MINIMUM HOP DOMINATING ENERGY OF A GRAPH

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Abstract

Let $G = (V, E)$ be a $(p, q)$ graph. A subset $S \subseteq V(G)$ is a minimum hop dominating set of a graph $G$ if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set (MHD) is called the hop dominating number. Motivated by the definition of minimum dominating energy by Rajeshkanna et al. [11], in this article the concept of minimum hop dominating energy $E_h(G)$ is introduced and minimum hop dominating energies $E_h(G)$ of some standard graphs and a few well-known families of graphs are computed. Also we establish an upper bound and lower bound for $E_h(G)$.

1. Introduction

In this article we consider finite, undirected and simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and number of edges are denoted by $p$ and $q$ respectively. A complete graph $K_p$ is a simple undirected graph in which every pair of distinct vertices is connected by a...
unique edge. The star graph $K_{1, p-1}$ of order $p$ is a tree on $p$ nodes with one node having vertex degree $p - 1$ and the other $p - 1$ having vertex degree $1$. The characteristic polynomials are computed using an online tool https://www.dcode.fr/matrix-eigenvalues. The distance $d(u, v)$ is the length of the minimum path from vertex $u$ to vertex $v$. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the eigenvalues of $G$. Then the spectrum of $G$ is denoted by

$$\text{spec } G = [\lambda_1 \lambda_2 \ldots \lambda_p]$$

where $m_i$ is the algebraic multiplicity of the eigenvalues $\lambda_i$, for $1 \leq i \leq p$.

The concept of energy $E(G)$ was proposed by I. Gutman in 1978 [3] as the sum of absolute values of the eigenvalues of the adjacency matrix of $G$. The basic properties including various upper and lower bounds for energy of a graph have been established in [7], [9]. Many researchers have introduced various concepts of energies. Motivated by the definition of minimum dominating energy by Rajeshkanna et al., [11], in this article the concept of minimum hop dominating energy $E_h(G)$ is introduced and minimum hop dominating energies $E_h(G)$ of some standard graphs and a few well-known families of graphs are computed. Also we establish an upper bound and lower bound for $E_h(G)$.

## 2. Minimum Hop Dominating Energy

Let $G = (V, E)$ be a $(p, q)$ graph. Let $S$ be a minimum hop dominating set of $G$. Then the minimum hop dominating matrix corresponding to $S$ is a square matrix of order $p$ and is defined as $H_S(G) = \begin{cases} 1 & \text{if } v_iv_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in S \\ 0 & \text{otherwise} \end{cases}$

The characteristic polynomial of $H_S(G)$ is denoted by $f_p(G, \lambda) = \det (H_S(G) - \lambda I)$. The MHD eigenvalues of the graph $G$ are the eigenvalues of $H_S(G)$ and is known as hop dominating spectra of $G$. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the eigenvalues of $H_S(G)$. Then the minimum hop
dominating energy of $G$ is defined as $E_h(G) = \sum_{i=1}^{P} |\lambda_i|$.

3. Example

![Figure 3.1](image)

(i) Consider the graph in Figure 3.1, $S_1 = \{v_1, v_4\}$ is a MHD-set.

Correspondingly,

$$H_{S_1}(G) = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

The characteristic polynomial of $H_{S_1}(G)$ is $f_p(G, \lambda) = \lambda^6 - 2\lambda^5 - 5\lambda^4 + 9\lambda^3 - 2\lambda^2 - 4\lambda$.

Hence the MHD spectra are $2.55099$, $-2.07073$, $1.47506$, $0.739171$, $-0.694499$, $0$.

Therefore the MHD energy of $G$ with respect to $S_1$ is $E_h(G) = 7.53045$

(ii) $S_3 = \{v_2, v_3\}$ is also a MHD-set of $G$.

Correspondingly,
The characteristic polynomial of $H_{S_2}(G)$ is $f_p(G, \lambda) = \lambda^6 - 2\lambda^5 - 5\lambda^4 + 6\lambda^3 + 4\lambda^2 - 2\lambda - 1$.

Hence the MHD spectra are 2.8794, -1.8019, 1.247, 0.6527, -0.53209, -0.44504.

Therefore the MHD energy of $G$ with respect to $S_2$ is $E_h(G) = 7.53045$

(iii) $S_3 = \{v_2, v_5\}$ is also a MHD-set of $G$.

Correspondingly,

$$H_{S_3}(G) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

The characteristic polynomial of $H_{S_3}(G)$ is $f_p(G, \lambda) = \lambda^6 - 2\lambda^5 - 5\lambda^4 + 7\lambda^3 + 3\lambda^2 - 3\lambda - 1$.

Hence the MHD spectra are $2, 1, -0.323, -0.575, -1.913$.

Therefore the MHD energy of $G$ with respect to $S_3$ is $E_h(G) = 7.62311$.

Remark. Though all the $\gamma_h$-set are of same cardinality, the MHD energy $E_h(G)$ need not be the same for all $\gamma_h$-set.

4. Main Results

**Theorem 4.1.** For the Peterson graph $G$, $E_h(G) = 16.94236$
Proof. Let $G$ be the Peterson graph and let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3, v'_4, v'_5\}$ be the vertex set of $G$. Then $S = \{v_1, v_2, v'_3\}$ be the vertex set of $G$. Therefore

$$H_S(G) = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}$$

The characteristic polynomial of $H_S(G)$ is $f_p(G, \lambda) = \lambda^{10} - 3\lambda^{9} - 12\lambda^{8} + 35\lambda^{7} + 47\lambda^{6} - 152\lambda^{5} - 54\lambda^{4} + 290\lambda^{3} - 52\lambda^{2} - 204\lambda + 104.$

Hence the MHD spectra are $3.38604, -2, -1.91999, 1.85122, -1.63698, -1.41421, 1.41421, 1.31971, 1, 1.$

Therefore the MHD energy of $G$ with respect to $S_1$ is $E_h(G) = 17.42236$

Theorem 4.2. For the bull graph $G$, $E_h(G) = 7.16843$

Proof. Let $G$ be the bull graph and let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of $G$. Then $S = \{v_1, v_5\}$ be the vertex set of $G$. Therefore

$$H_S(G) = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The characteristic polynomial of $H_S(G)$ is $f_p(G, \lambda) = \lambda^{5} - 2\lambda^{4} - 4\lambda^{3} + 6\lambda^{2} + 4\lambda - 4.$

Hence the MHD spectra are $2.481, 1.41421, 0.689, -1.41421, -1.170$
Therefore the MHD energy of $G$ with respect to $S_1$ is $E_h(G) = 7.16843$

**Theorem 4.3.** For any complete graph $K_p$ with $p \geq 2$, $E_h(K_p) = p$.

**Proof.** Let $K_p$ be the complete graph having vertex set $V(K_p) = \{v_1, v_2, ..., v_p\}$. Then $\gamma_h(K_p) = p$. So, the unique MHD-set of $K_p$ is $S = \{v_1, v_2, ..., v_p\}$. Therefore the matrix $H_S(K_p)$ is,

$$H_S(K_p) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

And the corresponding characteristic polynomial is $f_p(K_p, \lambda) = \det(H_S(K_p) - \lambda I_p)$

$$f_p(K_p, \lambda) = \lambda^{p-1}(\lambda - p)$$

Hence the MHD spectrum of $K_p$ is \( \left( \begin{array}{c} 0 \\ p \\ p-1 \\ 1 \end{array} \right) \) Thus the MHD energy is $E_h(K_p) = p$.

**Theorem 4.4.** For a star graph $K_{1,p-1}$ with $p \geq 3$, $E_h(K_{1,p-1}) \leq 2 + 2\sqrt{p-2}$.

**Proof.** Let $K_{1,p-1}$ be the star graph having vertex set $V(K_{1,p-1}) = \{v_0, v_1, ..., v_{p-1}\}$, where $v_0$ is the vertex of degree $p - 1$. Then the $\gamma_h$-set is $S = \{v_0, v_1\}$. Therefore
$$H_S(K_{1, p-1}) = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

And the corresponding characteristic polynomial is
$$f_p(K_{1, p-1}, \lambda) = \det(H_S(K_{1, p-1}) - \lambda I_p)$$

$$= \begin{vmatrix}
1 - \lambda & 1 & 1 & \cdots & 1 \\
1 & 1 - \lambda & 0 & \cdots & 0 \\
1 & 0 & -\lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & -\lambda
\end{vmatrix}$$

$$f_p(K_{1, p-1}, \lambda) = \lambda^{p-3}(\lambda^3 - 2\lambda^2 - (p - 2)\lambda + (p - 2))$$

$$\leq \lambda^{p-3}(\lambda - 2)(\lambda^2 + p - 2)$$

Hence the MHD spectrum of $K_{1, p-1}$ is $\approx \left\{ 0, 2 \sqrt{p - 2}, 1 \right\}$. Thus the MHD energy is $E_h(K_{1, p-1}) \leq 2 + 2\sqrt{p - 2}$.

**Theorem 4.5.** Let $G = (V, E)$ be any simple $(p, q)$ graph. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the MHD eigen values of the matrix $H_S(G)$ then the following condition holds.

(i) $\sum_{i=1}^{p} \lambda_i = \gamma_h(G)$.

(ii) $\sum_{i=1}^{p} \lambda_i^2 = 2q + \gamma_h(G)$.

**Proof.** (i) We know that, sum of eigen values of $H_S(G)$ is same as the trace of $H_S(G)$.

i.e., $\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} h_{ii} = |S| = \gamma_h(G)$. 

Advances and Applications in Mathematical Sciences, Volume 21, Issue 3, January 2022
Therefore \( \sum_{i=1}^{p} \lambda_i = \gamma_h(G) \).

(ii) Since the sum of squares of the eigen values of \( H_S(G) \) is the trace of \( [H_S(G)]^2 \),

\[
\sum_{i=1}^{p} \lambda_i^2 = \sum_{i=1}^{p} \sum_{i=1}^{p} h_{ij}^2 = \sum_{i=1}^{p} (h_{ij})^2 + \sum_{i<j} h_{ij}^2
\]

\[
= \sum_{i=1}^{p} (h_{ij})^2 + 2 \sum_{i<j} (h_{ij})^2 = \gamma_h(G) + 2q
\]

\[
\sum_{i=1}^{p} \lambda_i^2 = 2q + \gamma_h(G).
\]

**Theorem 4.6.** Let \( G = (V, E) \) be any simple \((p, q)\) graph. If the \( E_h(G) \) is a rational number then \( E_h(G) = \gamma_h(G) \mod 2 \).

**Proof.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_p \) be the MHD eigen values of the MHD matrix \( H_S(G) \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_t \) are positive and the rest are of negative sign, then

\[
\sum_{i=1}^{p} |\lambda_i| = (\lambda_1 + \lambda_2 + \ldots + \lambda_t) - (\lambda_{t+1} + \lambda_{t+2} + \ldots + \lambda_p)
\]

\[
\sum_{i=1}^{p} |\lambda_i| = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_t) - (\lambda_1 + \lambda_2 + \ldots + \lambda_p)
\]

\[
\Rightarrow E_h(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_t) - \sum_{i=1}^{p} \lambda_i
\]

\[
\Rightarrow E_h(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_t) - \gamma_h(G)
\]
since \( \sum_{i=1}^{p} \lambda_i = \gamma_h(G) \)

Hence \( E_h(G) = \gamma_h(G) \mod 2 \).

**Theorem 4.7.** Let \( G \) be a \((p, q)\) simple graph. Let ‘d’ be the absolute value of determinant of the MHD matrix \( H_{S}(G) \), i.e., \( d = | \det H_{S}(G) | \). Then \( \sqrt{(2q + \gamma_h(G)) + p(p - 1)d^{2/p}} \leq E_h(G) \leq \sqrt{p(2q + \gamma_h(G))} \) where \( \gamma_h(G) \) is the MHD number of the graph \( G \).

**Proof.** We know, Cauchy Schwarz inequality is

\[
\left( \sum_{i=1}^{p} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{p} a_i^2 \right) \left( \sum_{i=1}^{p} b_i^2 \right)
\]

Put \( a_i = 1 \) and \( b_i = | \lambda_i | \) then

\[
\left( \sum_{i=1}^{p} | \lambda_i | \right)^2 \leq \left( \sum_{i=1}^{p} 1 \right) \left( \sum_{i=1}^{p} \lambda_i^2 \right)
\]

\[ \Rightarrow [E_h(G)]^2 \leq p \sum_{i=1}^{p} \lambda_i^2 \]

\[ = p(2q + \gamma_h(G)) \]

\[ \Rightarrow [E_h(G)] \leq \sqrt{p(2q + \gamma_h(G))} \]  \( (I) \)

Consider \( [E_h(G)]^2 \left( \sum_{i=1}^{p} | \lambda_i |^2 \right) \)

\[ = \left( \sum_{i=1}^{p} | \lambda_i |^2 \right) + \sum_{i \neq j} | \lambda_i | | \lambda_j | \]

\[ [E_h(G)]^2 = (2q + \gamma_h(G)) + \sum_{i \neq j} | \lambda_i | | \lambda_j | \]  \( (II) \)

where \( 2q + \gamma_h(G) = \left( \sum_{i=1}^{p} | \lambda_i |^2 \right) \).
We know, the geometric mean cannot exceed the arithmetic mean, we have

\[
\frac{1}{p(p-1)} \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq \left[ \prod_{i \neq j} |\lambda_i| |\lambda_j| \right]^{1 \over p(p-1)}
\]

\[
= \left[ \prod_{i=1}^{p} |\lambda_i| ^{2(p-1)} \right] ^{1 \over p(p-1)}
\]

\[
= \left[ \prod_{i=1}^{p} |\lambda_i| \right] ^{2 \over p}
\]

\[
= \left[ \prod_{i=1}^{p} \lambda_i \right] ^{2 \over p}
\]

\[
= \left| \det E_h(G) \right| ^{2 \over p}
\]

\[
= d \left( \frac{2}{p} \right)
\]

i.e.,

\[
\frac{1}{p(p-1)} \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq d \left( \frac{2}{p} \right)
\]

\[
\Rightarrow \sum_{i \neq j} \geq |\lambda_i| |\lambda_j| \geq p(p-1)d \left( \frac{2}{p} \right)
\]

(III)

Put (III) in (II)

\[
\left[ E_h(G) \right]^2 \geq (2q + \gamma_h(G)) + p(p-1)d \left( \frac{2}{p} \right)
\]

\[
\left[ E_h(G) \right]^2 \geq \sqrt{(2q + \gamma_h(G)) + p(p-1)d \left( \frac{2}{p} \right)}
\]

(IV)

From (I) and (IV)
\( \sqrt{(2q + \gamma_h(G)) + p(p - 1)d^P} \leq E_h(G) \leq \sqrt{p(2q + \gamma_h(G))}. \)

References


