ldots,\ k-2,\ 3k+1,\ k-1,\ 3k-1,\ k,\ 3k-2,\ \ldots,\ 2k+1, 2k-2, 2k, 2k-1\}$ with missing value $x = 3k$.

(d) $\{0, 4k, 1, 4k-1, 2, 4k-2, \ldots, k-2, 3k+2, k-1, 3k+1, k, 3k-1, k+1, \ldots, 2k-2, 2k+1, 2k-1, 2k\}$ with missing value $x = 3k$.

Figure 8 shows the second of the standard $\alpha$-valuations of $C_{12}$:

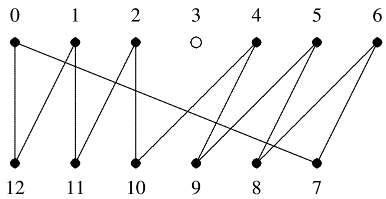


Figure 8. A standard $\alpha$-valuation of $C_{12}$

Now, we want to show how to extend from a standard $\alpha$-valuation of $C_{4k}$ to an $\alpha$-valuation of a disjoint union of cycles with order $4k$. For example, consider how we may obtain an $\alpha$-valuation of $2C_6$ from a standard $\alpha$-valuation of $C_{12}$. Suppose that we have the standard $\alpha$-valuation of $C_{12}$ as in Figure 8. By Theorems 1 and 3, we see that $2C_6$ has an $\alpha$-valuation, where the missing value is either $k = 3$ or $3k = 9$. If the missing value is 9, then we can obtain an $\alpha$-valuation of $2C_6$ with missing value 3 by replacing the vertex labeled with $y$ with $4k - y = 12 - y$. Thus, $2C_6$ has an $\alpha$-valuation with missing value 3. We may then replace $C_{12}$ with $2C_6$, by matching the vertex labels of the two valuations. Figure 9 below shows such a possible $\alpha$-valuation of $2C_6$.

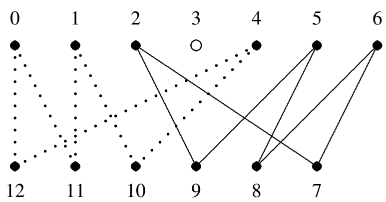


Figure 9. An $\alpha$-valuation of $2C_6$

Next, we define a standard valuation for the graph $C_{4k}$, as follows.

**Definition 2** A *standard valuation* (or *standard labeling*) of the graph $C_{4k}$ is a vertex labeling that can be generated as follows: We take a standard $\alpha$-valuation $\Psi$ of $C_{4k}$ with the corresponding $\gamma$, and then add a constant $a$ to the values of the small vertices $\{v \in V(G) : \Psi(v) \leq \gamma\}$, and add a constant $b$ to the values of the large vertices $\{v \in V(G) : \Psi(v) > \gamma\}$. Note that the edge values of the standard valuation of $C_{4k}$ are $[b-a+1, b-a+2, \ldots, b-a+4k]$.

Having a standard labeling for a part of a graph enables us to replace it with a modified $\alpha$-valuation, where the small values and large values are shifted by two additive constants.

For example $C_{12}$ in the $\alpha$-valuation of $C_{12} \cup C_{20}$, shown in Figure 10, has a standard valuation because it can be generated from a standard $\alpha$-valuation of $C_{12}$ with $a = 0$ and $b = 20$.

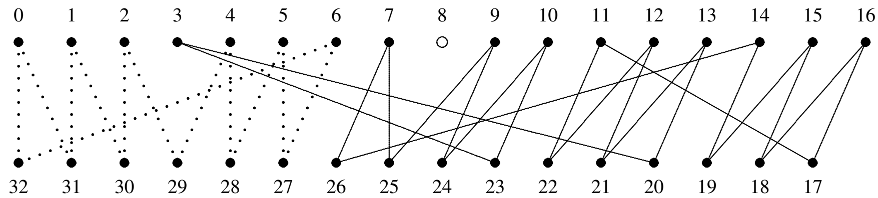


Figure 10. An $\alpha$-valuation of $C_{12} \cup C_{20}$

Thus, in view of the procedure described earlier when $C_{12}$ was replaced by $2C_6$, the standard valuation of $C_{12}$ can be replaced by an $\alpha$-valuation of $2C_6$ to form an $\alpha$-valuation of $2C_6 \cup C_{20}$ in Figure 11. We may simply take the $\alpha$-valuation of $2C_6$, increase the small values by $a = 0$ and the large values by $b = 20$, and then match the vertex values of this new valuation of $2C_6$ to that of the standard valuation of $C_{12}$.

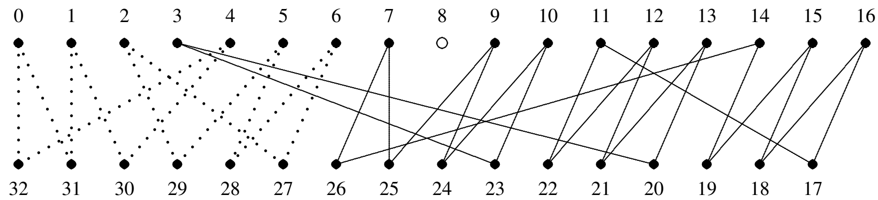


Figure 11. An $\alpha$-valuation of $2C_6 \cup C_{20}$

In general, if a graph $C_{4k}$ has a standard labeling then it can be replaced by any existing $\alpha$-valuation of disjoint union of cycles with the same number of vertices and missing value by considering the additive constants. In other words, if a graph $C_{4k}$ has a standard valuation it can be replaced by any $\alpha$-valuation of $C_{k_1} \cup C_{k_2} \cup \cdots \cup C_{k_n}$ where $k_1 + k_2 + \cdots + k_n = 4k$, by considering the additive constants $a$ and $b$. Using this approach, we obtain Theorem 4 as follows, where we obtain a larger class of 2-regular graphs with $\alpha$-valuations.

Let $A(k)$ denote the family of 2-regular graphs of the form $C_{4k_1} \cup C_{4k_2} \cup \cdots \cup C_{4k_p}$, where $p \geq 1$, $k_1 + \cdots + k_p = k$, and $k_i \geq \sum_{j=i+1}^{p} k_j$ for $i = 1, 2, \ldots, p-1$. Let $B(k)$ denote the family which consists of $C_{4k}$ and all 2-regular graphs of the form

$$C_{4k_1} \cup C_{4\ell_1} \cup C_{4k_2} \cup C_{4\ell_2} \cup \cdots \cup C_{4k_q} \cup C_{4\ell_q} \cup C_{4k_q},$$

where $q \geq 1$, $\sum_{i=1}^{q}(k_i + \ell_i) + k_q = k$, and $k_i = 2k_{i+1} + \ell_{i+1}$ for $i = 1, 2, \ldots, q-1$.

**Theorem 4** *The following graphs have $\alpha$-valuations.*

*(a)* $G_1 \cup G_2 \cup G_3 \cup G_4 \cup Q(3,4k)$*, where* $G_1, G_2, G_3, G_4 \in A(k)$.

*(b)* $H_1 \cup H_2 \cup H_3 \cup H_4 \cup Q(3,4k)$*, where* $H_1, H_2, H_3, H_4 \in B(k)$.

Before we prove Theorem 4, we recall the following proposition, which contains results of Abrham and Kotzig [2], and Eshghi [5].

**Proposition 5** [2, 5]

*(a) Let $1 \leq p \leq q$ be positive integers. Then the graph $C_{4p} \cup C_{4q}$ has an $\alpha$-valuation, where $C_{4p}$ has a standard valuation.*

*(b) Let $p, q \geq 1$ be positive integers. Then the graph $2C_{4p} \cup C_{4q}$ has an $\alpha$-valuation, where at least one of the copies of $C_{4p}$ has a standard valuation.*

*Proof of Theorem 4.* We know that in the construction of the $\alpha$-valuation of $Q(7,4k)$, the first four cycles have standard valuations. In order to obtain the two parts of the Theorem 4, we replace these four cycles with 2-regular graphs having the same vertex values, as follows:

(a) Consider one of the first four cycles $C_{4k}$. It has a standard valuation, which is obtained by taking one of the four standard $\alpha$-valuations of $C_{4k}$ as given in Definition 1, say $\Psi$, and adding some constant $a$ to the small values, and some constant $b$ to the large values. The missing value of $\Psi$ is either $k$ or $3k$. Now suppose that $k = k_1 + \ell_1$ with $k_1 \geq \ell_1 \geq 1$. By Proposition 5(a), $C_{4k_1} \cup C_{4\ell_1}$ has an $\alpha$-valuation $\Psi'$, with $C_{4\ell_1}$ having a standard valuation. By Theorem 1, the missing value of $\Psi'$ is either $k$ or $3k$. If the missing values of $\Psi$ and $\Psi'$ are different, then we may replace $\Psi'$ with $\Psi'' = 4k - \Psi'$, which is still an $\alpha$-valuation of $C_{4k_1} \cup C_{4\ell_1}$ where $C_{4\ell_1}$ has a standard valuation, but now $\Psi$ and $\Psi''$ have the same missing value. We may thus assume that $\Psi$ and $\Psi'$ have the same missing value of either $k$ or $3k$. Note also that $\Psi$ and $\Psi'$ have the same sets of small values and large values, with $\gamma = 2k$ if the common missing value is $k$, and $\gamma = 2k-1$ if the common missing value is $3k$. Therefore, we may replace the $C_{4k}$ with $C_{4k_1} \cup C_{4\ell_1}$ by first adding $a$ to the small values of $\Psi'$ and adding $b$ to the large values of $\Psi'$, and then match the resulting values of the vertices to the vertex values of the standard valuation of $C_{4k}$. Now since $C_{4\ell_1}$ still has a standard valuation, we are able to similarly replace it again by $C_{4k_2} \cup C_{4\ell_2}$, where $\ell_1 = k_2 + \ell_2$ and $k_2 \geq \ell_2 \geq 1$. In the next steps, we continue to replace each $C_{4\ell_i}$ recursively by $C_{4k_{i+1}} \cup C_{4\ell_{i+1}}$, where $\ell_i = k_{i+1} + \ell_{i+1}$ and $k_{i+1} \geq \ell_{i+1} \geq 1$ for $i = 2, 3, \ldots, p-2$, and $\ell_{p-1} = k_p$. The result follows by performing this procedure on the first four cycles of $Q(7, 4k)$.

(b) This part can be proved similarly. Again, we consider one of the first four cycles. Using Proposition 5(b) and Theorem 1, we may similarly replace one of the first four cycles $C_{4k}$ with $2C_{4k_1} \cup C_{4\ell_1}$, where $k = 2k_1 + \ell_1$. Since at least one of the $C_{4k_1}$ has a standard valuation, we can replace $C_{4k_1}$ in the next step by $2C_{4k_2} \cup C_{4\ell_2}$, where $k_1 = 2k_2 + \ell_2$. In next steps, we recursively replace $C_{4k_i}$ by $2C_{4k_{i+1}} \cup C_{4\ell_{i+1}}$, where $k_i = 2k_{i+1} + \ell_{i+1}$ for $i = 1, 2, \ldots, q-1$. □

**Remark** This theorem can be extended to generate $\alpha$-valuations of many general classes of 2-regular graphs, by replacing each standard valuation of $C_{4k}$ by any other 2-regular graphs which have $\alpha$-valuation with $4k$ vertices. Therefore, it can be easily shown that an infinite number of graphs in the class of disjoint union of cycles have $\alpha$-valuations.

## 4 Conclusions and future research

In this paper, it is proved that there is an $\alpha$-valuation for the quadratic graph $Q(7, 4k)$, which consists of seven cycles of length $4k$, for every $k \geq 1$. Furthermore the standard valuation for $C_{4k}$, is defined and used to determine the $\alpha$-labelings for general classes of graphs that are derived from $Q(7, 4k)$.

The main idea in this paper can be used to generate $\alpha$-labelings for the open problems in quadratic graphs. Finding an $\alpha$-labeling for other quadratic graphs like $Q(6, 4k)$ is interesting. Furthermore, generalization of the main theorem of this paper, can be investigated to extend the classes of disjoint union of cycles having $\alpha$-labelings.

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