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FIXED POINT THEOREM AND ITS APPLICATION IN *M*-FUZZY METRIC SPACES

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Abstract

In this paper, we prove a fixed point theorem for a new class of self-maps in M-complete fuzzy metric spaces. Our result generalizes the result of Yonghong et al. [8]. We justify our result by a suitable example. Some applications are also given in support of our results.

1. Introduction

The conception of fuzzy sets was introduced by Zadeh [11] in 1965. Kramosil and Michalek [6] introduced the concept of fuzzy metric space in 1975, which can be regarded as a generalization of the statistical metric space, clearly this work plays essential role for the construction of fixed point theory in fuzzy metric spaces. Subsequently, in 1988, M. Grabiec [2] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces. Following Grabiec's work, many authors introduced and generalized the different types of fuzzy contractive mappings and investigated some fixed point theorem in fuzzy metric spaces. In 1994, George and Veeramani [1] modified the notion of M-complete fuzzy metric spaces with the help of continuous t-norms. From the above analysis, we can see that there are many

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studies related to fixed point theory based on the two kinds of fuzzy metric spaces. In 2012, Y. Shen et al. [8] established several fixed point theorems in M-complete fuzzy metric spaces and compact fuzzy metric spaces by using self-maps. Vishal Gupta et al. [4] proved some fixed point theorems in fuzzy metric spaces through rational inequality and given some applications also in 2013.

In 2019, Lukman Zicky et al. [12] provided the concept of fuzzy metric is developed based on fuzzy concepts. This fuzzy metric is then applied to convergence problems and fixed point problems.

2. Preliminaries

Now, we begin with some basic concepts.

Definition 2.1 (Schweizer and Sklar [7]). A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (in short, continuous t-norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative.

(TN-2) $*$ is continuous.

(TN-3) $a * 1 = a$ for every $a \in [0, 1]$.

(TN-4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2 (George and Veeramani [1]). An ordered triple $(X, M, *)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions, for all $x, y, z \in X, s, t > 0$.

(FM-1) $M(x, y, t) > 0$.

(FM-2) $M(x, y, t) = 1$ iff $x = y$.

(FM-3) $M(x, y, t) = M(y, x, t)$.

(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$.

(FM-5) $M(x, y, *) : (0, \infty) \to (0, 1]$ is left continuous.

Definition 2.3 (George and Veeramani [1], Gregori and Sapena [3]). Let $(X, M, *)$ be a fuzzy metric space. Then:

(i) A sequence $\{x_n\}$ is said to converge to x in X, denoted by $x_n \to x$, if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.

(ii) A sequence $\{x_n\}$ in X is an M-Cauchy sequence if and only if for each $\epsilon \in (0, 1)$, $t > 0$, there exists $n_0 \in N$ such that $M(x_m, x_n, t) > 1 - \epsilon$ for any $m, n \geq n_0$.

(iii) The fuzzy metric space $(X, M, *)$ is called M-complete if every M-Cauchy sequence is convergent.

Definition 2.4. A fuzzy metric space $(X, M, *)$ is compact if every sequence in M has a convergent subsequence.

Definition 2.5. A function $\phi : [0, 1] \to [0, 1]$ is called an altering function, if the following properties are satisfied:

(A1) ϕ is strictly decreasing and left continuous.

(A2) $\phi(\lambda) = 0$ if and only if $\lambda = 1$.

Certainly, we obtain that $\lim_{\lambda\to 1-} \phi(\lambda) = \phi(1) = 0$.

Definition 2.6 [5]. A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied:

(B1) ψ is monotone increasing and continuous.

(B2) $\psi(t) = 0$ if and only if $t = 0$.

The objective of this work is to introduce a new class of self-maps by using ϕ-function and ψ-function in M-fuzzy metric spaces. We furnish an example to validate our result. Some applications are also given.

3. Main Result

In this section, we establish fixed point theorem in M-fuzzy metric spaces.

Theorem 3.1. *Let* $(X, M, *)$ *be an M-complete fuzzy metric space and T a self-map of X and suppose that* $\phi : [0, 1] \to [0, 1]$ *satisfies the above properties* *of Definition* 2.5 *and* $\psi : [0, \infty) \to [0, \infty)$ *satisfies the above properties of Definition* 2.6, *let* $k : (0, \infty) \to (0, 1)$ *be a function. If for any* $t \in [0, 1]$, T *satisfies the following condition*:

$$\phi(M(Tx, Ty, t)) \leq k(t)(\phi(M(x, y, t)) - \psi(M(x, y, t))), \tag{3.1}$$

where $x, y \in X$ *and* $x \neq y$, *then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \geq 0$ and $\tau_n(t) = M(x_n, x_{n+1}, t)$, for all $n \in \mathbb{N} \cup \{0\}$, $t \geq 0$.

Now we first prove that T has a fixed point. The proof is divided into two cases.

Case (I)**.** If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, i.e. $Tx_{n_0} = x_{n_0}$ then x_{n_0} is a fixed point of T.

Case (II)**.** We assume that $0 < \tau_n(t) < 1$ for each n. That is to say, the relationship $x_n \neq x_{n+1}$ holds true for each n. From (3.1), for every $t > 0$, $n \in \mathbb{N} \cup \{0\}$ we get

$$\begin{aligned} \phi(\tau_n(t)) &= \phi(M(x_n, x_{n+1}, t)) - \psi(M(x_n, x_{n+1}, t)), \\ &= \phi(M(Tx_{n-1}, x_n, t)) - \psi(M(Tx_{n-1}, x_n, t)), \\ &\leq k(t)(\phi(\tau_{n-1}(t)) - (\psi(\tau_{n-1}(t)), \\ &< \phi(\tau_{n-1}(t) - \psi(\tau_{n-1}(t). \end{aligned} \tag{3.2}$$

Since ϕ is strictly decreasing and ψ is non-decreasing, it is easy to show that $\{\tau_n(t)\}$ is an increasing sequence for every $t > 0$ with respect to n. Put $\lim_{n\to\infty} \tau_n(t) = \tau(t)$ as $n \to \infty$ and suppose that $0 < \tau(t) < 1$. By equation (3.2) then $\tau_n(t) \leq \tau(t)$ implies:

$$(\phi(\tau_{n+1}(t)) \leq k(t)(\phi(\tau_n(t) - \psi(\tau_n(t)), \tag{3.3}$$

letting $n \to \infty$, for all t, since that ϕ and ψ are left continuous,

$$\phi(\tau(t)) \le k(t)(\phi(\tau(t) - \psi(\tau(t)))) < \phi(\tau(t)) - \psi(\tau(t)), \tag{3.4}$$

which is a contradiction. Hence $\tau(t) = 1$. Therefore the sequence $\tau_n(t)$ convergence to 1 for any $t > 0$. Now, we show that the sequence $\{x_n\}$ is *M*-Cauchy sequence. Suppose that it is not true. Then there exist $0 < \epsilon < 1$ and two sequence $\{p(n)\}$ and $\{q(n)\}$ such that for every $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, we obtain that $p(n) > q(n) \ge n$, $M(x_{p(n)}, x_{q(n)}, t) \le 1 - \epsilon$.

$$M(x_{p(n)-1}, x_{q(n)-1}, t) > 1 - \epsilon \text{ and } M(x_{p(n)-1}, x_{q(n)}, t) > 1 - \epsilon, \tag{3.5}$$

for each $n \in \mathbb{N} \cup \{0\}$. Suppose $s_n(t) = M(x_{p(n)}, x_{q(n)}, t)$, then we have

$$\begin{aligned} 1 - \epsilon \ge s_n(t) &= M(x_{p(n)}, x_{q(n)}, t), \\ &\ge M(x_{p(n)-1}, x_{p(n)}, t/2) * M(x_{p(n)-1}, x_{q(n)}, t/2), \\ &> \tau_{p(n)}(t/2) * (1 - \epsilon). \end{aligned} \tag{3.6}$$

Since $\tau_{p(n)}(t/2) \to 1$ as $n \to \infty$, for every t, letting $n \to \infty$ we see that $\{s_n(t)\}$ convergence to $1 - \epsilon$ for any $t > 0$. By equation (3.1), we have

$$\begin{aligned} \phi(M(x_{p(n)}, x_{q(n)}, t)) &\le k(t)(\phi(M(x_{p(n)-1}, x_{q(n)-1}, t)) - \psi(M(x_{p(n)-1}, x_{q(n)-1}, t))), \\ &< \phi(M(x_{p(n)-1}, x_{q(n)-1}, t)) - \psi(M(x_{p(n)-1}, x_{q(n)-1}, t)). \end{aligned} \tag{3.7}$$

According to properties of ϕ and ψ, we have $M(x_{p(n)}, x_{q(n)}, t)$ $> M(x_{p(n)-1}, x_{q(n)-1}, t)$ for each n. So on the basis of the equation (3.5) we can obtain

$$1 - \epsilon \ge M(x_{p(n)}, x_{q(n)}, t) > M(x_{p(n)-1}, x_{q(n)-1}, t) > 1 - \epsilon, \tag{3.8}$$

which is a contradiction.

We consider another case, there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $M(x_m, x_n, t) \le 1 - \epsilon$ for all $m, n \ge n_0$, for any $p \in \mathbb{N}$, we know that $M(x_{n_0+p(n)+2}, x_{n_0+p+1}, t) \le 1 - \epsilon, \phi$ is monotonic, so there exist $\alpha \in (0, 1 - \epsilon]$ such that $M(x_{n_0+p(n)+2}, x_{n_0+p+1}, t) = \alpha$ for all $t > 0$. Thus we obtain

$$\phi(M(x_{n_0+p(n)+2}, x_{n_0+p+1}, t)) \le k(t)(\phi(M(x_{n_0+p(n)+1}, x_{n_0+p}, t))$$
$$- \psi(M(x_{n_0+p(n)+1}, x_{n_0+p}, t))) \tag{3.9}$$

letting $p \to \infty$, $\phi(\alpha) \le k(t)(\phi(\alpha) - \psi(\alpha))$, $\phi(\alpha) \le 0$, which is a contradiction.

Hence by Definition 2.3, $\{x_n\}$ is M-Cauchy sequence in the M-complete metric space X.

We conclude that there exists a point $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Now we will show that x is a fixed point of T. Since $0 < \tau_{n(t)} < 1$, there exists a subsequence $x_{r(n)}$ of x_n such that $x_{r(n)} \ne x$ for every $n \in \mathbb{N}$. From (3.2), we have

$$0 \le \phi(M(x_{r(n)}, T(x), t)) = \phi(M(Tx_{r(n)}, T(x), t))$$
$$\le k(t)(\phi(M(x_{r(n)}, x, t)) - \psi(M(x_{r(n)}, x, t))), \tag{3.10}$$

letting $n \to \infty$ in (3.10), we obtain

$$0 \le \phi(M(x, T(x), t)) \le k(t)(\phi(M(x, x, t)) - \psi(M(x, x, t))),$$
$$= k(t)(\phi(1)) - \psi(0) = 0. \tag{3.11}$$

Thus we obtain $\phi(M(x, T(x), t)) = 0$. By Definition 2.1 it is easy to show that $M(x, Tx, t) = 1$ i.e. $Tx = x$. We claim that x is the unique fixed point of T. Let $y \ne x$ is another fixed point of T. Then we get

$$\phi(M(x, y, t)) = \phi(M(Tx, Ty, t)) \le k(t)(\phi(M(x, y, t))) - \psi(M(x, y, t))$$
$$< \phi(M(x, y. t)) - \psi(M(x, y. t)), \tag{3.12}$$

which is a contradiction. This competes the proof of the theorem.

Remark 3.2. For $\psi(M(x, y, t)) = 0$, we get Theorem 3.1 of Y. Shen et al. [8].

Example 3.3. Let X be the subset of $\mathbb{R}^2$ defined by $X = \{A, B, C, D, E\}$, where $A = (0, 0), B = (2, 0), C = (2, 1), D = (0, 1), E = (2, 3), \phi(\tau) = 1 - \tau$ and $\psi(\tau) = \sqrt{(\tau)}$ for all $\tau \in [0, 1]$ and $M(x, y, t) = e^{-2d(x, y)/t}$ for all $t > 0$, where $d(x, y)$ denotes the Euclidean distance of $\mathbb{R}^2$. Clearly $(X, M, *)$ is an M-complete fuzzy metric space with respect to the t-norm: $a * b = ab$.

Let $T : X \to X$ be given by $T(A) = T(B) = T(C) = T(D) = A, T(E) = B.$ Define function $k : (0, \infty) \to (0, 1)$ as

$$k = \begin{cases} 1 - e^{-9/t} & \text{if } 0 < t \leq 3. \\ t/(t+1) & \text{if } t > 3. \end{cases}$$

We show that x is fixed point of T with $A = (0, 0)$.

Solution. $(X, M, *)$ is an M-complete fuzzy metric space, we have to prove ϕ satisfy (A1) and (A2) of Definition 3.5, take any $\tau_i, \tau_j \in [0, 1]$ with $i \neq j$. If $i < j$, then

$$\tau_i < \tau_j$$

$$-\tau_i > -\tau_j$$

$$1 - \tau_i > 1 - \tau_j$$

$$\phi(\tau_i) > \phi(\tau_j).$$

So taking any elements on [0, 1], we obtain that ϕ is strictly decreasing. For left continuity, suppose there exist $r \in R^+$ and $r \to 0$ satisfy

$$\phi(\tau - r) = \phi(\tau)$$

$$\lim_{r \to 0^+} \phi(\tau - r) = \lim_{r \to 0^+} 1 - (\tau - r) = 1 - \tau = \phi(\tau).$$

Therefore (A1) is satisfied. For (A2), we see that

$$\phi(\lambda) = 0 \Leftrightarrow \lambda = 1.$$

So that (A2) is satisfied. Now we have to prove ψ satisfy conditions of Definition 3.6. So by taking any elements on $[0, \infty)$, we can see that ψ is increasing, take any $\tau_i, \tau_j \in [0, \infty)$ with $i \neq j$. If $i > j$, then

$$\tau_i < \tau_j$$

$$\sqrt{\tau_i} < \sqrt{\tau_j}$$

$$\psi(\tau_i) < \psi(\tau_j).$$

For left continuity, suppose there exists $r \in R^+$ and $r \to 0$ satisfy

$$\lim_{r \to 0} \psi(\tau - r) = \lim_{r \to 0^+} \sqrt{\tau - r} = \sqrt{\tau} = \psi(\tau).$$

Therefore first part is satisfied of Definition 3.6. For second part, we see that

$$\psi(\lambda) = 0 \Leftrightarrow \lambda = 0.$$

So that second part of Definition 3.6 is also satisfied. Similarly k also satisfies equation (3.1). Now, all the hypotheses of Theorem 3.1 are satisfied and thus T has a unique fixed point, that is $x = A$.

4. Application

In this section, we give an application related to our result. Let us define $\psi : [0, \infty) \to [0, \infty)$ as $\psi(t) = \int_0^t \varphi(t) dt$ for all $t > 0$, be a non-decreasing and continuous function and $\phi : [0, 1] \to [0, 1]$ as $\phi(t) = \int_0^t \varphi(t) dt$ for all $t > 0$, be a decreasing and continuous function. For each $\epsilon > 0$, $\varphi(\epsilon) > 0$. Which shows that $\varphi(t)$ and $\phi(t) = 0$ iff $t = 0$.

Theorem 4.1. *Let* $(X, M, *)$ *be an M-complete fuzzy metric space and* $T : X \to X$ *be a mapping satisfying* $M(x, y, t) = 1$ *and*

$$\int_0^{\phi(M(Tx, Ty, t)} \varphi(t) dt \leq k(t) \left(\int_0^{\phi(M(x, y, t))} \varphi(t) dt - \int_0^{\psi(M(x, y, t))} \varphi(t) dt \right).$$

for all $x, y \in k \in (0, 1)$. *Then T has a unique fixed point.*

Proof. By taking $\varphi(t) = 1$ and applying Theorem 3.1, we obtain the result.

For $\int_0^{\psi(M(x,\, y,\, t))} \varphi(t)dt = 0,$ we get the following corollary.

Corollary 4.3. *Let $(X, M, *)$ be a M-complete fuzzy metric space and $T : X \to X$ be a mapping satisfying $M(x, y, t) = 1$ and*

$$\int_0^{\phi(M(Tx,\, Ty,\, t))} \varphi(t)dt \le k(t)\left(\int_0^{\phi(M(x,\, y,\, t))} \varphi(t)dt\right)$$

for all $x, y \in X$, $k \in (0, 1)$. Then T has a unique fixed point.

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