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MINIMUM HOP DOMINATING ENERGY OF A GRAPH

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Abstract

Let $G = (V, E)$ be a (p, q) graph. A subset $S \subseteq V(G)$ is a minimum hop dominating set of a graph G if for every $v \in V - S,$ there exists $u \in S$ such that $d(u, v) = 2.$ The minimum cardinality of a hop dominating set (MHD) is called the hop dominating number. Motivated by the definition of minimum dominating energy by Rajeshkanna et al. [11], in this article the concept of minimum hop dominating energy $E_h(G)$ is introduced and minimum hop dominating energies $E_h(G)$ of some standard graphs and a few well-known families of graphs are computed. Also we establish an upper bound and lower bound for $E_h(G)$.

1. Introduction

In this article we consider finite, undirected and simple graph G with vertex set $V(G)$ and edge set $E(G).$ The number of vertices and number of edges are denoted by p and q respectively. A complete graph K_p is a simple undirected graph in which every pair of distinct vertices is connected by a

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unique edge. The star graph $K_{1,\, p-1}$ of order p is a tree on p nodes with one node having vertex degree $p-1$ and the other $p-1$ having vertex degree 1. The characteristic polynomials are computed using an online tool https://www.dcode.fr/matrix-eigenvalues. The distance $d(u, v)$ is the length of the minimum path from vertex u to vertex v. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the eigen values of G. Then the spectrum of G is denoted by

$$spec\ G = \begin{cases} \lambda_1 \lambda_2 \ldots \lambda_p \\ m_1 m_2 \ldots m_p \end{cases}$$

where m_i is the algebraic multiplicity of the eigen values λ_i, for $1 \leq i \leq p$.

The concept of energy $E(G)$ was proposed by I. Gutman in 1978 [3] as the sum of absolute values of the eigen values of the adjacency matrix of G. The basic properties including various upper and lower bounds for energy of a graph have been established in [7], [9]. Many researchers have introduced various concepts of energies. Motivated by the definition of minimum dominating energy by Rajeshkanna et al., [11], in this article the concept of minimum hop dominating energy $E_h(G)$ is introduced and minimum hop dominating energies $E_h(G)$ of some standard graphs and a few well-known families of graphs are computed. Also we establish an upper bound and lower bound for $E_h(G)$.

2. Minimum Hop Dominating Energy

Let $G = (V, E)$ be a (p, q) graph. Let S be a minimum hop dominating set of G. Then the minimum hop dominating matrix corresponding to S is a square matrix of order p and is defined as $H_S(G) = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in S \\ 0 & \text{otherwise} \end{cases}$

The characteristic polynomial of $H_S(G)$ is denoted by $f_p(G, \lambda) = \det(H_S(G) - \lambda I)$. The MHD eigen values of the graph G are the eigen values of $H_S(G)$ and is known as hop dominating spectra of G. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the eigen values of $H_S(G)$. Then the minimum hop dominating energy of G is defined as $E_h(G) = \sum_{i=1}^{p} | \lambda_i |$.

3. Example

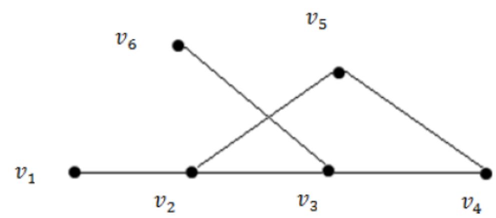

Figure 3.1

(i) Consider the graph in Figure 3.1, $S_1 = \{v_1, v_4\}$ is a MHD-set.

Correspondingly,

$$H_{S_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $H_{S_1}(G)$ is $f_p(G, \lambda) = \lambda^6 - 2\lambda^5 - 5\lambda^4 + 9\lambda^3 - 2\lambda^2 - 4\lambda$.

Hence the MHD spectra are 2.55099, -2.07073, 1.47506, 0.739171, -0.694499, 0.

Therefore the MHD energy of G with respect to S_1 is $E_h(G) = 7.53045$

(ii) $S_3 = \{v_2, v_3\}$ is also a MHD-set of G.

Correspondingly,

$$H_{S_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $H_{S_2}(G)$ is $f_p(G, \lambda) = \lambda^6 - 2\lambda^5 - 5\lambda^4 + 6\lambda^3 + 4\lambda^2 - 2\lambda - 1$.

Hence the MHD spectra are 2.8794, -1.8019, 1.247, 0.6527, -0.53209, -0.44504.

Therefore the MHD energy of G with respect to S_2 is $E_h(G) = 7.53045$.

(iii) $S_3 = \{v_2, v_5\}$ is also a MHD-set of G.

Correspondingly,

$$H_{S_3}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $H_{S_3}(G)$ is $f_p(G, \lambda) = \lambda^6 - 2\lambda^5 - 5\lambda^4 + 7\lambda^3 + 3\lambda^2 - 3\lambda - 1$.

Hence the MHD spectra are 2.812, 1, 1, -0.323, -0.575, -1.913.

Therefore the MHD energy of G with respect to S_3 is $E_h(G) = 7.62311$.

Remark. Though all the γ_h-set are of same cardinality, the MHD energy $E_h(G)$ need not be the same for all γ_h-set.

4. Main Results

Theorem 4.1. *For the Peterson graph* G, $E_h(G) = 16.94236$.

Proof. Let G be the Peterson graph and let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3, v'_4, v'_5\}$ be the vertex set of G. Then $S = \{v_1, v_2, v_4\}$ is a γ_h-set of G. Therefore

$$H_S(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $H_S(G)$ is $f_p(G, \lambda) = \lambda^{10} - 3\lambda^9 - 12\lambda^8 + 35\lambda^7 + 47\lambda^6 - 152\lambda^5 - 54\lambda^4 + 290\lambda^3 - 52\lambda^2 - 204\lambda + 104.$

Hence the MHD spectra are 3.38604, -2, -1.91999, 1.85122, -1.63698, -1.41421, 1.41421, 1.31971, 1, 1.

Therefore the MHD energy of G with respect to S_1 is $E_h(G) = 17.42236.$

Theorem 4.2. *For the bull graph* G, $E_h(G) = 7.16843.$

Proof. Let G be the bull graph and let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of G. Then $S = \{v_1, v_5\}$ is a γ_h-set of G. Therefore

$$H_S(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of $H_{S_1}(G)$ is $f_p(G, \lambda) = \lambda^5 - 2\lambda^4 - 4\lambda^3 + 6\lambda^2 + 4\lambda - 4.$

Hence the MHD spectra are 2.481, 1.41421, 0.689, -1.41421, -1.170

Therefore the MHD energy of G with respect to S_1 is $E_h(G) = 7.16843.$

Theorem 4.3. *For any complete graph* K_p *with* $p \geq 2$, $E_h(K_p) = p$.

Proof. Let K_p be the complete graph having vertex set $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Then $\gamma_h(K_p) = p$. So, the unique MHD-set of K_p is $S = \{v_1, v_2, \ldots, v_p\}$. Therefore the matrix $H_S(K_p)$ is,

$$H_S(K_p) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

And the corresponding characteristic polynomial is $f_p(K_p, \lambda) = \det(H_S(K_p) - \lambda I_P)$

$$= \begin{pmatrix} 1-\lambda & 1 & \cdots & 1 \\ 1 & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\lambda \end{pmatrix}$$

$$f_p(K_p, \lambda) = \lambda^{p-1}(\lambda - p)$$

Hence the MHD spectrum of K_p is $\begin{pmatrix} 0 & p \\ p-1 & 1 \end{pmatrix}$ Thus the MHD energy is $E_h(K_p) = p$.

Theorem 4.4. *For a star graph* $K_{1,\, p-1}$ *with* $p \geq 3$, $E_h(K_{1,\, p-1}) \leq 2 + 2\sqrt{p-2}$.

Proof. Let $K_{1,\, p-1}$ be the star graph having vertex set $V(K_{1,\, p-1}) = \{v_0, v_1, \ldots, v_{p-1}\}$, where v_0 is the vertex of degree $p-1$. Then the γ_h-set is $S = \{v_0, v_1\}$. Therefore

$$H_S(K_{1,\, p-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

And the corresponding characteristic polynomial is $f_p(K_{1,\, p-1}, \lambda)$ $= \det(H_S(K_{1,\, p-1}) - \lambda I_p)$

$$= \begin{vmatrix} 1-\lambda & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 0 & \cdots & 0 \\ 1 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -\lambda \end{vmatrix}$$

$$f_p(K_{1,\, p-1}, \lambda) = \lambda^{p-3}(\lambda^3 - 2\lambda^2 - (p-2)\lambda + (p-2))$$

$$\leq \lambda^{p-3}(\lambda - 2)(\lambda^2 + p - 2)$$

Hence the MHD spectrum of $K_{1,\, p-1}$ is $\approx \begin{pmatrix} 0 & 2 & -\sqrt{p-2} & \sqrt{p-2} \\ p-3 & 1 & 1 & 1 \end{pmatrix}$.

Thus the MHD energy is $E_h(K_{1,\, p-1}) \leq 2 + 2\sqrt{p-2}$.

Theorem 4.5. *Let* $G = (V, E)$ *be any simple* (p, q) *graph. If* $\lambda_1, \lambda_2, \ldots, \lambda_p$ *are the MHD eigen values of the matrix* $H_S(G)$ *then the following condition holds.*

(i) $\sum_{i=1}^{p} \lambda_i = \gamma_h(G)$.

(ii) $\sum_{i=1}^{p} \lambda_i^2 = 2q + \gamma_h(G)$.

Proof. (i) We know that, sum of eigen values of $H_S(G)$ is same as the trace of $H_S(G)$.

i.e., $\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} h_{ii} = |S| = \gamma_h(G)$.

Therefore $\sum_{i=1}^{p} \lambda_i = \gamma_h(G)$.

(ii) Since the sum of squares of the eigen values of $H_S(G)$ is the trace of $[H_S(G)^2]$,

$$\sum_{i=1}^{p} \lambda_i^2 = \sum_{i=1}^{p} \sum_{i=1}^{p} h_{ij} h_{ji}$$

$$= \sum_{i=1}^{p} (h_{ij})^2 + \sum_{i \neq j} h_{ij} h_{ji}$$

$$= \sum_{i=1}^{p} (h_{ij})^2 + 2 \sum_{i<j} (h_{ij})^2$$

$$= \gamma_h(G) + 2q$$

$\sum_{i=1}^{p} \lambda_i^2 = 2q + \gamma_h(G)$.

Theorem 4.6. *Let* $G = (V, E)$ *be any simple* (p, q) *graph. If the* $E_h(G)$ *is a rational number then* $E_h(G) \equiv \gamma_h(G) (\text{mod } 2)$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the MHD eigen values of the MHD matrix $H_S(G)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_t$ are positive and the rest are of negative sign, then

$$\sum_{i=1}^{p} | \lambda_i | = (\lambda_1 + \lambda_2 + \ldots + \lambda_t) - (\lambda_{t+1} + \lambda_{t+2} + \ldots + \lambda_p)$$

$$\sum_{i=1}^{p} | \lambda_i | = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_t) - (\lambda_1 + \lambda_2 + \ldots + \lambda_p)$$

$$\Rightarrow E_h(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_t) - \sum_{i=1}^{p} \lambda_i$$

$$\Rightarrow E_h(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_t) - \gamma_h(G)$$

since $\sum_{i=1}^{p} \lambda_i = \gamma_h(G)$

Hence $E_h(G) \equiv \gamma_h(G) \pmod 2$.

Theorem 4.7. *Let G be a* (p, q) *simple graph. Let 'd' be the absolute value of determinant of the MHD matrix* $H_S(G)$, *i.e.,* $d = |\det H_S(G)|$. *Then* $\sqrt{(2q + \gamma_h(G)) + p(p-1)d^{(2/p)}} \leq E_h(G) \leq \sqrt{p(2q + \gamma_h(G))}$ *where* $\gamma_h(G)$ *is the MHD number of the graph G.*

Proof. We know, Cauchy Schwarz inequality is

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\lambda_i|$ then

$$\left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} \lambda_i^2\right)$$

$$\Rightarrow [E_h(G)]^2 \leq p\sum_{i=1}^{p} \lambda_i^2$$

$$= p(2q + \gamma_h(G))$$

$$\Rightarrow [E_h(G)] \leq \sqrt{p(2q + \gamma_h(G))} \tag{I}$$

Consider $[E_h(G)]^2(\sum_{i=1}^{p} |\lambda_i|^2)$

$$= \left(\sum_{i=1}^{p} |\lambda_i|^2\right) + \sum_{i \neq j} |\lambda_i||\lambda_j|$$

$$[E_h(G)]^2 = (2q + \gamma_h(G)) + \sum_{i \neq j} |\lambda_i||\lambda_j| \tag{II}$$

where $2q + \gamma_h(G) = \left(\sum_{i=1}^{p} |\lambda_i|^2\right)$.

We know, the geometric mean cannot exceed the arithmetic mean, we have

$$\frac{1}{p(p-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left[\prod_{i\neq j}|\lambda_i||\lambda_j|\right]^{\frac{1}{p(p-1)}}$$

$$= \left[\prod_{i=1}^{p}|\lambda_i|^{2(p-1)}\right]^{\frac{1}{p(p-1)}}$$

$$= \left[\prod_{i=1}^{p}|\lambda_i|\right]^{\frac{2}{p}}$$

$$= \left|\prod_{i=1}^{p}\lambda_i\right|^{\frac{2}{p}}$$

$$= |\det E_h(G)|^{\frac{2}{p}}$$

$$= d^{\left(\frac{2}{p}\right)}$$

i.e., $\frac{1}{p(p-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq d^{\left(\frac{2}{p}\right)}$

$$\Rightarrow \sum_{i\neq j} \geq |\lambda_i||\lambda_j| \geq p(p-1)d^{\left(\frac{2}{p}\right)} \quad \text{(III)}$$

Put (III) in (II)

$$[E_h(G)]^2 \geq (2q + \gamma_h(G)) + p(p-1)d^{\frac{2}{p}}$$

$$[E_h(G)]^2 \geq \sqrt{(2q + \gamma_h(G)) + p(p-1)d^{\frac{2}{p}}} \quad \text{(IV)}$$

From (I) and (IV)

$$\sqrt{(2q+\gamma_h(G))+p(p-1)d^{\frac{2}{p}}} \leq E_h(G) \leq \sqrt{p(2q+\gamma_h(G))}.$$

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