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HYERS-ULAM-RASSIAS STABILITY FOR FIRST ORDER NEUTRAL DELAY DIFFERENTIAL EQUATION

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Abstract

This paper discusses the Hyers-Ulam and Hyers-Ulam-Rassias stability for first order neutral delay differential equation with piecewise constant deviating argument. The initial value problem for this equation is solved by the method of steps. Gronwall type inequality is obtained and is used to discuss the stability of the solution of the equation. An example is given to support the results.

1. Introduction

In a Mathematical Colloquium at the University of Wisconsin, Stanislaw M. Ulam [21] discussed a couple of unsolved mathematical problems. One of the problems was about the stability of homomorphism [21]. Donal H. Hyers [4] in 1941 gave the first solution by solving it for a pair of Banach spaces using direct method. This stability phenomenon is now called Hyers-Ulam Stability. The next breakthrough came in the year 1978 when Themistocles

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M. Rassias [19] extended the result of Hyer's theorem. Rassias weaken the condition for boundedness of the norm for the Cauchy difference. This stability phenomenon is called "Hyers-Ulam-Rassias" stability. For more details the readers can refer to [5, 10].

Many papers are published since then, with the various generalization of Ulam problem and Hyers, Rassias theorem. Obloza [14] studied Hyers stability for linear differential equations and later [15] gave the connection between Hyers and Lyapunov stability of ordinary differential equations. A year later Alsina and Ger [1] studied the stability of $y'(t) = y(t)$. Miura et al. [12] generalized the study to the differential equation $y'(t) = \lambda y(t)$, where λ is a complex number, while Jung [8] proved a similar result for the differential equation of the type $\phi(t)y' = y$. Many mathematicians investigated the Hyers-Ulam and Hyers-Ulam-Rassias stability for different types of differential equations [[7], [13], [18], and references therein]. Recently Masakazu Onitsuka and Tomohiro Shoji [16] studied the Hyers-Ulam stability of the first-order linear differential equation $x' - ax = 0$ where a is a non-zero real number.

Delay differential equations (DDEs) are the simplest form of Functional differential equations (FDEs). They are also known as differential equations with the retarded argument as the derivative of the unknown function depends on the past history. Neutral delay differential equations (NDDEs) are natural extensions of the DDEs which involves the derivative of the unknown function at the delayed argument. Most of the realistic models in natural sciences, economics, and engineering are well defined by DDEs and NDDEs and details can be read in the monograph by Kuang [11]. As a result, many researchers are attracted to study Hyers-Ulam and Hyers-Ulam-Rassias stabilities of these equations.

Jung et al. [9] studied Hyers-Ulam stability of the DDE $y'(t) = y(t - \tau)$. D. Otrocol et al. [17] studied Ulam stability for a DDE of the type $x'(t) = f(t, x(t), x(g(t)))$, while J.H. Huang et al. [2] studied Hyers-Ulam stability of linear functional differential equations, and a year later for DDE of first order [3]. C. Tunc et al. [20] studied Hyers-Ulam-Rassias stability for a first-order functional differential equation.

Recently, A. Zada, S. Faisal, and Y. Li [22, 23] studied Hyers-Ulam stability of first-order impulsive DDEs and Hyers-Ulam-Rassias stability of nonlinear DDEs. Most of the research in last decades is on DDEs but not much is done on functional differential equations of neutral type. In this paper, we study the NDDE with piecewise constant argument.

The aim of this paper is to investigate Hyers-Ulam and Hyers-Ulam-Rassias stability of the equation,

$$x'(t) = f(t, x(t), x([t]), x'([t]))'; \; t \in J, \tag{1}$$

where $J = [0, T], T > 0, f \in C(J \times \mathbb{R}^3, \mathbb{R})$ and $[\cdot]$ is the greatest integer function.

Let $\mathcal{D}$ denote the class of all functions $x : J \to \mathbb{R}$ satisfying

1. $x(t)$ is continuous, for $t \in J$.

2. $x'(t)$ exists and is continuous on the intervals $[n, n+1)$, for $n = 0, 1, 2, \ldots, \tilde{T} - 2$ and on $[\tilde{T} - 1, T)$,

where

$$\tilde{T} = \begin{cases} [T] + 1, & T \neq [T], \\ T, & T = [T]. \end{cases}$$

A function $x : J \to \mathbb{R}$ is said to be solution of (1) if $x \in \mathcal{D}$ and satisfies (1) with $x'(t) = x'_+(t)$ on $t = 1, 2, \ldots, \tilde{T} - 1$.

2. Preliminaries

In this section, we present some preliminaries required for our discussion. We consider the equation (1) with initial condition

$$x(0) = x_0. \tag{2}$$

Here $(t, x(t), x([t]), x'([t])) \in K \subset \mathbb{R}^4$ a closed bounded set, $x \in \mathcal{D}$, and $t \in J$. We now define Hyers-Ulam and Hyers-Ulam-Rassias stability for (1).

Definition 1. Equation (1) is said to have Hyers-Ulam stability, if there exists a constant $c > 0$ such that for each $\epsilon > 0$ and for each solution x of the equation (1) satisfying

$$| x'(t) - f(t, x(t), x(t), x([t]), x'([t]), x'([t]) | \leq \epsilon, \tag{3}$$

with initial condition (2), there exists a solution $x_\alpha(t)$ of the equation (1), such that $| x(t) - x_\alpha(t) | \leq c\epsilon,$ where c is independent of ϵ and $x(t)$.

Let $\beta(t) : J \to [0, \infty)$ and $\gamma(t) : J \to [0, \infty)$ be continuous functions. We have the following definition.

Definition 2. Equation (1) is said to have Hyers-Ulam-Rassias stability, if there exists $\beta(t) > 0$ such that for each $\gamma(t) > 0$ and for each solution $x(t)$ of the equation (1) satisfying

$$| x'(t) - f(t, x(t), x([t]), x'([t])) | \leq \gamma(t), \tag{4}$$

with initial condition (2), there exists a solution $x_\alpha(t)$ of the equation (1) such that $| x(t) - x_\alpha(t) | \leq \beta(t),$ where $\gamma(t), \beta(t)$ are independent of $x(t)$ and $x_\alpha(t)$.

We now prove Gronwall type inequality.

Lemma 3. *Let* $x_0 > 0$ *be a constant. Let* $x \in \mathcal{D}$ *and* $x \geq 0$. *Let* $a_1, a_2, a_3 \in C(J, \mathbb{R}^+)$. *If*

$$x(t) < x_0 + \int_0^t [a_1(s)x(s) + a_2(s)x([s]) + a_3(s)x'([s])]ds, \; t \in J, \tag{5}$$

then

$$x(t) \leq x_0 \prod_{i=0}^{[t]-1} \left\{ e^{\int_i^{i+1} a_1(u)du} + \int_i^{i+1} e^{\int_s^{i+1} a_1(u)du} [a_2(s) + a_3(s)k(i)]ds \right\}$$

$$\times \left\{ e^{\int_{[t]}^{t} a_1(u)du} + \int_i^{i+1} e^{\int_s^{i+1} a_1(u)du} [a_2(s) + a_3(s)k([t])]ds \right\}, \; t \in J, \tag{6}$$

where $k([t]) = \dfrac{a_1([t]) + a_2([t])}{1 - a_3([t])}$ and $a_3([t]) \neq 1$.

Proof. We set $z(t) = x_0 + \int_0^t [a_1(s)x(s) + a_2(s)x([s]) + a_3(s)x'([s])]ds, \; t \in J$.

Then, $z(0) = x_0, x(t) \leq z(t), z(t)$ is positive and nondecreasing for $t \in J$, and $x'([t]) \leq k([t])x([t]) \leq k([t])z([t])$.

Then

$$z(t) \leq x_0 + \int_0^t [a_1(s)z(s) + a_2(s)z([s]) + a_3(s)k([s])z([s])]ds,$$

$$\leq x_0 + \int_0^t [a_1(s) + (a_2(s) + a_3(s)k([s])z([s])]ds.$$

By Theorem 4.1 in [6], we obtain the desired result. □

Lemma 4. *Let* $h(t)$ *be a nondecreasing function in* $\mathbb{R}$, $x \in \mathcal{D}$, $x \geq 0$, $a_1, a_2, a_3 \in C(J, \mathbb{R}^+)$. *If*

$$x(t) < h(t) + \int_0^t [a_1(s)x(s) + a_2(s)x([s]) + a_3(s)x'([s])]ds, \; t \in J. \tag{7}$$

Then

$$x(t) \leq h(t) \prod_{i=0}^{[t]-1} \left\{ e^{\int_i^{i+1} a_1(u)du} + \int_s^{i+1} e^{\int_s^{i+1} a_1(u)du} [a_2(s) + a_3(s)k(i)]ds \right\}$$

$$\times \left\{ e^{\int_{[t]}^t a_1(u)du} + \int_{[t]}^t e^{\int_s^t a_1(u)du} [a_2(s) + a_3(s)k([t])]ds \right\}, \; t \in J. \tag{8}$$

where $k([t]) = \dfrac{a_1([t]) + a_2([t])}{1 - a_3([t])}$ *and* $a_3([t]) \neq 1$.

Proof. Dividing (7) by $h(t)$ we get

$$\frac{x(t)}{h(t)} < 1 + \int_0^t \left[\frac{a_1(s)x(s)}{h(s)} + \frac{a_2(s)x([s])}{h(s)} + \frac{a_3(s)x'([s])}{h(s)} \right] ds.$$

Noticing that $h(t) \geq h([t])$ is nondecreasing. We get

$$\frac{x(t)}{h(t)} < 1 + \int_0^t \left[\frac{a_1(s)x(s)}{h(s)} + \frac{a_2(s)x([s])}{h(s)} + \frac{a_3(s)x'([s])}{h(s)} \right] ds.$$

Now taking $p(t) = \frac{x(t)}{h(t)}$ and $x'([t]) = k([t])x([t])$, we obtain

$$p(t) < 1 + \int_0^t [a_1(s)p(s) + a_2(s)p([s]) + a_3(s)k([s])p([s])]ds$$

$$< 1 + \int_0^t \{a_1(s)p(s) + [a_2(s) + a_3(s)k([s])]p([s])\}ds$$

$$< 1 + \int_0^t [a_1(s)p(s) + r(s)p([s])]ds, \tag{9}$$

where $r(t) = a_2(t) + a_3(t)k([t])$. Using Lemma 3 with $a_3(s) = 0$ in (5), we obtain

$$p(t) \le \prod_{i=0}^{[t]-1} \left\{ e^{\int_i^{i+1} a_1(u)du} + \int_i^{i+1} e^{\int_s^{i+1} a_1(u)du} r(s)ds \right\}$$

$$\times \left\{ e^{\int_{[t]}^{i+1} a_1(u)du} + \int_i^t e^{\int_s^t a_1(u)du} r([s])ds \right\},\ t \in J.$$

Now, on substituting for $r(t)$ we obtain

$$p(t) \le \prod_{i=0}^{[t]-1} \left\{ e^{\int_i^{i+1} a_1(u)du} + \int_i^{i+1} e^{\int_s^{i+1} a_1(u)du} [a_2(s) + a_3(s)k(i)]ds \right\}$$

$$\times \left\{ e^{\int_{[t]}^{t} a_1(u)du} + \int_{[t]}^t e^{\int_s^t a_1(u)du} [a_2(s) + a_3(s)k([t])]ds \right\},\ t \in J,$$

from which (8) follows. □

Theorem 5. *The unique solution of* IVP (1), (2) *on* $[0, T]$ *is given by*

$$x(t) = x_0 + \sum_{i=0}^{\tilde{T}-2} \int_i^{i+1} f(s, x(s), x(i), x'(i))ds$$

$$+ \int_{\tilde{T}-1}^t f(s, x(s), x([s]), x'([s]))ds. \tag{10}$$

Proof. Consider equations (1) and (2). For $t \in [0, 1)$ we obtain,

$$x(t) = x_0 + \int_0^t f(s, x(s), x(0), x'(0))ds.$$

As $t \to 1$, we get

$$x(1) = x_0 + \int_0^1 f(s, x(s), x_0, x'_0)ds.$$

For $t \in [1, 2)$, we get

$$x(t) = x(1) + \int_1^t f(s, x(s), x(1), x'(1))ds$$

$$= x_0 + \int_0^1 f(s, x(s), x_0, x'_0)ds + \int_1^t f(s, x(s), x(1), x'(1))ds.$$

As $t \to 2$

$$x(2) = x_0 + \int_0^1 f(s, x(s), x_0, x'_0)ds + \int_1^2 f(s, x(s), x(1), x'(1))ds$$

$$= x_0 + \sum_{i=0}^{1} \int_i^{i+1} f(s, x(s), x(i), x'(i))ds.$$

Continuing this way and as $t \to n$, we obtain

$$x(n) = x_0 + \sum_{i=0}^{n-1} \int_i^{i+1} f(s, x(s), x(i), x'(i))ds.$$

Consequently the unique solution for (1) and (2) on $[0, T]$ is given by (10). □

3. Hyers-Ulam and Hyers-Ulam-Rassias Stability

In this section we investigate Hyers-Ulam and Hyers-Ulam-Rassias stability of (1) and (2).

Theorem 6. *Suppose that* $x \in \mathcal{D}$, $(t, x(t), x([t]), x'([t])) \in K \subset \mathbb{R}^4$ *a closed bounded set and f satisfies the following conditions*:

(H1) $f(t, x(t), x([t]), x'([t]))$ *be continuous function on* K,

(H2) f *is bounded on* K *i.e.* $\sup_{(t,\, x(t),\, x([t]),\, x'([t])) \in K} |f| \leq M$,

(H3) *For* $x_1 \geq x_2$, $y_1 \geq y_2$, $z_1 \geq z_2$ *and* L_1, L_2, L_3 *are positive constant with* $L_3 \neq 1$,

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq L_1 |x_1 - x_2| + L_2 |y_1 - y_2| + L_3 |z_1 - z_2|.$$

(H4) *for* $\epsilon > 0, t \in J$

$$|x'(t) - f(t, x(t), x([t]), x'([t])) \leq \epsilon,$$

then (1) *and* (2) *have Hyers-Ulam stability*.

Proof. Let $x(t)$ be a solution of (1) and (2). Then for $\epsilon > 0$ on J,

$$|x'(t) - f(t, x(t), x([t]), x'([t])) \leq \epsilon.$$

From Theorem 5, we have for $t \in [0, T]$,

$$|x(t) - x(0) - \sum_{i=0}^{[t]-1} \int_i^{i+1} f(s, x(s), x(i), x'(i)) ds - \int_{[t]}^t f(s, x(s), x([s]), x'([s])) | \leq \epsilon t.$$

Now for $t \in J$ and $x, y \in \mathcal{D}$, satisfying the equations (1) and (2), we obtain

$$\begin{aligned} |x(t) - y(t)| = |x(t) - x_0 - \sum_{i=0}^{[t]-1} \int_i^{i+1} f(s, x(s), x(i), x'(i)) ds \\ - \int_{[t]}^t f(s, x(s), x([s]), x'([s]) ds - y(t) + y_0 \\ + \sum_{i=0}^{[t]-1} \int_i^{i+1} f(s, y(s), y(i), y'(i)) ds + \int_i^t f(s, y(s), y([s]), y'([s])) ds \\ + \sum_{i=0}^{[t]-1} \int_i^{i+1} [f(s, x(s), x(i), x'(i)) - f(s, y(s), y(i), y'(i))] ds \end{aligned}$$

$$+\int_i^t [f(s,\ x(s),\ x([s]),\ x'([s])) - f(s,\ y(s),\ y([s]),\ y'([s]))]ds\ |,$$

$$\leq \epsilon t + \sum_{i=0}^{[t]-1}\int_i^{i+1}[L_1|\ x(s) - y(s)\ | + L_2|\ x(i) - y(i)\ | + L_3|\ x'(i) - y'(i)\ |]ds$$

$$+\int_{[t]}^t [L_1|\ x(s) - y(s)\ | + L_2|\ x([s]) - y([s])\ | + L_3|\ x'([s]) - y'([s])\ |]ds.$$

Using Lemma 4, we can write

$$|\ x(t) - y(t)\ | \leq \epsilon t \times \prod_{i=0}^{[t]-1}\left\{e^{\int_i^{i-1} L_1 du} + \int_i^{i+1} e^{\int_s^{i+1} L_1 du}[L_2 + L_3 L^*]ds\right\}$$

$$\times\left[e^{\int_{[t]}^t L_1 du} + \int_{[t]}^t e^{\int_i^t L_1 du}[L_2 + L_3 L^*]ds\right]$$

$$\leq \epsilon T \times \prod_{i=0}^{[t]-1}\left\{e^{\int_i^{i-1} L_1 du} + \int_i^{i+1} e^{\int_s^{i+1} L_1 du}[L_2 + L_3 L^*]ds\right\}$$

$$\times\left\{e^{\int_{[t]}^t L_1 du} + \int_{[t]}^t e^{\int_i^t L_1 du}[L_2 + L_3 L^*]ds\right\}$$

$$\leq c\epsilon,$$

where $L^* = \dfrac{L_1 + L_2}{1 - L_3}$, $L_3 \neq 1$, and c is a constant given by

$$c = T \times \prod_{i=0}^{[t]-1}\left\{e^{\int_i^{i+1} L_1 du} + \int_i^{i+1} e^{\int_s^{i+1} L_1 du}[L_2 + L_3 L^*]ds\right\}$$

$$\times\left\{e^{\int_{[t]}^t L_1 du} + \int_{[t]}^t e^{\int_s^t L_1 du}[L_2 + L_3 L^*]ds\right\}.$$

Therefore,

$$\max_{0 \leq t \leq T}|\ x(t) - y(t)\ | \leq c\epsilon.$$

Consequently, (1) and (2) have Hyers-Ulam stability. □

In the following theorem, we state without proof the Hyers-Ulam-Rassias stability of (1) and (2) on J. The proof is similar to Theorem 6.

Theorem 7. *Suppose that* $x \in \mathcal{D}, (t, x(t), x([t]), x'([t])) \in K \subset \mathbb{R}^4$ *a closed bounded set and along with* (H1), (H2), (H3) f *satisfies the following condition*:

(H5) *for* $t \in J$

$$| x'(t) - f(t, x(t), x([t]), x'([t])) | \leq \gamma(t),$$

then IVP (1) *and* (2) *have Hyers-Ulam-Rassias stability*.

4. An Example

Consider the equation

$$x'(t) = -\frac{\cos t}{2}[2x(t) + x([t])] + \left[\frac{\sin t}{2} x([t])\right]', \tag{11}$$

with initial conditions

$$x(0) = 1. \tag{12}$$

Then $f(t, x(t), x([t]), x'([t])) = -\frac{\cos t}{2}[2x(t) + x([t])] + \left[\frac{\sin}{2} x([t])\right]'$.

It is easy to see that $f(t, x(t), x([t]), x'([t]))$ satisfies (H1)-(H4).

Now from (H4), we get

$$| x'(t) - f(t, x(t), x([t]), x'([t])) | \leq \epsilon, \tag{13}$$

where $t \in [0, T], T > 0$. Integrating (13), we obtain

$$| x(t) - x(0) - \left\{\int_0^t \frac{\cos(s)}{2}[2x(s) + x([s])]ds + \int_0^t \left[\frac{\sin(s)}{2} x([s])\right]' ds\right\} | \leq \epsilon t.$$

Let us consider $y(t)$ which satisfies (11) and (12) with $y(0) = x(0) = 1$. Let L_1, L_2, L_3 be positive constants and $L_3 \neq 1$.

Then,

$$|x(t)-y(t)| \leq \epsilon T \prod_{i=0}^{[t]-1}\left\{e^{\int_i^{i+1} L_1 du} + \int_i^{i+1} e^{\int_s^{i+1} L_1 du}[L_2 + L_3 L^*]ds\right\},$$

$$\times \left\{e^{\int_{[t]}^{t} L_1 du} + \int_{[t]}^{t} e^{\int_s^{t} L_1 du}[L_2 + L_3 L^*]ds\right\},$$

$$\leq c\epsilon,$$

where $L^* = \frac{L_1 + L_2}{1 - L_3}$. As a result Hyers-Ulam stability of (11) follows. Also, it is easy to see that if (H5) is satisfied, that is

$$|x'(t) - f(t, x(t), x([t]), x'([t]))| \leq \gamma(t), \tag{14}$$

where $\gamma(t) = e^{\left(\frac{-\sin t}{24}\right)}$, then,

$$|x(t) - y(t)| \leq \int_0^t c\gamma(s)ds \leq \beta(t),$$

which shows that (11) has Hyers-Ulam-Rassias stability.

5. Conclusion

The following are the conclusions:

1. We prove Gronwall type inequality for first order NDDE with piecewise constant deviating argument.

2. We use the method of steps to get the existence of a solution of (1).

3. We prove Hyers-Ulam and Hyers-Ulam-Rassias stability of a solution of (1) using Gronwall type inequality.

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