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COMMON FIXED POINT THEOREMS ON (φ, ψ)-TYPE MULTI VALUED MAPPINGS IN *b*-METRIC SPACES

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Abstract

In this paper, we prove a fixed point theorem and a common fixed point theorem for new type of generalized multi-valued contractive mappings, via the class functions Φ and Ψ. The main theorem is an extension of the common fixed point theorem for (φ, ψ)-type multi-valued mappings on complete *b*-metric spaces. The conditions for existence and uniqueness of the common fixed point are investigated.

Introduction

In 1922, Stefan Banach [3] proved a fixed point theorem for contractive mappings in complete metric spaces. In 1969, Nadler [13] Introduce the concept of Multivalve function. Later, Czerwik [5, 6] initiate the concept of *b*-metrics which generalized usual metric spaces. After his contribution, many results were presented in β-generalized weak contractive multifunction's and b-metric spaces. In 2012, Aydi et al. [2]. Reformulate the *b*-metric space. Many researcher work in this area of research of multivalued function and *b*-metric spaces [1, 4, 7, 8, 9-12, 14]. The following definitions will be needed in the sequel:

Definition 1 [13]**.** Let X and Y be nonempty sets. T is said to be multi-valued mapping from X to Y if T is a function for X to the power set of Y. We

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denote a multi-valued map by $T : X \to 2^Y$.

Definition 2 [13]. A point of $x_0 \in X$ is said to be a fixed point of the multi-valued mapping T if $x_0 \in Tx_0$.

Definition 3 [13]. Let (X, D_b) be a metric space. A map $T : X \to X$ is called contraction if there exists $0 \leq s < 1$ such that $d(Tx, Ty) \leq s\, d(x, y)$, for all $x, y \in X$.

Definition 4 [13]. Let (X, D_b) be a metric space. We define the Hausdorff metric on $CB(X)$ induced by D_b. That is $H(A, B) = \max\{\sup x \in AD_b(x, B), \sup y \in BD_b(y, A)\}$ for all $A, B \in CB(X)$, where $CB(X)$ denotes the family of all non-empty closed and bounded subsets of X and $D_b(x, B) = \inf\{D_b(x, b) : b \in B\}$, for all $x \in X$.

Definition 5 [13]. Let (X, D_b) be a metric space. A map $T : X \to CB(X)$ is a_i to be multi-valued contraction if there exists $0 \leq s < 1$ such that $H(Tx, Ty) \leq s\ D_b(x, y)$, for all $x, y \in X$.

Lemma 6 [13]. *If* $A, B \in CB(X)$ *and* $a \in A$, *then for each* $\epsilon > 0$, *there exists* $b \in B$ *such that* $D_b(a, b) \leq H(A, B) + \epsilon$.

Definition 7 [2]. Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $D_b : X \times X \to R^+$ is called a b-metric provide that, for all $x, y, z \in X$,

(d1) $D_b(x, y) = 0$ if and only if $x = y$,

(d2) $D_b(x, y) = D_b(y, x)$;

(d3) $D_b(x, z) \leq s[D_b(x, y) + D_b(y, z)]$.

A pair (X, D_b) is called a b-metric space.

Example 8 [9]. The space $l^p = \{(x_n : \sum_{n=1}^{\infty} | x_n |^p < \infty, 0 < p < 1)\}$, together with the function $D_b : l^p \times l^p \to R^+$.

Definition 9 [2]**.** Let (X, D_b) be a b-metric space. Then a sequence (x_n) in X is called Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in R$ such that for each $m, n \geq n(\epsilon)$ we have $D_b(x_n, x_m) < \epsilon$.

Definition 10 [2]**.** Let (X, D_b) be a b-metric space. Then a sequence (x_n) in X is called convergent sequence if and only if there exists $x \in X$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in R$ such that for all $n \geq n(\epsilon)$ we have $D_b(x_n, x) < \epsilon$. In this case we write $\lim_{n \to \infty} x_n = x$.

Definition 11 [7]**.** Let ψ be the family of all functions $\psi : R_0^+ \to R_0^+$ such that

(1) ψ is non-decreasing

(2) $\lim_{n \to \infty} \psi_n(t) = 0$ for all $t \geq 0$, where ψ^n stands for the n^{th} iterate of ψ.

Example 12 [7]**.** Let

1. $\psi_1(t) = \alpha t,\ 0 < \alpha < 1,$ for all $t \geq 0$,

2. $\psi_2(t) = \dfrac{t}{1+t}$, for all $t \geq 0$,

Definition 13 [7]**.** Let Φ be the family of all functions $\varphi : R_0^+ \to R_0^+$ such that:

(1) φ is non-decreasing and continuous;

(2) For each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n \to \infty} \varphi(t_n)$ if and only if $\lim_{n \to \infty} t_n = 0$.

(3) $\varphi(kt) = k\varphi(t)$, for some $k > 0$.

Example 14 [7]**.** Let

1. $\varphi_1(t) = t$

2. $\varphi_2(t) = ef(t),\ f \in F,\ \forall t > 0$.

Main Results

We prove a unique fixed point for generalised multi-valued (φ, ψ)-contraction mapping via the class functions Φ and Ψ.

Theorem 15. *Let* (X, D_b) *be a complete b-metric space with constant* $s \geq 1$. *Let* $f : X \to CB(X)$ *be define a generalised multi-valued* (ψ, φ)-*contraction mapping if there exist* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that for all* $x, y \in X$,

$$\varphi(H(fx, fy)) \leq \psi(\varphi(M(x, y))) \tag{1.1}$$

where,

$$M(x, y) = \alpha_1 D_b(x, fx) + \alpha_2 D_b(y, fy) + \alpha_3 D_b(x, fy) + \alpha_4 D_b(y, fx)$$

$$+ \alpha_5 D_b(x, y) + \alpha_6 \frac{D_b(x, fx)(1 + D_b(x, fx))}{1 + D_b(x, y)}$$

for all $x, y \in X$ *and* $\alpha_i \geq 0, i = 1, 2, 3, \ldots, 6$ *with* $\alpha_1 + \alpha_2 + 2s\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 < 1$. *Then f has a unique fixed point.*

Proof. Fix any $x \in X$. Define $x_0 = x$ and let $x_1 \in fx_0$. By Lemma 6, we may choose $x_2 \in fx_1$ such that

$$\varphi(D_b(x_1, x_2)) \leq \varphi(H(fx_0, fx_1) + (\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)).$$

Now,

$$\varphi(D_b(x_1, x_2)) \leq \varphi(H(fx_0, fx_1))$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_0, fx_0) + \alpha_2 D_b(x_1, fx_1) + \alpha_3 D_b(x_0, fx_1) + \alpha_4 D_b(x_1, fx_0) \\ + \alpha_5 D_b(x_0, x_1) + \alpha_6\left(\frac{D_b(x_0, fx_0)(1 + D_b(x_0, fx_0))}{1 + D_b(x_0, x_1)}\right)\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_0, x_1) + \alpha_2 D_b(x_1, x_2) + \alpha_3 D_b(x_0, x_2) + \alpha_4 D_b(x_1, x_1) \\ + \alpha_5 D_b(x_0, x_1) + \alpha_6\left(\frac{D_b(x_0, x_1)(1 + D_b(x_0, x_1))}{1 + D_b(x_0, x_1)}\right)\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_0, x_1) + \alpha_2 D_b(x_1, x_2) + \alpha_3 D_b(x_0, x_2) + \alpha_4 D_b(x_1, x_1) \\ + \alpha_5 D_b(x_0, x_1) + \alpha_6 (D_b(x_0, x_1))\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + \alpha_5 + \alpha_6)D_b(x_0, x_1) + \alpha_2 D_b(x_1, x_2) \\ + \alpha_3 s[D_b(x_0, x_1) + D_b(x_1, x_2)]\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_0, x_1) \\ + \alpha_2 D_b(x_1, x_2) + s\alpha_3 D_b(x_1, x_2)\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_0, x_1) \\ + (\alpha_2, s\alpha_3) + D_b(x_1, x_2)\end{pmatrix}\right)$$

$$\leq \psi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_0, x_1) \\ + (\alpha_2, s\alpha_3)\varphi D_b(x_1, x_2)\end{pmatrix}$$

$$\leq \psi\left(\frac{\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6}{1 + (\alpha_2, s\alpha_3)}\varphi(D_b(x_0, x_1))\right).$$

There exist $x_3 \in fx_2$ such that

$$\varphi(D_b(x_2, x_3)) \leq \varphi\left(H(fx_1, fx_2) + \frac{\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6}{1 + (\alpha_2, s\alpha_3)}\right)$$

Now,

$$\varphi(D_b(x_2, x_3)) \leq \varphi\left(H(fx_1, fx_2) + \frac{\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6}{1 + (\alpha_2, s\alpha_3)}\right)$$

$$\leq \varphi(H(fx_1, fx_2))$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_1, fx_1) + \alpha_2 D_b(x_2, fx_2) + \alpha_3 D_b(x_1, fx_2) + \alpha_4 D_b(x_2, fx_1) \\ + \alpha_5 D_b(x_1, x_2) + \alpha_6\left(\frac{D_b(x_1, fx_1)(1 + D_b(x_1, fx_1))}{1 + D_b(x_1, x_2)}\right)\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_1, x_2) + \alpha_2 D_b(x_2, x_3) + \alpha_3 D_b(x_1, x_3) + \alpha_4 D_b(x_2, x_2) \\ + \alpha_5 D_b(x_1, x_2) + \alpha_6\left(\frac{D_b(x_1, x_2)(1 + D_b(x_1, x_2))}{1 + D_b(x_1, x_2)}\right)\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_1, x_2) + \alpha_2 D_b(x_2, x_3) + \alpha_3 D_b(x_1, x_3) + \alpha_4 D_b(x_2, x_2) \\ + \alpha_5 D_b(x_1, x_2) + \alpha_6(D_b(x_1, x_2))\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + \alpha_5 + \alpha_6)D_b(x_1, x_2) + \alpha_2 D_b(x_2, x_3) \\ + \alpha_3 s[D_b(x_1, x_2) + D_b(x_2, x_3)]\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_1, x_2)\\ + \alpha_2 D_b(x_2, x_3) + s\alpha_3 D_b(x_2, x_3)\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_1, x_2)\\ + (\alpha_2, s\alpha_3) + D_b(x_2, x_3)\end{pmatrix}\right)$$

$$\leq \psi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_1, x_2)\\ + (\alpha_2, s\alpha_3)\varphi D_b(x_2, x_3)\end{pmatrix}$$

$$\leq \psi\left(\frac{\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6}{1 + (\alpha_2, s\alpha_3)}\varphi(D_b(x_1, x_2))\right).$$

Continuing this process, we obtain by induction a sequence $\{x_n\}$ such that $x_n \in fx_{n-1}$, $x_{n-1} \in fx_n$ such that

$$\varphi(D_b(x_n, x_{n-1})) \leq \varphi\left(H(fx_{n-1}, fx_n) + \frac{\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6}{1 + (\alpha_2, s\alpha_3)}\right)$$

$$\leq \varphi(H(fx_{n-1}, fx_n))$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{n-1}, fx_{n-1}) + \alpha_2 D_b(x_n, fx_n) + \alpha_3 D_b(x_{n-1}, fx_n) + \alpha_4 D_b(x_n, fx_{n-1})\\ + \alpha_5 D_b(x_{n-1}, x_n) + \alpha_6\left(\dfrac{D_b(x_{n-1}, fx_{n-1})(1 + D_b(x_{n-1}, fx_{n-1}))}{1 + D_b(x_{n-1}, x_n)}\right)\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{n-1}, x_n) + \alpha_2 D_b(x_n, x_{n+1}) + \alpha_3 D_b(x_{n-1}, x_{n+1}) + \alpha_4 D_b(x_n, x_n)\\ + \alpha_5 D_b(x_{n-1}, x_n) + \alpha_6\left(\dfrac{D_b(x_{n-1}, x_n)(1 + D_b(x_{n-1}, x_n))}{1 + D_b(x_{n-1}, x_n)}\right)\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{n-1}, x_n) + \alpha_2 D_b(x_n, x_{n+1}) + \alpha_3 D_b(x_{n-1}, x_{n+1}) + \alpha_4 D_b(x_n, x_n)\\ + \alpha_5 D_b(x_{n-1}, x_n) + \alpha_6(D_b(x_{n-1}, x_n))\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + \alpha_5 + \alpha_6)D_b(x_{n-1}, x_n) + \alpha_2 D_b(x_n, x_{n+1})\\ + \alpha_3 s[D_b(x_{n-1}, x_n) + D_b(x_n, x_{n+1})]\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_{n-1}, x_n)\\ + \alpha_2 D_b(x_n, x_{n+1}) + s\alpha_3 D_b(x_n, x_{n+1})\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_{n-1}, x_n)\\ + (\alpha_2, s\alpha_3) + D_b(x_n, x_{n+1})\end{pmatrix}\right)$$

$$\leq \psi\begin{pmatrix}(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)D_b(x_{n-1}, x_n) \\ + (\alpha_2, s\alpha_3)\varphi D_b(x_n, x_{n+1})\end{pmatrix}$$

$$\leq \psi\left(\frac{(\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6)}{1 + (\alpha_2, s\alpha_3)}\varphi(D_b(x_{n-1}, x_n))\right)$$

for all $n \in N$ and let

$$\delta = \left[\frac{\alpha_1 + s\alpha_3 + \alpha_5 + \alpha_6}{1 + (\alpha_2, s\alpha_3)}\right]$$

$$\begin{aligned}\varphi(D_b(x_n, x_{n+1})) &\leq \delta(\psi(\varphi(D_b(x_{n-1}, x_n)))) \\ &\leq \delta[\delta(\psi(\varphi(D_b(x_{n-2}, x_{n-1}))))] \\ &= \delta^2(\psi(\varphi(D_b(x_{n-2}, x_{n-1})))) \\ &\vdots \\ &\leq \delta^n(\psi(\varphi(D_b(x_0, x_1)))).\end{aligned}$$

Since $\delta < 1, \Sigma\delta^n$ have a radius of convergence. Then, $\{x_n\}$ is a Cauchy sequence. But (X, D_b) is a complete b-metric space, it follows that $\{x_n\}_{n=1}^{\infty}$ is convergent $u = \lim_{n\to\infty} x_n$.

Now,

$$\begin{aligned}\varphi(D_b(q, fq)) &\leq \psi(\varphi(s[D_b(q, x_{n+1}) + D_b(x_{n+1}, fq)])) \\ &\leq \varphi(s[D_b(q, x_{n+1}) + D_b(fx_n, fq)])\end{aligned}$$

Using (1.1), we obtain,

$$\varphi(D_b(q, fq)) \leq \psi\left(\varphi\left(\begin{matrix} s[D_b(q, x_{n+1})] \\ + \left(s\begin{bmatrix}\alpha_1 D_b(x_n, fx_n) + \alpha_2 D_b(q, fq) + \alpha_3 D_b(x_n, fq) \\ + \alpha_4 D_b(q, fx_n) + \alpha_5 D_b(x_n, q) + \alpha_6 D_b(x_n, q)\end{bmatrix}\right)\end{matrix}\right)\right)$$

As $n \to \infty$, $\varphi(D_b(q, fq)) \leq \psi(\varphi(s[\alpha_2 D_b(q, fq) + \alpha_3 D_b(q, fq)]))$

$$(1 - (\alpha_2 s + \alpha_3 s))D_b(q, fq) \leq 0.$$

Which is not true, $\varphi(D_b(q, fq)) = 0$. Thus, $fq = q$.

Now, show that q is the unique fixed point of f. Assume that r is another fixed point of f. Then we have $fr = r$ and

$$\varphi(D_b(q, r)) = \varphi(D_b(fq, fr))$$
$$\leq \varphi(s[D_b(q, fr) + D_b(r, fq)])$$

Which is implies that, $\varphi(D_b(q, r)) \leq 2sD_b(q, r)$. This implies $q = r$.

Corollary 16. *Let* (X, D_b) *be a complete b-metric space with constant* $s \geq 1$. *Let* $f : X \to CB(X)$ *be define a generalised multi-valued* (ψ, φ)-*contraction mapping if there exist* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that for all* $x, y \in X$,

$$\varphi(H(fx, fy)) \leq \psi\left(\varphi\left(\alpha D_b(x, fx) + \beta \frac{D_b(x, fy)(1 + D_b(x, fx))}{1 + D_b(x, y)}\right)\right)$$

for all $x, y \in X$ *and* $\alpha, \beta \geq 0$, *with* $\alpha + s\beta < 1$. *Then f has a unique fixed point.*

Next, we need to prove a unique common fixed point for generalised multi-valued (ψ, φ)-contraction mappings via the class functions Φ and Ψ.

Theorem 17. *Let* (X, D_b) *be a complete b-metric space with constant* $s \geq 1$. *Let* $f, g : X \to CB(X)$ *be generalized multi-valued* (ψ, φ)-*contraction mapping, satisfies the condition*:

$$\varphi(H(fx, gy)) \leq \psi(\varphi(\alpha_1 D_b(x, fx) + \alpha_2 D_b(y, gy)$$
$$+ \alpha_3 D_b(x, gy) + \alpha_4 D_b(y, fx) + \alpha_5 D_b(x, y)),$$

Where, there exists $\psi \in \Psi, \varphi \in \Phi$ *such that for all* $x, y \in X$ *and* $\alpha_i \geq 0, i = 1, 2, \ldots, 5$, *with* $(\alpha_1 + \alpha_2)(s + 1) + (\alpha_3 + \alpha_4)(s_2 + s) + 2s\alpha_5 < 2$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$. *Then f and g have a unique common fixed point.*

Proof. Fix any $x \in X$. Define $x_0 = x$ and let $x_1 \in fx_0, x_2 \in gx_1$ such that $x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}$, choose $x_2 \in gx_1$ such that

$$\varphi(D_b(x_1, x_2)) \leq \varphi(H(fx_0, gx_1) + (\alpha_1 + \alpha_5 + s\alpha_3))$$

$$\leq \varphi(H(fx_0, gx_1))$$

$$\varphi(D_b(x_1, x_2)) \leq \psi\left(\varphi\left(\begin{array}{c}\alpha_1 D_b(x_0, fx_0) + \alpha_2 D_b(x_1, gx_1) + \alpha_3 D_b(x_0, gx_1) \\ + \alpha_4 D_b(x_1, fx_0) + \alpha_5 D_b(x_0, x_1)\end{array}\right)\right)$$

$$= \psi\left(\varphi\left(\begin{array}{c}\alpha_1 D_b(x_0, x_1) + \alpha_2 D_b(x_1, x_2) + \alpha_3 D_b(x_0, x_2) \\ + \alpha_4 D_b(x_2, x_2) + \alpha_5 D_b(x_0, x_1)\end{array}\right)\right)$$

$$\leq \psi\left(\varphi\left(\begin{array}{c}\alpha_1 D_b(x_0, x_1) + \alpha_2 D_b(x_1, x_2) \\ + \alpha_3 s[D_b(x_0, x_1) + D_b(x_1, x_2)] + \alpha_5 D_b(x_0, x_1)\end{array}\right)\right)$$

$$\leq \psi(\varphi((\alpha_1 + s\alpha_3 + \alpha_5) D_b(x_0, x_1) + (\alpha_2 + s\alpha_3) D_b(x_1, x_2)))$$

$$\leq \psi\left(\frac{\alpha_1 + \alpha_5 + s\alpha_3}{1 - (\alpha_2 + s\alpha_3)} \varphi(D_b(x_0, x_1))\right) \tag{1.2}$$

Then,

$$\varphi(D_b(x_2, x_1)) = \varphi(D_b(gx_1, fx_0))$$

$$\leq \varphi(H(gx_1, fx_0) + (\alpha_2 + \alpha_5 + s\alpha_4))$$

$$\leq \varphi(H(gx_1, fx_0))$$

$$\leq \psi\left(\varphi\left(\begin{array}{c}\alpha_1 D_b(x_1, gx_1) + \alpha_2 D_b(x_0, fx_0) + \alpha_3 D_b(x_1, fx_0) \\ + \alpha_4 D_b(x_0, gx_1) + \alpha_5 D_b(x_1, x_0)\end{array}\right)\right)$$

$$= \psi\left(\varphi\left(\begin{array}{c}\alpha_1 D_b(x_1, x_2) + \alpha_2 D_b(x_0, x_1) + \alpha_3 D_b(x_1, x_1) \\ + \alpha_4 D_b(x_0, x_2) + \alpha_5 D_b(x_0, x_1)\end{array}\right)\right)$$

$$\leq \psi\left(\varphi\left(\begin{array}{c}\alpha_1 D_b(x_1, x_2) + \alpha_2 D_b(x_0, x_1) \\ + \alpha_4 s[D_b(x_0, x_1) + D_b(x_1, x_2)] + \alpha_5 D_b(x_0, x_1)\end{array}\right)\right)$$

$$= \psi(\varphi((\alpha_2 + \alpha_5 + s\alpha_4) D_b(x_0, x_1) + (\alpha_1 + s\alpha_4) D_b(x_2, x_1)))$$

$$\leq \psi\left(\frac{\alpha_2 + \alpha_5 + s\alpha_4}{1 - (\alpha_1 + s\alpha_4)} \varphi(D_b(x_0, x_1))\right) \tag{1.3}$$

Adding inequalities (1.2) and (1.3), we obtain $D_b(x_1, x_2)$ where,

$$\varphi(D_b(x_1, x_2)) \leq \left(\frac{(\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + 2\alpha_5)}{2 - (\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4)} \varphi(D_b(x_0, x_1))\right)$$

where,

$$\delta = \frac{(\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + 2\alpha_5)}{2 - (\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4)} < \frac{1}{s}.$$

Similarly, it can be shown that, there exists $x_3 \in fx_2$ such that

$$\varphi(D_b(x_3, x_2)) \le \varphi(H(fx_2, gx_1)) \le \psi(\delta\varphi(D_b(x_1, x_0)))$$

Continuing this process, we obtain by induction a sequence $\{x_n\}$ such that $x_{2n+1} \in fx_{2n}, x_{2n+2} \in gx_{2n+1}$ such that

$$\varphi(D_b(x_{2n+1}, x_{2n+2})) \le \varphi(H(fx_{2n}, gx_{2n+1}))$$

$$\le \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{2n}, fx_{2n}) + \alpha_2 D_b(x_{2n+1}, gx_{2n+1}) + \alpha_3 D_b(x_{2n}, gx_{2n+1}) \\ + \alpha_4 D_b(x_{2n+1}, fx_{2n}) + \alpha_5 D_b(x_{2n}, x_{2n+1})\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{2n}, x_{2n+1}) + \alpha_2 D_b(x_{2n+1}, x_{2n+2}) + \alpha_3 D_b(x_{2n}, x_{2n+2}) \\ + \alpha_4 D_b(x_{2n+1}, x_{2n+1}) + \alpha_5 D_b(x_{2n}, x_{2n+1})\end{pmatrix}\right)$$

$$\le \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{2n}, x_{2n+1}) + \alpha_2 D_b(x_{2n+1}, x_{2n+2}) \\ + \alpha_3 s[D_b(x_{2n}, x_{2n+1}) + D_b(x_{2n+1}, x_{2n+2})] + \alpha_5 D_b(x_{2n}, x_{2n+1})\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}(\alpha_1 + \alpha_5 + s\alpha_3) D_b(x_{2n}, x_{2n+1}) \\ + (\alpha_2 + s\alpha_3) D_b(x_{2n+1}, x_{2n+2})\end{pmatrix}\right)$$

$$\le \psi\left(\frac{\alpha_1 + \alpha_5 + s\alpha_3}{1 - (\alpha_2 + s\alpha_3)} \varphi(D_b(x_{2n}, x_{2n+1}))\right) \tag{1.4}$$

also,

$$\varphi(D_b(x_{2n+2}, x_{2n+1})) \le \varphi(H(gx_{2n+1}, fx_{2n}))$$

$$\le \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{2n+1}, gx_{2n+1}) + \alpha_2 D_b(x_{2n}, fx_{2n}) + \alpha_3 D_b(x_{2n+1}, fx_{2n}) \\ + \alpha_4 D_b(x_{2n}, gx_{2n+1}) + \alpha_5 D_b(x_{2n+1}, x_{2n})\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{2n+1}, x_{2n+2}) + \alpha_2 D_b(x_{2n}, x_{2n+1}) + \alpha_3 D_b(x_{2n+1}, x_{2n+1}) \\ + \alpha_4 D_b(x_{2n}, x_{2n+2}) + \alpha_5 D_b(x_{2n+1}, x_{2n})\end{pmatrix}\right)$$

$$\le \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{2n+1}, x_{2n+2}) + \alpha_2 D_b(x_{2n}, x_{2n+1}) \\ + \alpha_3 s[D_b(x_{2n}, x_{2n+1}) + D_b(x_{2n+1}, x_{2n+2})] + \alpha_5 D_b(x_{2n+1}, x_{2n})\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}(\alpha_2 + \alpha_5 + s\alpha_4)D_b(x_{2n}, x_{2n+1})\\ + (\alpha_1 + s\alpha_4)D_b(x_{2n+1}, x_{2n+2})\end{pmatrix}\right)$$

$$\leq \psi\left(\frac{\alpha_2 + \alpha_5 + s\alpha_4}{1 - (\alpha_1 + s\alpha_4)}\varphi(D_b(x_{2n}, x_{2n+1}))\right) \tag{1.5}$$

From (1.4) and (1.5)

$$\varphi(D_b(x_{2n+1}, x_{2n+2})) \leq \psi(\delta\varphi(D_b(x_{2n}, x_{2n+1}))).$$

Therefore,

$$\varphi(D_b(x_n, x_{n+1})) \leq \varphi(H(fx_{n-1}, gx_n))$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{n-1}, fx_{n-1}) + \alpha_2 D_b(x_n, gx_n) + \alpha_3 D_b(x_{n-1}, gx_n)\\ + \alpha_4 D_b(x_n, fx_{n-1}) + \alpha_5 D_b(x_{n-1}, x_n)\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{n-1}, x_n) + \alpha_2 D_b(x_n, x_{n+1}) + \alpha_3 D_b(x_{n-1}, x_{n+1})\\ + \alpha_4 D_b(x_n, x_n) + \alpha_5 D_b(x_{n-1}, x_n)\end{pmatrix}\right)$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{n-1}, x_n) + \alpha_2 D_b(x_n, x_{n+1})\\ + \alpha_3 s[D_b(x_{n-1}, x_n) + D_b(x_n, x_{n+1})] + \alpha_5 D_b(x_{n-1}, x_n)\end{pmatrix}\right)$$

$$= \psi\left(\varphi\begin{pmatrix}(\alpha_1 + \alpha_5 + s\alpha_3)D_b(x_{n-1}, x_n)\\ + (\alpha_2 + s\alpha_3)D_b(x_n, x_{n+1})\end{pmatrix}\right)$$

$$\leq \psi\left(\frac{\alpha_1 + \alpha_5 + s\alpha_3}{1 - (\alpha_2 + s\alpha_3)}\varphi(D_b(x_{n-1}, x_n))\right) \tag{1.6}$$

Therefore,

$$\varphi(D_b(x_{n-1}, x_n)) \leq \left(\frac{(\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + 2\alpha_5)}{2 - (\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4}\varphi(D_b(x_{n-1}, x_n))\right)$$

$n \in N$ and let

$$\delta = \frac{(\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + 2\alpha_5)}{2 - (\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4}$$

$$\varphi(D_b(x_n, x_{n+1})) \leq \delta(\psi(\varphi(D_b(x_{n-1}, x_n))))$$

$$\leq \delta[\delta(\psi(\varphi(D_b(x_{n-2}, x_{n-1}))))]$$

$$= \delta^2(\psi(\varphi(D_b(x_{n-2}, x_{n-1}))))$$

$$\vdots$$

$$\leq \delta^n(\psi(\varphi(D_b(x_0, x_1)))).$$

Since $0 < \delta < 1, \Sigma\delta^n$ have a radius of convergence. Then, $\{x_n\}$ is a Cauchy sequence. Since (x, D_b) is complete, there exists $u \in X$ such that $x_n \to u.$ We shall prove that u is a common fixed point of f and g.

$$\varphi(D_b(u, fu)) \leq \psi(\varphi(s[D_b(u, x_{2n+1}) + D_b(x_{2n+1}, fu)]))$$

$$\leq \psi(\varphi(s[D_b(u, x_{2n+1}) + H(x_{2n+1}, fu)]))$$

$$\varphi(D_b(u, gu)) \leq \psi(\varphi(s[D_b(u, x_{2n+1}) + D_b(x_{2n+1}, gu)]))$$

$$\leq \psi(\varphi(s[D_b(u, x_{2n+1}) + H(x_{2n}, gu)])) \tag{1.7}$$

Where,

$$\varphi(H(x_{2n}, gu)) \leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(x_{2n}, fx_{2n}) + \alpha_2 D_b(u, gu) + \alpha_3 D_b(x_{2n}, gu) \\ + \alpha_4 D_b(u, fx_{2n}) + \alpha_5 D_b(x_{2n}, u)\end{pmatrix}\right) \tag{1.8}$$

Using (1.8) in (1.7) and letting as $n \to \infty,$ we obtain,

$$\varphi(D_b(u, gu)) \leq (\varphi(s\, D_b(u, u))$$

$$\leq \psi\left(\varphi\left(s\begin{bmatrix}\alpha_1 D_b(u, u) + \alpha_2 D_b(u, gu) + \alpha_3 D_b(u, gu) \\ + \alpha_4 D_b(u, u) + \alpha_5 D_b(u, u)\end{bmatrix}\right)\right)$$

$$= \psi(\varphi(s[\alpha_2 D_b(u, gu) + \alpha_3 D_b(u, gu)]))$$

$$\leq \psi(\varphi(s(\alpha_2 + \alpha_3) D_b(u, gu)))$$

$$[1 - s(\alpha_2 + \alpha_3)] D_b(u, gu) \leq 0$$

Implies that, $1 - s(\alpha_2 + \alpha_3) \leq 0$ and $g(u)$ is closed. Thus, $g(u) = u.$

Similarly, $f(u) = u.$ Now, show that u is the unique fixed point of g and f. Now,

$$\varphi(D_b(u, v)) \leq \varphi(H(fu, gv))$$

$$\leq \psi\left(\varphi\begin{pmatrix}\alpha_1 D_b(u, fu) + \alpha_2 D_b(v, gv) + \alpha_3 D_b(u, gv) \\ + \alpha_4 D_b(v, fu) + \alpha_5 D_b(u, v)\end{pmatrix}\right)$$

$$\leq \psi(\varphi(\alpha_3 D_b(u, v) + \alpha_4 D_b(u, v) + \alpha_5(u, v)))$$

This is not true for $[1 - (\alpha_3 + \alpha_4 + \alpha_5)] \geq 0$, $D_b(u, v) = 0$. Hence, g and f have a unique common fixed point.

Corollary 18. *Let* (x, D_b) *be a complete b-metric space with constant* $s \geq 1$. *Let* $f, g : X \to CB(X)$ *be generalized multi-valued* (ψ, φ)*-contraction mapping, satisfies the condition*:

$$\varphi(H(fx, gy)) \leq (\varphi(\alpha D_b(x, gy) + \beta D_b(y, fx) + \gamma D_b(x, y)))$$

Where, there exists $\psi \in \Psi$, $\varphi \in \Phi$ *such that for all* $x, y \in X$ *and* $\alpha, \beta, \gamma \geq 0$, *with* $(\alpha + \beta)(s^2 + s) + 2s\gamma < 2$, $\alpha + \beta + \gamma < 1$. *Then* f *and* g *have a unique common fixed point*.

Example 19. Let $X = [0, 1]$. Define a function $D_b : X \times X \to R^+$ by $D_b(x, y) = |x - y|$. Clearly, (x, D_b) is a complete b-metric space. Define $\phi : R_0^+ \to R_0^+$ by $\phi(t) = t$ for all $t > 0$. Then $\phi \in \Phi$. Also define $\psi : R_0^+ \to R_0^+$ by $\psi(t) = ut$ for all $t > 0$. Then ψ is a continuous comparison function.

Define the mapping $f : X \to CB(X)$ by $fx = \left[0, \frac{x}{6}\right]$, for all $x, y \in X$. Then,

$$\varphi(H(fx, fy)) \leq \psi(\varphi(M(x, y)))$$

where,

$$M(x, y) = \alpha_1 D_b(x, fx) + \alpha_2 D_b(y, fy) + \alpha_3 D_b + \alpha_3 D_b(x, fy) + \alpha_4 D_b(y, fx)$$

$$+ \alpha_5 D_b(x, y) + \alpha_6 \frac{D_b(x, fx)(1 + D_b(x, fx))}{1 + D_b(x, y)}$$

$$\varphi(H(fx,\ fy)) \leq \psi\left(\varphi\left(\begin{array}{l}\alpha_1\left|x-\frac{x}{6}\right|+\alpha_2\left|y-\frac{y}{6}\right|+\alpha_3\left|x-\frac{y}{6}\right|+\alpha_4\left|y-\frac{x}{6}\right| \\ +\alpha_5|x-y|+\alpha_6\left(\frac{\left|x-\frac{x}{6}\right|\left(1+\left|x-\frac{x}{6}\right|\right)}{1+|x-y|}\right)\end{array}\right)\right)$$

$$\leq \psi\left(\phi\left(\frac{1}{6}|x-y|\right)\right), \text{ where } \left[\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_6 = 0,\ \alpha_5 = \frac{1}{6}\right]$$

$$\leq \psi\left(\frac{1}{6}|x-y|\right)$$

$$\leq \frac{\mu}{6}|x-y|, \text{ for } 0 < \mu < 1$$

$$\leq \frac{\mu}{6}M(x,\ y) = \frac{\mu}{6}\phi(M(x,\ y))$$

$$\leq \psi(\varphi(M(x,\ y))).$$

Therefore, $0 \in X$ is a unique fixed point of f.

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