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CHANNEL ASSIGNMENT OF TRIANGULAR GRID AND LADDER RELATED GRAPHS USING RADIO LABELING

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Abstract

Let $G(V, E)$ be a graph with vertex set V and edge set E. Let u, v be any two distinct vertices of graph G. A radio labeling of a graph G is a function $f : V(G) \to N$ (set of natural numbers) such that, $d(u, v) + |f(u) - f(v)| \geq diam(G) + 1$, where $d(u, v)$ represents the shortest distance between the vertices u and v and $diam(G)$ represents the diameter of the graph G. The span of a radio labeling f is $\max\{|f(u) - f(v)| : u, v \in V(G)\}$. The radio number of G is the minimum span of all radio labeling of G and is denoted by $rn(G)$. In this paper, the bounds of radio number of certain types of graphs have been determined.

1. Introduction

In telecommunication network, an important challenging problem in designing of radio networks is to assign channels (frequencies) to all the transmitters in such a way that there is no interference between any two transmitters [16]. This problem is studied as a graph theory problem. The given network is modeled as a graph where the vertices represents transmitters and the adjacent transmitters are connected by an edge. Now the vertices are assigned different colours or non-negative integers (frequencies) in such a way that the adjacent vertices are assigned different frequencies to avoid interference [6].

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This assignment of channels to the transmitters is popularly known as channel assignment problem which was introduced by Hale [8] in 1980. The channel assignment problem motivated Gary Chartrand et al. [7] to introduce a new type of labeling known as radio labeling (or multilevel distance labeling).

The radio labeling of a graph G is an injection from the set of vertices of G to the set of natural numbers such that, $d(u, v) + | f(u) - f(v) | \geq diam(G) + 1$, where $d(u, v)$ represents the shortest distance between every distinct pair of vertices u and v of G. The span of a radio labeling f is $\max\{| f(u) - f(v) | : u, v \in V(G)\}$. The radio number of G is the minimum span of all radio labeling of G and it is denoted by $rn(G)$. A graph G with n vertices is called radio graceful if $rn(G) = n$ [1].

A radio labeling is also known as radio k-labeling or multi-level distance labeling for some integer k with $1 \leq k \leq d$ such that, $d(u, v) + | f(u) - f(v) | \geq 1 + k$ where d represents the diameter of the graph G. Based on the value of k, this labeling is classified as follows. When $k = 2$, the labeling is said to be distance two labeling or $L(2, 1)$-labeling. When $k = d - 1$, the labeling is called radio antipodal labeling. When $k = d$, the labeling is called radio labeling.

The problem of finding the radio number is NP complete for general graphs [11]. However the lower bound of radio number of general graphs can be obtained by using the following result.

Result 1.1. Ahmad Ali et al. [1] obtained a formula for finding the lower bound of radio number of graphs with small diameter. Suppose in a graph G there are at most x pairs $\{u, v\}$ such that $| f(u) - f(v) | = 1$, then

$$rn(G) \geq 1 + x + 2(k - 1 - x) \tag{1}$$

where k is the number of vertices in G.

The radio number of certain graphs like paths and cycles [5], hypercubes [15], square paths [4], circulant graphs $G(4k + 2 : 1, 2)$ [12], generalized prism graphs [13], toroidal grids [17], extended mesh [22], uniform theta graphs [14], some thorn graphs [9], some cycle related graphs [19], Cartesian product of graphs [10], gear graphs [3], complete m-ary trees [21], some ladder related graphs [1], total graph of path [18] were already investigated.

In this paper, the bounds of radio number of certain types of graphs like triangular grid graphs, triangular ladder graphs and pagoda graphs were determined.

2. Radio number of Triangular grid graphs

In this section, the radio number of triangular grid graphs have been studied. Triangular grid graph is a graph obtained from an infinite graph which is formed by the arrangement of transmitters in a network. These transmitters are assumed to be located like the vertices of a triangular lattice in a plane so that it gives a good coverage [6]. The triangular grid graph is defined based on this pattern of arrangements of transmitters.

Definition 2.1 [20]**.** The infinite graph T^{∞} associated with the two dimensional triangular grid graph or triangular tiling graph is a graph drawn in the plane with straight line edges and vertices defined as follows.

The vertices of T^{∞} are represented by the linear combination $xp + uq$ of two vectors $p = (1, 0)$ and $q = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ with integers x and y. Thus the vertices of T^{∞} are points with Cartesian coordinates $\left(x + \frac{y}{2}, y\frac{\sqrt{3}}{2}\right)$. Two vertices of T^{∞} are adjacent if and only if the Euclidean distance between them is equal to 1.

A triangular grid graph is a finite induced sub-graph of T^{∞}. The n^{th} dimension of triangular grid graph is denoted by $TG(n)$. See Figure 1.

Remark 2.1. $TG(n)$ has $\frac{1}{2}(n^2 + 3n + 2)$ vertices and $\frac{3}{2}n(n+1)$ edges. Its diameter is n. Assuming that the top single vertex of $TG(n)$ is at level 1, it has $n + 1$ levels. See Figure 2.

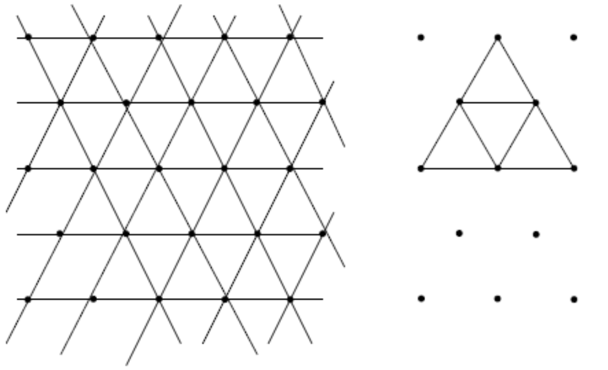

Figure 1. Construction of $TG(n)$ from T^{∞}.

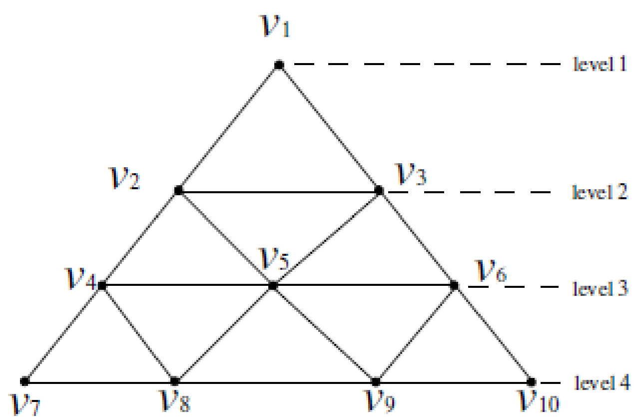

Figure 2. Different levels of $TG(3)$.

Theorem 2.1. *Radio number of* $TG(1)$, $rn(TG(1)) = 3$.

Proof. $TG(1)$ has 3 vertices and 3 edges.

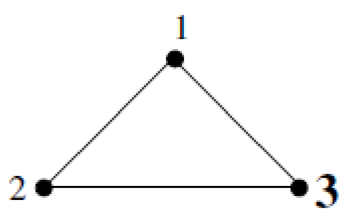

Figure 3. $TG(1)$.

By Result 1.1, $rn(TG(1)) \geq 3.$

It can be seen from Figure 3 that,

$rn(TG(1)) \leq 3.$

Hence, $rn(TG(1)) = 3.$ □

Note 2.1. $TG(1)$ has 3 vertices and $rn(TG(1)) = 3.$ Hence, $TG(1)$ is a radio graceful labeling.

Theorem 2.2. *Radio number of* $TG(2)$, $rn(TG(2)) = 8.$

Proof. $TG(2)$ has 6 vertices and 9 edges. The diameter of $TG(2)$ is 2.

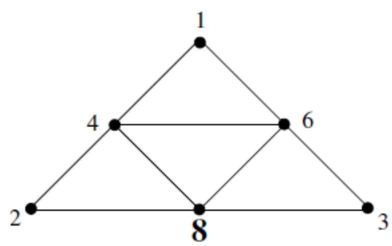

Figure 4. $TG(2)$.

By Result 1.1, $rn(TG(2)) \geq 8.$

Also it is observed that from Figure 4,

$rn(TG(2)) \leq 8.$

Hence, $rn(TG(2)) = 8.$

Theorem 2.3. *For* $n \geq 3$, $rn(TG(n)) \geq n^2 + 3n - 2.$

Proof. We can prove this theorem by using Result 1.1.

Case (i). When $n = 3.$ For $TG(3)$ there are five pair of vertices at diametric distance. In these pairs three pairs $\{u, v\}$ are in such a way that, $| f(u) - f(v) | = 1.$ Therefore by Result 1.1, $rn(TG(n)) \geq n^2 + 3n - 2.$

Case (ii). When $n > 3, TG(n)$ has five pair of vertices at diametric distance but only two pairs $\{u, v\}$ are in such a way that, $| f(u) - f(v) | = 1.$

Therefore by Result 1.1, $rn(TG(n)) \geq n^2 + 3n - 1$. Hence, from Case (i) and Case (ii), $rn(TG(n)) \geq n^2 + 3n - 2$, $n \geq 3$. □

Theorem 2.4. *For* $n \geq 3$, $rn(TG(n)) \leq \frac{1}{2}(n^3 + 3n^2 - 4n + 4)$.

Proof. Let $v_1, v_2, \ldots, v_{\frac{1}{2}(n^2+3n+2)}$ be the vertices of $TG(n)$. These vertices are labeled as follows. Let $f(v_1) = 1$. As the vertices v_1 and $v_{\frac{1}{2}(n^2+3n+2)}$ are at diametric distance, take $f(v_{\frac{1}{2}(n^2+3n+2)}) = 2$. The remaining vertices of $TG(n)$ are labeled by the mapping,

$$f(v_i) = \begin{cases} n(i-1)+1, \; 1 < i < i \leq \left\lfloor \frac{1}{2}(n^2 + n + 1) \right\rfloor \\ n(i-2)+2, \; \left\lceil \frac{1}{2}(n^2 + n + 1) \right\rceil \leq i \leq \frac{1}{2}(n^2 + 3n) \end{cases} \tag{2}$$

Claim. The mapping (2) is a valid radio labeling.

Let u, v be any two vertices of $TG(n)$. We must show that the radio labeling condition is satisfied for all pairs of vertices $\{u, v\}$ in $TG(n)$.

Case (i). Suppose the vertices u, v are in the same level

Subcase(i): $d(u, v) = 1$

By mapping (2), $|f(u) - f(v)| \geq n$.

Subcase(ii): $d(u, v) > 1$.

By (2), $|f(u) - f(v)| > n$.

Case (ii). Suppose the vertices u, v are at different levels

Subcase(i). $d(u, v) = 1$

By (2), $|f(u) - f(v)| \geq n$.

Subcase (ii). Suppose $d(u, v) > 1$.

By (2), $|f(u)-f(v)| \geq n$.

Case (iii). Suppose $u = v_{\left\lfloor \frac{1}{2}(n^2+n+1) \right\rfloor}$ and $v = v_{\left\lfloor \frac{1}{2}(n^2+n+1) \right\rfloor}$

By (2), $|f(u)-f(v)| = 1$ and $d(u, v) = n$.

Case (iv). Suppose $u = v_1$ and $v = v_{\frac{1}{2}(n^2+3n+2)}$. Then, $d(u, v) = n$ and by our assumption $|f(u)-f(v)| = 1$. Hence, $d(u, v) + |f(u)-f(v)| \geq n+1$.

Therefore these four cases establish the claim that the mapping (2) is a valid radio labeling of $TG(n)$.

Here the vertex $v_{\frac{(n^2+3n)}{2}}$ receives the maximum label and its label is $\frac{1}{2}(n^3+3n^2-4n+4)$. Hence, $rn(TG(n)) \leq \frac{1}{2}(n^3+3n^2-4n+4)$, $n \geq 3$. □

Theorem 2.5. *The radio number of* $TG(n)$ *lies between* n^2+3n-2 *and* $\frac{1}{2}(n^3+3n^2-4n+4)n \geq 3$.

Proof. From Theorem 2.3 $rn(TG(n)) \geq n^2+3n-2$ and From Theorem 2.4 $rn(TG(n)) \leq \frac{1}{2}(n^3+3n^2-4n+4)$. Therefore, $n^2+3n-2 \leq rn(TG(n)) \leq \frac{1}{2}(n^3+3n^2-4n+4)$, $n \geq 3$. □

3. Radio number of Triangular Ladder Graph

In this section, the radio number of Triangular ladder graphs have been investigated.

Definition 3.1 [1]**.** The ladder graph L_n is a graph obtained by the Cartesian product of two path graphs P_2 and P_n, $n \geq 2$.

Remark 3.1. The n^{th} dimension of L_n has $2n$ vertices, in which $v_1, v_2, \ldots, v_n$ forms the top row (say) R_1 and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ forms the bottom row (say) R_2. It has $3n-2$ edges and its diameter is n.

Definition 3.2 [2]**.** A Triangular ladder graph denoted by $TLG(n)$, is a ladder graph obtained by adding the edges $\{(v_i, v_{n+i-1}), i = 2, \ldots, n\}$. See Figure 5.

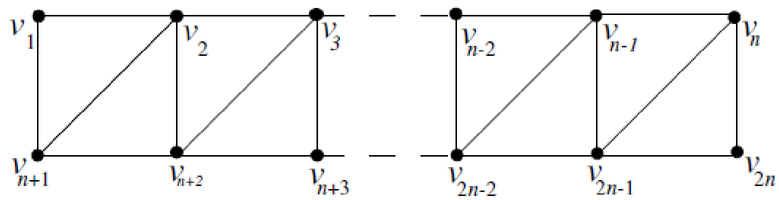

Figure 5. $TLG(n)$.

Remark 3.2. $TLG(n)$ has $2n$ vertices and $4n - 3$ edges. Its diameter is n.

Theorem 3.1. *The radio number of* $TLG(2)$, $rn(TLG(2)) = 6$.

Proof. $TLG(2)$ has 4 vertices and 5 edges. The diameter of $TLG(2)$ is 2.

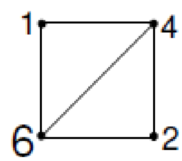

Figure 6. $TLG(2)$.

There is only one pair of vertices at diametric distance. Here $d(v_1, v_4) = 2$. Hence these vertices v_1 and v_4 are labeled in such a way that $| f(v_1) - f(v_4) | = 1$. Then by Result 1.1, $rn(TLG(2)) \geq 6$. From Figure 6, it is observed that, $rn(TLG(2)) \leq 6$. Hence, $rn(TLG(2)) = 6$. □

Theorem 3.2. *For* $n \geq 3$, $rn(TLG(n)) \geq 4n - 2$.

Proof. In order to prove this theorem, we make use of the Result 1.1. For $TLG(n)$, there is only one pair $\{v_1, v_{2n}\}$ at diametric distance and $| f(v_1) - f(v_{2n}) | = 1$. Therefore, by Result 1.1 we have, $rn(TLG(n)) \geq 1 + 1 + 2(2n - 1 - 1)$. Hence, $rn(TLG(n)) \geq 4n - 2$, $n \geq 2$. □

Theorem 3.3. *For* $n \geq 3$, $rn(TLG(n)) \leq 2n^2 - 3n + 3$.

Proof. Consider the vertices $v_1, v_2, \ldots, v_{2n}$ of $TLG(n)$. These vertices are labeled as follows. The vertices v_1 and v_{2n} are at diametric distances. Take $f(v_1) = 1$ and $f(v_{2n}) = 2$.

The remaining vertices of $TLG(n)$ are labeled by the following mapping

$$f(v_i) = \begin{cases} n(i-1)+1, \ 1 < i < n, \ n \geq 3 \\ n(i-1)-(n-3), \ n+1 \leq i \leq 2n-1, \ n \geq 3. \end{cases} \tag{3}$$

Claim. The mapping (3) is a valid radio labeling.

Let u, v be any two vertices of $TLG(n)$. We must show that the radio labeling condition is satisfied for all the pairs of vertices $\{u, v\} \in TLG(n)$.

Consider the following cases.

Case (i). Suppose $u, v \in R_i$, $i = 1, 2$.

Subcase(i). $d(u, v) = 1$.

By mapping (3), $|f(u) - f(v)| \geq n$.

Subcase (ii). $d(u, v) > 1$.

By (3), $|f(u) - f(v)| > n$.

Case (ii). Suppose $u \in R_i$ and $v \in R_j$, $i \neq j$.

Subcase(i). $d(u, v) = 1$.

By (3), $|f(u) - f(v)| \geq n$.

Subcase(ii): $d(u, v) > 1$

By (3), $|f(u) - f(v)| > n$.

Case (iii). Suppose the vertices u and v are extreme vertices

Subcase (i). Suppose $u = v_1$ and $v = v_{2n}$.

Then by our assumption, $|f(u) - f(v)| = 1$ and $d(u, v) = n$.

Subcase (ii). Suppose $u = v_n$ and $v = v_{n+1}$.

Then by (3), $|f(u) - f(v)| = 2$ and $d(u, v) = n - 1$.

Hence in all the cases, $d(u, v) + |f(u) - f(v)| \geq n + 1$.

Therefore, the mapping (3) is a valid radio labeling.

The vertex v_{2n+1} receives the maximum labeling and its label is $2n^2 - 3n + 3$.

Hence, $TLG(n) \leq 2n^2 - 3n + 3$, $n \geq 3$. □

Theorem 3.4. *The radio number of* $TLG(n)$ *lies between* $4n - 2$ *and* $2n^2 - 3n + 3$, $n \geq 3$.

Proof. From Theorem 3.2, $rn(TLG(n)) \geq 4n - 2$ and

From Theorem 3.3, $rn(TLG(n)) \leq 2n^2 - 3n + 3$, $n \geq 3$.

Hence, $4n - 2 \leq rn(TLG(n)) \leq 2n^2 - 3n + 3$, $n \geq 3$. □

4. Radio number of Pagoda graphs

In this section, the radio number of pagoda graphs have been determined.

Definition 4.1. A Pagoda graph is a graph obtained from ladder graph L_n in which a vertex v_a is added in such a way that it is adjacent to the vertices v_1 and v_2. See Figure 7.

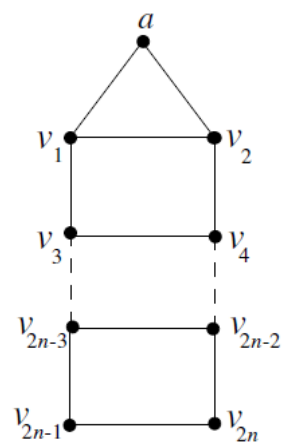

Figure 7. $PG(n)$.

As this graph resembles the structure of Pagoda, Buddhists temples in South-East Asia, this graph is named as pagoda graphs.

Remark 4.1. The n^{th} dimension of the pagoda graph is denoted by $PG(n)$. It has $2n+1$ vertices and $3n$ edges. The diameter of $PG(n)$ is n.

Theorem 4.1. *For* $n \geq 2$, $rnPG(n) \geq 4n+1$.

Proof. In order to prove this theorem we will use Result 1.1.

In $PG(n)$, there are four pairs of vertices for which $d(u, v) = diam(G)$. Among these pairs only two vertex pairs are such that, $|f(u) - f(v)| = 1$.

Therefore, by Result 1.1, we have

$rn(PG(n)) \geq 1 + 2 + 2(2n + 1 - 2)$

Hence, $rn(PG(n)) \geq 4n + 1$, $n \geq 2$ □

Theorem 4.2. *For* $n \geq 2$, $rn(PG(n)) \leq 2n^2 - n + 1$.

Proof. Let $v_a, v_1, v_2, \ldots, v_{2n}$ be the vertices of $PG(n)$.

These vertices are labeled as follows.

Take $f(v_a) = 1$ and $f(v_{2n}) = 2$.

The remaining $2n-1$ vertices of $PG(n)$ are labeled by the mapping,

$$f(v_i) = ni + 1, 1 \leq i < 2n. \tag{4}$$

Claim. The mapping (4) is a valid radio labeling.

In order to prove this claim, consider the following cases.

Let u, v be any two vertices of $PG(n)$.

Case (i). Suppose $u = v_a$ and $v = v_i$, $1 \leq i \leq 2n-1$.

Then by mapping (4), $|f(u) - f(v)| \geq n$ and $d(u, v) \geq 1$.

Case (ii). Suppose $u = v_a$ and $v = v_{2i}$, $1 \leq i \leq n$.

The proof is similar to Case (i).

Case (iii). Suppose $u = v_{2i-1}$ and $v = v_{2i}$, $1 \leq i \leq n-1$.

Then by mapping (4), $|f(u) - f(v)| \geq n$ and $d(u, v) = 1$.

Case (iv). Suppose $u = v_{2i}$ and $v = v_{2i+1}$, $1 \leq i \leq n-1$.

Then by (4) $|f(u) - f(v)| \geq n-1$ and $d(u, v) > 1$.

Case (v). Suppose $u = v_{2i}$ and $v = v_{2j}$, $i \neq j$, $1 \leq i, j \leq n$.

Then by (4) $|f(u) - f(v)| \geq n$ and $d(u, v) \geq 1$.

Case (vi). Suppose $u = v_{2i-1}$ and $v = v_{2j-1}$, $i \neq j$, $1 \leq i, j \leq n$.

The proof is similar to Case (iv).

Case (vii). Suppose $u = v_a$ and $v = v_{2n}$.

Then by our assumption, $|f(u) - f(v)| = 1$ and $d(u, v) = n$.

Hence, in all the cases, $d(u, v) + |f(u) - f(v)| \geq n + 1$. Therefore, the mapping (4) is a valid radio labeling.

By mapping (4), the vertex v_{2n-1} receives the maximum labeling and is given by $f(v_{2n-1}) = 2n^2 - n + 1$.

Hence, $rn(PG(n)) \leq 2n^2 - n + 1$. □

Theorem 4.3. *The bounds of* $PG(n)$ *lies between* $4n + 1$ *and* $2n^2 - n + 1$, $n \geq 2$.

Proof. From Theorem 4.1, $rn(PG(n)) \geq 4n + 1$ and

By Theorem 4.2,

$rn(PG(n)) \leq 2n^2 - n + 1$.

Hence, $4n + 1 \leq rn(PG(n)) \leq 2n^2 - n + 1$, $n \geq 2$. □

Conclusion

The radio labeling is used in communication engineering to assign frequencies (channels) to different transmitters such that there is no interference between the transmitters with minimum bandwidth. In this paper, the bounds of radio number of certain graphs like Triangular grid graph, Triangular ladder graph and Pagoda graphs have been determined. This study can be extended further to other networks also.

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References

[1] Ahmad Ali and Ruxandra Marinescu-Ghemeci, Radio labeling of some ladder related graphs, Mathematical Report (Bucuresti) 19(69) (2017), 107-119.

[2] A. Lourdusamy and F. Patrick, Vertex equitable labeling for ladder and snake related graphs, Journal of Prime Research in Mathematics 13 (2017), 01-07.

[3] Fernandez Christina, America Flores, Maggy Tomova and Cindy Wyels, The radio number of gear graphs, arXiv preprint arXiv:0809.2623, 2008.

[4] Liu Daphne Der-Fen and Melanie Xie, Radio number for square paths, Ars Combinatoria 90 (2009), 307-319.

[5] Liu Daphne Der-Fen and Xuding Zhu, Multilevel distance labelings for paths and cycles, SIAM Journal on Discrete Mathematics 16(3) (2005), 610-621.

[6] F. Havet, Channel assignment and multicolouring of the induced subgraphs of the triangular lattice, Discrete Mathematics 233(1-3) (2001), 219-231.

[7] G. Chartrand, D. Erwin and P. Zhang, Radio labeling of Graphs, Bull. Inst. Combin. Appl. 33 (2001), 77-85.

[8] William K. Hale, Frequency assignment: theory and applications, Proceedings of the IEEE 68(12), 1980.

[9] Ruxandra Marinescu-Ghemeci, Radio number for some thorn graphs, Discussiones Mathematicae Graph Theory 30(2) (2010), 201-222.

[10] Kchikech, Mustapha, Riadh Khennoufa and Olivier Togni, Radio k-labelings for Cartesian products of graphs, Discussiones Mathematicae Graph Theory 28(1) (2008), 165-178.

[11] M. Kchikech, R. Khennoufa and O. Togni, Linear and cyclic radio k-labelings of trees, Discussiones Mathematicae Graph Theory 130(1) (2007), 105-123.

[12] Saima Nazeer, Imrana Kousar and Waqas Nazeer, Radio and radio antipodal labelings for circulant graphs $G(4k+2; 1; 2)$, Journal of Applied Mathematics and Informatics 33 (2015), 173-183.

[13] P. Martinez, J. Ortiz, M. Tomova and C. Wyels, Radio numbers for generalized prism graphs, arXiv preprint arXiv:1007.5346, 2010.

[14] Bharathi Rajan and Kins Yenoke, Radio number of uniform theta graphs, Journal of Computer and Mathematical Sciences 2(6) (2011), 874-881.

[15] R. Khennoufa and O. Togni, The radio antipodal and radio numbers of the hypercube, Ars Combinatoria 102 (2011), 447-461.

[16] Kang Shin Min, Saima Nazeer, Waqas Nazeer, Imrana Kousar and Y. C. Kwun, Radio labeling and radio Number of caterpillar related graphs, Mitteilungen Klosterneuburg 65 (2015), 149-159.

[17] L. Saha and P. Panigrahi, On the radio number of toroidal grids, Australasian Journal of Combinatorics 55 (2013), 273-288.

[18] S. K. Vaidya and D. D. Bantva, Radio number for total graph of paths, ISRN Combinatorics, 2013.

[19] S. K. Vaidya and P. L. Vihol, Radio labeling for some cycle related graphs, International Journal of Mathematics and Soft Computing 2(2) (2012), 11-24.

[20] V. S. Gordon, Y. L. Orlovich and F. Werner, Hamiltonian properties of triangular grid graphs, Discrete Mathematics 308(24) (2008), 6166-6188.

[21] Li Xiangwen, Vicky Mak and Sanming Zhou, Optimal radio labellings of complete m-ary trees, Discrete Applied Mathematics 158(5) (2010), 507-515.

[22] Kins Yenoke, On the Radio number of extended mesh, Journal of Computer and Mathematical Sciences 5(4) (2014), 358-366.