Advances and Applications in Mathematical Sciences ISSN 0974-6803
Volume 22, Issue 6, April 2023, Pages 1059-1078

FINSLER HYPERSURFACE WITH GENERALISED Z. SHEN'S (α, β) METRIC

DEEPAK K S and AVEESH S T*

Department of Mathematics
Alva's College, Moodbidire 574227
Mangalore University, India
E-mail: deepakpatel2124@gmail.com

Department of Mathematics
PES Institute of Technology and Management
Shivamogga 577204, Visvesvaraya
Technological University, Belagavi, India

Abstract

The (α, β)-metric is a Finsler metric made up of a Riemannian metric α and a differential *l*-form β, that has been studied in theoretical physics. In this paper a special (α, β) metric is considered known as generalised Z. Shen's metric. Here, the hypersurface of generalised Z. Shen's (α, β) metric on hyperplane of 1st, 2nd and 3rd kind is discussed. The results indicated that the hypersurfaces obtained are hyperplanes 1st and 2nd kind, but not of the 3rd kind for Z. Shen's (α, β) metric.

1. Introduction

Definition 1.1[5]. Let M^n be an n-dimensional smooth manifold and $L(x, y)$ is a fundamental function, which satisfies the following conditions.

(1) $L(x, y) > 0$ for $(x, y) \in D$

(2) $L(x, \lambda y) = |\lambda| L(x, y)$ for any $(x, y) \in D$ and $\lambda \in R(x, \lambda y) \in D$

2020 Mathematics Subject Classification: 53B40, 53C60.
Keywords: Finsler space, Finsler hypersurface (FH), angular metric tensor, hyperplane.
*Correspondence author; E-mail: aveeshst@gmail.com
Received January 1, 2023; Accepted February 23, 2023

(3) The D tensor field $g_{ij}(x, y) = \frac{1}{2}\partial^i \partial^j L^2(x, y)$

Then $F = (M^n, L)$ is called Finsler space furnished with fundamental function $L(x, y)$ on M^n, where $\partial^j = \frac{\partial}{\partial y^j}$ is non degenerated on D.

Definition 1.2 (Finsler Space with the (α, β)-metric) [15, 22]**.** The fundamental function L of a Finsler space $F^n = (M^n, L)$ is called a n-dimensional Finsler space with an (α, β) metric, here L is a positive homogeneous function of two arguments $\alpha(x, y) = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$, $\beta(x, y) = b_i(x)y^i$, where α is a Rimaninian fundamental function and β is a differential 1-form.

Let $L(\alpha, \beta) = \frac{(\alpha + \beta)^2}{\alpha}$ is called Z. Shen's square metric. In this paper, we have considered a generalized Z. Shen's (α, β) metric, $L(\alpha, \beta) = \frac{(\alpha + \beta)^{n+1}}{\alpha^n}$ $(n > 0)$.

Definition 1.3 (Hypersurface of Finsler Space) [6]**.** Finsler hypersurface $F^{n-1} = (M^{n-1}, L(u, v))$ of a Finsler space $F^n = (M^n, L(u, v))$ can be parameterized as an equation, $x^i = x^i(u^\alpha)$, here u^α are Gaussian coordinates of F^{n-1} and Greek indices take values from 1 to $n-1$.

Hyperplanes of the 1st, 2nd and 3rd kind [7].

Definition 1.4. If each path of a F^{n-1} with respect to the induced connection is besides a path of the enclosing space F^n then Finsler hypersurface F^{n-1} is a hyperplane of 1st kind.

Definition 1.5. If each path of a F^{n-1} with respect to the induced connection is also a h-path of the enclosing space F^n then Finsler hypersurface F^{n-1} is a hyperplane of 2nd kind.

Definition 1.6. The unit normal vector of F^{n-1} is parallel along each curve of F^{n-1} then it is a hyperplane of the 3rd kind.

Table 1. Summary of various types of (α, β) metrics and their results in hyperplanes.

Sl No	Types of Metrics	Hyperplane of 1st kind	Hyperplane of 2nd kind	Hyper plane of 3rd kind	Reference
1.	$L(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha} + \frac{\beta^2}{(\beta-\alpha)}$ (Deformed Berwald-Infinite Series Metric)	yes	yes	no	[1]
2.	$L(\alpha, \beta) = \mu_1\alpha + \mu_2\beta + \mu_3 \frac{\beta^2}{\alpha}$ (Generalized (α, β)-Metric)	yes	yes	no	[2]
3.	$L(\alpha, \beta) = \alpha + \epsilon\beta + \kappa \frac{\beta^2}{\alpha} + \beta$ (Randers Change of Generalized (α, β) Metric)	yes	yes	no	[3]
4.	$L(\alpha, \beta) = \frac{\alpha^2}{(\alpha-\beta)} + \beta$ (Matsumoto Metric)	yes	yes	no	[4]
5.	$L(\alpha, \beta) = \alpha + \beta + \frac{\beta^{N+1}}{\alpha^N}$ (Special (α, β)-Metric)	yes	yes	no	[5]
6.	$\overline{L} = Le^{\frac{\beta}{L}}$ (exponential metric)	yes	yes	yes	[6]
7.	$F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ (Finsler Square Metric)	yes	yes	no	[7]
8.	$L = \frac{\alpha^2}{(v\alpha - \omega\beta)}$ (Matsumoto Metric)	yes	yes	no	[8]
9.	$L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}$ (exponential form of (α, β))	yes	yes	yes	[9]
10.	$L(\alpha, \beta) = \alpha + \beta + \frac{\alpha^2}{(\alpha-\beta)}$	yes	yes	no	[10]

	(Special (α, β)-metric)				
11.	$L = \beta + \frac{\alpha^3 + \beta^3}{\alpha(\alpha - \beta)}$ (Special (α, β)-metric)	yes	yes	no	[11]
12.	$L^* = \alpha + \mu\beta + \nu\frac{\beta^5}{\alpha^4} + \beta$ (Special (α, β)-metric)	yes	yes	yes	[12]
13.	$L(x, y) = \alpha \cosh\frac{\beta}{\alpha} + \beta$ (Special (α, β)-metric)	yes	yes	no	[13]
14.	$L(\alpha, \beta) = \frac{(\alpha + \beta)^2}{\alpha}$ (Square Metric)	yes	yes	no	[14]
15.	$L(\alpha, \beta) = \frac{(\alpha + \beta)^2}{\alpha} + \frac{\alpha}{(\beta - \alpha)}$	yes	yes	no	[15]
16.	$L(\alpha, \beta) = A_1(\alpha e^{\frac{\beta}{\alpha}}) + (\alpha + \beta)$ (special (α, β)-metric)	yes	yes	no	[16]
17.	$L + \alpha + \frac{\beta^{n+1}}{(\alpha - \beta)^n}$ (special (α, β)-metric)	yes	yes	yes	[17]
18.	$L = \alpha e^{\frac{\beta}{\alpha}} + \beta$ (Exponential (α, β)-Metric)	yes	yes	no	[18]
19.	$L(\alpha, \beta) = \alpha + \sqrt{\alpha^2 + \beta^2}$ (special (α, β)-metric)	yes	yes	yes	[19]

In this paper we intend to consider the generalised Z. Shen's (α, β) metric [14] and verify the results in the hyperplane.

2. Preliminaries

We take into account the Finsler space (M^n, L), where L is the generalised Z. Shen's metric that is given by

$$L(\alpha, \beta) = \frac{(\alpha + \beta)}{\alpha^n} \tag{2.1}$$

Differentiating equation (2.1) partially with respect to α and β up to second order, we get

$$L_\alpha = \frac{(n+1)(\alpha+\beta)^n \alpha^n - n(\alpha+\beta)^{n+1}\alpha^{n+1}}{(\alpha^n)^2},$$

$$L_\beta = \frac{1}{\alpha^n}(n+1)(\alpha+\beta)^n,$$

$$L_{\alpha\alpha} = \frac{\begin{matrix}[n(n+1)(\alpha+\beta)^{n-1}\alpha^n + (n+1)(\alpha+\beta)^n\alpha^{n-1} - n(n+1)(\alpha+\beta)^n\alpha^{n-1} \\ -(n-1)n(\alpha+\beta)^{n+1}\alpha^{n-2}] - [(n+1)(\alpha+\beta)^n\alpha^n - n(\alpha+\beta)^n\alpha^{n-1}]2n\alpha^{2n-1}\end{matrix}}{\alpha^{4n}}$$

$$L_{\beta\beta} = \frac{1}{\alpha^n} n(n+1)(\alpha+\beta)^{n-1},$$

$$L_{\alpha\beta} = \frac{1}{\alpha^{2n}}[n(n+1)(\alpha+\beta)^{n-1}\alpha^n - n(n+1)(\alpha+\beta)^n\alpha^{n-1}] \tag{2.2}$$

In Finsler space (M^n, L), the $l_i = \frac{\partial L}{\partial y_i}$ and h_{ij} [21] are given as,

$$l_i = \frac{L_\alpha y_i}{\alpha} + L_\beta b_i \tag{2.3}$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i y_j + b_j y_i) + q_2 y_i y_j, \tag{2.4}$$

where the coefficients are defined and calculated as follows

$$y_i = a_{ij} y^j,$$

$$p = \frac{LL_\alpha}{\alpha} = \frac{(n+1)(\alpha+\beta)^{2n+1}\alpha^n - n(\alpha+\beta)^{n+2}\alpha^{n-1}}{\alpha^{3n+1}},$$

$$q_0 = LL_{\beta\beta} = \frac{1}{\alpha^{2n}} n(n+1)(\alpha+\beta)^{2n}, \tag{2.5}$$

$$q_1 = \frac{LL_{\alpha\beta}}{\alpha} = \frac{n(n+1)(\alpha+\beta)^{2n}\alpha^n - n(n+1)(\alpha+\beta)^{2n+1}\alpha^{n-1}}{\alpha^{3n+1}},$$

$$q_2 = \frac{L\left(L_{\alpha\alpha} - \frac{L_\alpha}{\alpha}\right)}{\alpha^2}$$

$$= \frac{1}{\alpha^2} \frac{\left(\frac{(\alpha+\beta)^{n+1}}{\alpha^n}\right)\begin{bmatrix} n(n+1)(\alpha+\beta)^n\alpha^n \\ +(n+1)(\alpha+\beta)^n\alpha^{n-1} \\ -n(n+1)(\alpha+\beta)^n\alpha^{n-1} \\ -(n-1)n(\alpha+\beta)^{n+1}\alpha^{n-2}\end{bmatrix} - [(n+1)(\alpha+\beta)^n\alpha^n - n(\alpha+\beta)^{n+1}\alpha^{n-1}]2n\alpha^{nn-1}}{\alpha^{4n}} \tag{2.6}$$

The fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\delta^2 L}{\delta y^i \delta y^j}$ of Finsler space (M^n, L), is defined by [21]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j \tag{2.7}$$

Where the coefficients p, p_0, p_1 and p_2 are defined and calculated as follows:

$$p = \frac{LL_\alpha}{\alpha} = \frac{(n+1)(\alpha+\beta)^{2n+1}\alpha^n - n(\alpha+\beta)^{n+2}\alpha^{n-1}}{\alpha^{3n+1}}, \tag{2.8}$$

$$p_0 = q_0 + L_\beta^2 = \frac{1}{\alpha^{2n}} n(n+1)(\alpha+\beta)^{2n} + (n+1)^2\frac{(\alpha+\beta)^{2n}}{\alpha^{2n}}, \tag{2.9}$$

$$p_1 = q_1 + \frac{pL_\beta}{L}$$

$$= \frac{\begin{matrix} n(n+1)(\alpha+\beta)^{2n+1}\alpha^n + (n+1)(\alpha+\beta)^{2n+1}\alpha^{n-1} - n(n+1)(\alpha+\beta)^{2n+1}\alpha^{n-1} \\ -(n-1)(\alpha+\beta)^{2n+2}\alpha^{n-2} + 2n(n+1)(\alpha+\beta)^{2n+1}\alpha^{3n-1} + 2n^2(\alpha+\beta)^{2n+2}\alpha^{3n-2}\end{matrix}}{\alpha^{5n+2}}, \tag{2.10}$$

$$p_2 = q_2 + \frac{p^2}{L^2}$$

$$= \left[\frac{\begin{array}{l} n(n+1)(\alpha+\beta)^{2n+1}\alpha^{n} + (n+1)(\alpha+\beta)^{2n+1}\alpha^{n-1} - n(n+1)(\alpha+\beta)^{2n+1}\alpha^{n-1} \\ -(n-1)(\alpha+\beta)^{2n+2}\alpha^{n-2} + 2n(n+1)(\alpha+\beta)^{2n+1}\alpha^{3n-1} + 2n^{2}(\alpha+\beta)^{2n+2}\alpha^{3n-2} \end{array}}{\alpha^{5n+2}} \right]$$

$$+ \frac{[(n+1)(\alpha+\beta)^{2n+1}\alpha^{n} - n(\alpha+\beta)^{2n+2}\alpha^{n}]}{(\alpha^{3n+1})^{2}} \times \frac{\alpha^{2n}}{(\alpha+\beta)^{2n+2}} \tag{2.11}$$

In addition, the reciprocal tensor g^{ij} of fundamental metric tensor g_{ij} is given by the relation [21]

$$g^{ij} = \frac{a^{ij}}{p} - S_0 b^i b^j - S_{-1}(b^i y^j + b^j y^i) - S_{-2} y^i y^j, \tag{2.12}$$

where the coefficients b^i, S_0, S_{-1} and S_{-2} are defined as follows:

$$b^i = a^{ij} b_j,$$

$$S_0 = \frac{pp_0 + (p_0 p_2 - p_1^2)\alpha^2}{p\zeta}, \tag{2.13}$$

$$S_{-1} = \frac{pp_1 + (p_0 p_2 - p_1^2)\beta}{p\zeta}, \tag{2.14}$$

$$S_{-2} = \frac{pp_2 + (p_0 p_2 - p_1^2)b^2}{p\zeta}, \tag{2.15}$$

$$\zeta = p(p + p_0 b^2 + p_1\beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2), \tag{2.16}$$

where $b^2 = a^{ij} b^i b^j$.

Now, the $h\nu$-torsion tensor $C_{ijk} = \frac{1}{2}\frac{\delta g_{ij}}{g y^k}$ is defined by [21]

$$C_{ijk} = \frac{p_1(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k}{2p}, \tag{2.17}$$

where the coefficients γ_1 and m_i are defined as

$$\gamma_1 = p\frac{\delta p_0}{\delta\beta} - 3p_1q_0,\ m_i = b_i - \frac{y_i\beta}{\alpha^2} \tag{2.18}$$

where m_i is non-zero covariant vector orthogonal to y^i.

Now consider,

$$2E_{ij} = b_{ij} + b_{ji} \tag{2.19}$$

$$2F_{ij} = b_{ij} - b_{ji},\ \text{where } b_{ij} = \nabla_j b_i. \tag{2.20}$$

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be the Cartan connection of F^n.

The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ of the Finsler space F^n is given by [21]

$$D^i_{jk} = B^iE_{jk} + F^i_kB_j + F^i_jB_k + B^i_jb_{0k} + B^i_kb_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k$$

$$- C^i_{km}A^m_j - C_{jkm}A^m_sg^{is} + \lambda^s(C^i_{jm}Csk^m + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}) \tag{2.21}$$

where

$$\left.\begin{aligned}
B_k &= p_0b_k + p_1y_k,\\
B_{ij} &= \frac{p_1\left(a_{ij} - \frac{y_iy_j}{\alpha^2}\right) + \frac{\delta p_0}{\delta\beta}m_im_j}{2},\\
B^i &= g^{ij}B_j,\\
B^k_i &= g^{kj}B_{ji},\\
A^m_k &= B^m_kE_{00} + B^mE_{k0} + B_kF^m_0 + B_0F^m_k,\\
\lambda^m &= B^m + E_{00} + 2B_0F^m_0,\\
F^k_i &= g^{kj}F_{ji},\\
B_0 &= B_ib^i.
\end{aligned}\right]$$

Here 0 will be denoted the tensorial contraction with y^i except the quantities q_0, p_0 and S_0.

3. Induced Cartan Connection of $F^{n-1}(c)$

Let $F^n = (M^n, L)$ be a Finsler space, and $F^{n-1}(c)$ be its hypersurface having equation $x^i = x^i(u^\alpha), i = 1, 2, 3, \ldots, (n-1)$. Let the matrix of projection factor be $B^i_\alpha = \dfrac{\delta x^i}{\delta u^\alpha}$ and its rank is $(n-1)$. The tangential component of element of support y^i of Finsler space F^n along its hypersurface $F^{n-1}(c)$ is given by

$$y^i = B^i_\alpha(u)v^\alpha. \tag{3.1}$$

Here v^α is the component of support of $F^{n-1}(c)$ at u^α. The tensor $g_{\alpha\beta}$ and hv-torsion tensor $C_{\alpha\beta\gamma}$ of $F^{n-1}(c)$ are given by $g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta$, $C_{\alpha\beta\gamma} = C_{ijk}B^j_\alpha B^k_\gamma$.

Now, the unit normal vector $N^i(u, v)$ at an arbitrary point u^α of $F^{n-1}(c)$ is defined by the following property

$$g_{ij}(x(u, v), y(u, v))B^i_\alpha N^j = 0,\ g_{ij}(x(u, v), y(u, v))N^iN^j = 1 \tag{3.2}$$

the angular metric tensor h_{ij} is defined as

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta,\ h_{ij}B^i_\alpha N^j = 0,\ h_{ij}N^iN^j = 1 \tag{3.3}$$

Let (B^α_i, N_i) be the inverse of (B^i_α, N^i), then we have

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta,\ B^i_\alpha B^\beta_i = \delta^\beta_\alpha,\ B^\alpha_i N^i = 0,$$

$$B^\alpha_i N_i = 0,\ N_i = g_{ij}N^i,\ B^k_i = g^{kj}B_{ji},\ B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j$$

The induced Cartan connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on $F^{n-1}(c)$ induced from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, G^i_{0k}, C^i_{jk})$ is given by [3] (3.4)

$$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_{i}(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}) + M^{\alpha}_{\beta}H_{\gamma},$$

$$G^{\alpha}_{\beta} = B^{\alpha}_{i}(B^{i}_{0\beta} + \Gamma^{*i}_{0j}B^{j}_{\beta}),$$

$$C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}C^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma},$$

where

$$M_{\beta\gamma} = N_i C^{i}_{jk}B^{j}_{\beta}\gamma^{k},$$

$$M^{\alpha}_{\beta} = g^{\alpha\beta}M_{\beta\gamma},$$

$$H_{\beta} = N_i(B^{i}_{0\beta} + \Gamma^{*i}_{0j}B^{j}_{\beta}),$$

$$B^{i}_{\beta\gamma} = \frac{\delta B^{i}_{\beta}}{\delta U^{\gamma}}, \tag{3.5}$$

$$B^{i}_{0\beta} = B^{i}_{\alpha\beta}v^{\alpha} \tag{3.6}$$

The quantities $M_{\beta\gamma}$ and H_{β} appeared in above equations are called the second fundamental v-tensor and normal curvature [20].

The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [20]

$$H_{\beta\gamma} = N_i(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}) + M_{\beta}H_{\gamma}, \tag{3.7}$$

Where $M_{\beta} = N_i C^{i}_{jk}B^{j}_{\beta}N^{k}$. (3.8)

The relative h-covariant derivative and ν-covariant derivative of projection factor B^{i}_{α} with respect to induced Cartan connection $IC\Gamma$ are respectively given by

$$B^{i}_{\alpha|\beta} = H_{\alpha\beta}N^{i}, \tag{3.9}$$

$$B^{i}_{\alpha|\beta} = M_{\alpha\beta}N^{i}$$

It is obvious that $H_{\beta\gamma}$ is not always symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta}H_{\gamma} - M_{\gamma}H_{\beta}, \tag{3.10}$$

Implying that

$$H_{0\gamma} = H_{\gamma},\ H_{\gamma 0} = H_{\gamma} + M_{\gamma}H_{0}. \tag{3.11}$$

We utilize the following lemma to prove our hypothesis. The following lemma has already been derived by M. Matsumoto [20]:

Lemma 3.1. *The normal curvature* $H_0 = H_{\beta}v^{\beta}$ *vanishes iff normal curvature vector* H_{β} *vanishes.*

Lemma 3.2. $F^{n-1}(c)$ *is a hyperplane of* 1st *kind iff* $H_{\alpha} = 0$.

Lemma 3.3. $F^{n-1}(c)$ *is a hyperplane of* 2nd *kind with respect to* $C\Gamma$ *iff* $H_{\alpha} = 0$ *and* $H_{\alpha\beta} = 0$.

Lemma 3.4. $F^{n-1}(c)$ *is a hyperplane of* 3rd *kind with respect to* $C\Gamma$ *iff* $H_{\alpha} = 0$, $H_{\alpha\beta} = 0$ *and* $M_{\alpha\beta} = 0$.

4. Hypersurface $F^{n-1}(c)$ of the Generalised Z. Shen's Finsler Metric

Let $F^{n-1}(c)$ whose equation is given by $b(x) = c$, where c is a fixed constant. Thus the gradient of the function representing $F^{n-1}(c)$ is given by in tensor notation $b_i(x) = \delta_i b$ and parametric equation $x^i = x^i(u^{\alpha})$ of $F^{n-1}(c)$. Differentiating the equation of hypersurface $b(x(u)) = c$ with respect to parameter u^{α}, we get $\delta_{\alpha} b(x(u)) = 0 = b_i B^i_{\alpha}$. It is clear that $b_i(x)$ are the covariant component of normal vector field of $F^{n-1}(c)$.

Furthermore, we have

$$b_i B^i_{\alpha} = 0 \tag{4.1}$$

$$b_i y_i = 0 \tag{4.2}$$

The induced metrics $L(u, v)$ from the special Finsler space (M^n, L) on the $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta}v^{\alpha}v^{\beta}, \text{ where } a_{\alpha\beta} = a_{ij}B^i_{\alpha}B^j_{\beta}. \tag{4.3}$$

The induced metric in equation (4.3) do not have β component, (i.e. $\beta = b_i y^i = 0$), of the Finsler metric of the original space (M^n, L), therefore induced metric in equation (4.3) is a Riemannian metric. Therefore at any point on $F^{n-1}(c)$ equations (2.5), (2.6), (2.8), (2.9), (2.10), (2.11), (2.13), (2.14), (2.15) and (2.16) reduce to

$$p = 1, q_0 = n(n+1), q_1 = 0, q_2 = \frac{1}{\alpha^2}, p_0 = n(n+1)(n+1)^2,$$

$$p_1 = \frac{n(n+1)}{\alpha}, p_2 = 0 \tag{4.4}$$

$$S_0 = \frac{n(n+1)}{1+n(n+1)b^2}, S_{-1} = \frac{n(n+1)}{\alpha(1+n(n+1)b^2)}, S_{-2}$$

$$= \frac{(n+1)^2}{\alpha^2(1+n(n+1))}, \Gamma = 1+n(n+1)b^2 \tag{4.5}$$

Using the values of p, S_0, S_{-1}, S_{-2} substitute in equation (2.12), we have

$$g_{ij} = a_i^{ij} - \frac{n(n+1)}{1+n(n+1)b^2}b^ib^j - \frac{2}{\alpha(1+n(n+1)b^2)}(b^iy^j + b^iy^j)$$

$$+ \frac{(n+1)^2}{12(1+n(n+1)b^2)}y^iy^j \tag{4.6}$$

Multiplying equation (4.6) by b_ib_j and using the fact that $\beta = b_iy^i = 0$, it becomes

$$g^{ij}b_ib_j = \frac{b^2}{1+n(n+1)b^2}.$$

Thus we get

$$b_i x(u) = \sqrt{\frac{b^2}{1 + n(n+1)b^2}} N_i \tag{4.7}$$

Where b is the length of the vector b^i. Now from (4.6) and (4.7) we get

$$b_i = a^{ij} b_j = \sqrt{b^2(1 + n(n+1)b^2)} N^i + \frac{b^2}{\alpha} y^i. \tag{4.8}$$

Theorem 4.1. *In* $F^{n-1}(c)$ *the induced metric is a Riemannian metric given by* (4.3) *and the scalar function* $b(x)$ *is given by* (4.7) *and* (4.8) *for the generalised Z. Shen's* (α, β) *metric.*

Using the values of p, p_0, p_1 and p_2 from equation (4.4) into equation (2.7), Finsler metric tensor of F^n reduces to

$$g_{ij} = a_{ij} + n(n+1)(n+1)^2 b_i b_j + \frac{n(n+1)}{\alpha}(b_i y_i + b_j y_j) \tag{4.9}$$

and using the value of p, q_0, q_1 and q_2 in equation (2.4), angular metric tensor of F^n reduces to

$$h_{ij} = a_{ij} + n(n+1) b_i b_j - \frac{1}{\alpha^2} y_i y_j \tag{4.10}$$

From equations (4.10), (4.2) and (3.3) it follows that if $h^{(a)}_{\alpha\beta}$ denotes the angular metric tensor of Riemannian $a_{ij}(x)$, then along hypersurface $F^{n-1}(c)$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. Thus along the hypersurface $F^{n-1}(c)$, above equation reduces to $\frac{\delta p_0}{\delta\beta} = \frac{(n(n+1) + (n+1)^2)2n(\alpha+\beta)^{2n-1}}{\alpha^{2n}}$ and (2.18) also reduces to $\gamma_1 = 0$, $m_i = b_i$. Using the values of p, p_1, γ_1 and m_i in equation (2.17), $h\nu$-torsion tensor in hypersurface $F^{n-1}(c)$, becomes

$$C_{ijk} = \frac{h_{ij} b_k + h_{jk} b_i + h_{ki} b_j}{\alpha} \tag{4.11}$$

Using equations (4.1) and (4.11) in equation (3.4), we get

$$M_{\alpha\beta} = \frac{1}{\alpha}\sqrt{\frac{b^2}{1+n(n+1)b^2}}h_{\alpha\beta} \tag{4.12}$$

Again, using equations (4.1) and (4.11) in (3.5), we get

$$M_{\alpha} = 0 \tag{4.13}$$

Using equation (4.13) in equation (3.10), it shows that $H_{\alpha\beta} = H_{\beta\alpha}$ i.e., $H_{\alpha\beta}$ is symmetric. As a result, we can derive the following theorem.

Theorem 4.2. *The second fundamental v-tensor of* $F^{n-1}(c)$ *is considered in the equation* (4.12) *and* (4.13) *and the second fundamental h-tensor* $H_{\alpha\beta}$ *is symmetric for the generalised Z. Shen's* (α, β) *metric. Now differentiating equation* (4.1) *with respect to* β, *we get*

$$b_{i|j}B^i_{\alpha} + b_i B^i_{\alpha|\beta} = 0 \tag{4.14}$$

Using equation (3.6) and $b_{i|j} = B^j_{\beta} + b_{i|j}N^j H_{\beta}$, equation (4.14) reduces to

$$b_{i|j}B^j_{\beta}B^i_{\alpha} + b_{i|j}N^j H_{\beta}B^i_{\alpha} + b_i H_{\alpha\beta}N^i = 0. \tag{4.15}$$

Since $b_{i|j} = -b_h C^h_{ij}$, from (3.5), (4.7) and (4.13) we get

$$b_{i|j}B^i_{\alpha}N^j = \sqrt{\frac{b^2}{1+n(n+1)b^2}}M_{\alpha} = 0.$$

Using the above expression in equation (4.15), we get

$$b_{i|j}B^j_{\alpha}B^i_{\alpha} + \sqrt{\frac{b^2}{1+n(n+1)b^2}}H_{\alpha\beta} = 0. \tag{4.16}$$

It is obvious that $b_{i|j}$ is symmetric. Contracting now (4.16) with v^{β} first and then with v^{α} respectively and using the equations (3.1), (3.9), and (4.13) we get

$$b_{i|j}B^j_{\alpha}y^i + \sqrt{\frac{b^2}{1+n(n+1)b^2}}H_{\alpha} = 0 \tag{4.17}$$

$$b_{i|j}y^i y^i + \sqrt{\frac{b^2}{1+n(n+1)b^2}}H_\alpha = 0. \tag{4.18}$$

According to Lemma (3.1), and Lemma (3.2), a hypersurface $F^{n-1}(c)$ is a hyperplane of 1st kind if and only if the normal curvature vanishes i.e., $H_0 = 0$. Using the value $H_0 = 0$ in equation (4.18) we find that hypersurface $F^{n-1}(c)$ is again a hyperplane of 1st kind if and only if $b_{i|j}y^i y^j = 0$. This $b_{i|j}$ is the covariant derivative with respect to Cartan connection $C\Gamma$ of Finsler space F^n. Since b_i is a gradient vector, from equations (2.19) and (2.20), we have

$$E_{ij} = b_{ij},\ F_{ij} = 0,\ F_j^i = 0 \tag{4.19}$$

Using equation (4.19) into equation (2.21), we get

$$D^j_{jk} = b_{jk}B^i + b_{0k}B^i_j + b_{0j}B^j_k - b_{0m}g^{im}B_{jk} - A^m_k C^i_{jm} - A^m_j C^i_{km} - A^m_s C_{jkm}g^{is}$$

$$+ \lambda^s(C^m_{sk}C^i_{jm} + C^m_{sj}C^i_{km} - C^i_{ms}C^m_{jk}) \tag{4.20}$$

Using the equations (4.2), (4.4), (4.5), and (4.6) into equation (2.22), we get

$$B_k = n(n+1)(n+2)b_k + \frac{n(n+1)}{\alpha}y_k,\ B^i = \frac{n(n+1)}{1+n(n+1)b^2}b^i$$

$$+ \frac{n(n+1)}{\alpha(1+n(n+1)b^2)}y^i \tag{4.21}$$

$$B_{ij} = \frac{1}{\alpha}a_{ij} - \frac{1}{\alpha^3}y_i y_j + \frac{n(n+1)}{\alpha}b_i b_j \tag{4.22}$$

$$B_{ij} = \frac{1}{\alpha}\left(\delta^i_j - \frac{1}{\alpha^2}y^i y_j\right) + \frac{(n+1)^2}{\alpha(1+n(n+1)b^2)}b^i b_j$$

$$- \frac{(n(n+1)+(n+2)(n+3)b^2)}{\alpha^2(1+n(n+1)b^2)}b_j y^i \tag{4.23}$$

$$A_k^m = B_k^m b_{00} + B^m b_{k0},\ \lambda^m = B^m b_{00}. \tag{4.24}$$

Using the tensor contraction operation on equations (4.24) and (4.25) by y_j, we get $B_{i0} = 0$, $B_0^i = 0$. Further contracting equation (4.26) by y^k and using the fact that $B_0^i = 0$, we get $A_0^m = B^m b_{00}$. Contracting equation (4.20) by y^k and using the facts $B_{i0} = 0$, $B_0^i = 0$, $A_0^m = B^m b_{00}$ and $C_{s0}^m = 0$, $C_{0m}^i = 0$, $C_{j0}^m = 0$ obtained by contracting (3.2) and (3.7), we get

$$D_{j0}^i = B^i b_{j0} B_j^i b_{00} - b_{00} B^m C_{jmi}^i \tag{4.25}$$

$$D_{00}^i = \frac{n(n+1)}{1+n(n+1)b^2} b^i b_{00} + \frac{n(n+1)}{\alpha(1+n(n+1)b^2)} y^i b_{00} \tag{4.26}$$

Multiplying equation (4.25) by b_i and then using equations (4.2), (4.21), (4.22), and (4.23), we get

$$b_i D_{j0}^i = \frac{n(n+1)b^2}{1+n(n+1)b^2} b_{j0} + \frac{1+(n+2)(n+3)b^2}{\alpha(1+n(n+1)b^2)} b_i b_{00} - \frac{n(n+1)}{1+b^2} + b_i b^m C_{jm}^i b_{00} \tag{4.27}$$

Now multiplying (4.26) by b_i and then using equation (4.2) we get

$$b_i D_{00}^i = \frac{n(n+1)b^2}{1+(n+1)b^2} b_{00} \tag{4.28}$$

From equation (4.11), it is clear that

$$b^m b_i C_{jm}^i B_\alpha^j = b^2 M_\alpha = 0 \tag{4.29}$$

Contracting the expression $b_{i|j} = b_{ij} - b_r D_{ij}^r$ by y^i and y^j respectively and then using equation (4.28) we get

$$b_{i|j} y^i y^j = b_r D_{00}^r = \frac{1}{1+n(n+1)b^2} b_{00}$$

Using equations (4.27) and (4.29), equations (4.17) and (4.18) can be expressed as

$$\frac{1}{\sqrt{1+n(n+1)b^2}}\, b_{i0}B^i_\alpha + \sqrt{b^2}H_\alpha = 0 \tag{4.30}$$

$$\frac{1}{\sqrt{1+n(n+1)b^2}}\, b_{00} + \sqrt{b^2}H_\alpha = 0 \tag{4.31}$$

From the equation (4.31), it is clear that the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} is independent of y^i. Since y^i satisfy equation (4.2), the condition can be expressed as $b_{ij}y^iy^i = (b_iy^i)(c_jy^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_ic_j + b_jc_i. \tag{4.32}$$

Contracting (4.32) and using the fact that $b_iy^j = 0$, we get $b_{00} = 0$. Multiplying equation (4.3) by B^i_α and then B^j_β and using equation (4.1) gives $b_{ij}B^i_\alpha B^j_\beta = 0$. Similarly we get $b_{ij}B^i_\alpha y^i = 0$. This further gives $b_{i0}B^i_\alpha y^j = 0$. Using this in equation (4.30) gives $H_\alpha = 0$. Again contracting (4.32) and then using equation (4.2) gives $b_{i0} = \frac{b^2c_0}{2}$. Now using (4.23) and (4.24) and using $b_{00} = 0$ and $b_{ij}B^i_\alpha B\beta^j = 0$ we get $\lambda^m = 0$, $A^i_jB^j_\beta = 0$ and $B_{ij}B^i_\alpha B^j_\beta = \frac{1}{n(n+1)\alpha}h_{\alpha\beta}$ thus using equations (4.6), (4.7), (4.8), (4.12) and (4.20) we get

$$b_rD^r_{ij}B^i_\alpha B^j_\beta = \frac{c_0b^2}{(n+1)^2\alpha(1+n(n+1)b^2)^2}\, h_{\alpha\beta} \tag{4.33}$$

Thus using the relation $b_{i|j} = b_{ij} - b_rD^r_{ij}$ and equations (4.33) and (4.16) reduces to

$$\frac{c_0b^2}{n(n+1)\alpha(1+n(n+1)b^2)^2}\, h_{\alpha\beta} + \sqrt{\frac{b^2}{1+n(n+1)b^2}}H_{\alpha\beta} = 0. \tag{4.34}$$

Hence $F^{n-1}(c)$ is umbilic. Thus, we have the following result.

Theorem 4.3. *Equation* (4.32) *fulfils both the essential and satisfactory condition for* $F^{n-1}(c)$ *to be a* 1st *kind of hyperplane and its second fundamental tensor is proportional to its angular metric tensor*.

From the Lemma 3.3, $F^{n-1}(c)$ is a hyperplane of 2nd kind, iff $H_\alpha = 0$ and $H_{\alpha\beta} = 0.$ Therefore from equation (4.32), we get $c_0 = c_i(x)y^i = 0.$ Thus there exists a function $e(x)$ such that $c_i(x) = e(x) = b_i(x).$ Therefore, from equation (4.32) we have

$$b_{ij} = eb_i b_j. \tag{4.35}$$

Theorem 4.4. *Equation* (4.35) *satisfies both the necessary and sufficient condition for* $F^{n-1}(c)$ *to be a* 2nd *kind of hyperplane*.

Finally, from equation (4.12) and Lemma 3.4, we deduce that $F^{n-1}(c)$ is not a hyperplane of 3rd kind.

Theorem 4.5. *The* $F^{n-1}(c)$ *of Finsler space with generalised Z. Shen's metric is not a hyperplane of* 3rd *kind*.

Conclusion

In this paper, we explore the diverse kinds of hypersurfaces of Finsler space using generalized Z. Shen's metric $(\alpha, \beta) = \frac{(\alpha + \beta)^{n+1}}{\alpha^n}$. Additionally, the hypersurfaces we obtained are hyperplanes of the 1st and 2nd kind, but not of the 3rd kind.

Data Availability. No data were used to support this paper.

Conflicts of Interest. The authors declare that there are no conflicts of interest.

References

[1] Brijesh Kumar Tripathi, Hypersurfaces of a Finsler space with deformed Berwald-infinite series metric, J. App. Eng. Math. 10(2) (2020), 296-304.

[2] Pradeep Kumar and T. S. Madhu, On Finslerian hypersurface with generalized (α, β)-metric, Waffen-Und Kostumkunde Journal, ISSN NO:0042-9945 XI(V) (2020), 55-68.

[3] Gauree Shanker and Vijeta Singh, On the hypersurface of a Finsler space with Randers change of generalized (α, β) metric, International Journal of Pure and Applied Mathematics 105(2) (2015), 223-234.

[4] M. K. Gupta, Abhay Singh and P. N. Pandey, On a hypersurface of a Finsler space with Randers change of Matsumoto metric, Hindawi Publishing Corporation Geometry vol. 2013 pp. 6.

[5] Gauree Shanker and Ravindra Yadav, On the hypersurface of a Finsler space with the special (α, β)-metric $\alpha + \beta + \frac{\beta^{N+1}}{\alpha^N}$, Journal of the Indian Math. Soc. 80(3-4) (2013), 329-339.

[6] M. K. Gupta and Anil K. Gupta, Hypersurface of a Finsler space subjected to an h-exponential change of metric, International Journal of Geometric Methods in Modern Physics 13(10) (2016), 1650129.

[7] N. Natesh, S. K. Narasimhamurthy and M. K. Roopa, On the hypersurface of randers change of Finsler square metric, Asian Journal of Science and Technology 10(08) (2019), 9929-9940.

[8] U. P. Singh and Bindu Kuari, On a hypersurface of matsumuto surafce, Indian J. Pure Appl. Math 32(4) (2001), 521-531.

[9] Brijshe Kumar Tripathi, Hypersurfaces of a Finsler space with exponential form of (α, β)-metric, Annals of the University of Craiova, Mathematics and Computer Science Series 47(1) (2020), 1223-6934.

[10] V. K. Chaubey and A. Mishra, Hypersurfaces of a Finsler space with a special (α, β)-metric, Journal of Contemporary Mathematical Analysis 52(1) (2017), 1-7.

[11] Kumar Vineet and Rajesh Kumar Gupta, Hypersurface of a special Finsler space with metric $L = \beta + \frac{\alpha^3 + \beta^3}{\alpha(\alpha - \beta)}$, Journal of Computer and Mathematical Sciences 9(6) (2018), 579-587.

[12] Ganga Prasad Yadav and P. N. Pandey, Hypersurface of a Finsler space with randers change of (α, β)-metric, Journal Academy of International Physical Sciences 22(3) (2018), 179-193.

[13] Suraj Kumar Shukla and T. N. Pandey, Hypersurface of special finsler space, International Journal of Mathematics Trends and Technology (IJMTT) 66(6) (2020).

[14] Mohammad Rafee, Avdhesh Kumar and G. C. Chaubey, On the hypersurface of a Finsler space with the square metric, International Journal of Pure and Applied Mathematics 118(3) (2018), 723-735.

[15] V. K. Chaubey and Brijesh Kumar Tripathi, Hypersurface of a Finsler space with deformed Berwald-Matsumoto metric, Bullen of the Transilvania University of Braşov Series III: Mathemacs, Informacs, Physics 11(60) No. 1 (2018), 37-48.

[16] H. S. Shukla, O. P. Pandey and A. K. Mishra, Hypersurface of a Finsler space with a special (α, β)-metric, South Asian Journal of Mathematics 6(2) (2016), 82-91.

[17] H. Wosoughi, On a hypersurface of a special Finsler space with an exponential (α, β)-metric, Int. J. Contemp. Math. Sciences 6 (2011), 1969-1980.

[18] Pradeep Kumar, S. K. Narasimhamurthy, H. G. Nagaraja and S. T. Aveesh, On a special hypersurface of a Finsler space with (α, β)-metric, Tbilisi Mathematical Journal, Tbilisi Centre for Mathematical Sciences and College Publications 2 (2009), 51-60.

[19] M. Kitayama, On Finslerian hypersurfaces given by β-Change, Balkan J. of Geometry and It's Applications 7(2) (2002), 49-55.

[20] M. Matsumoto, Foundation of finsler geometry and special Finsler spaces, Kaiseisha Press, Saikawa, Otsu, Japan (1986), 520.

[21] G. Shanker and Ravindra, On the hypersurface of a second approximate Matsumoto metric $\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha_2}$ International J. of Contemp. Math. Sciences 7(3) (2012), 115-124.

[22] G. Shanker and Vijeta Singh, On the hypersurface of a Finsler space with Randers change of generalized (α, β)-metric, International Journal of Pure and Applied Mathematics 105(2) (2015), 223-234.