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GRADE SIZES IN TWO-GRADE MODELS

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Abstract

Employees in hierarchical organizations are ideally placed in grades depending upon their skill levels, seniority, and the relevance of their role in the organizations. The control of grade sizes is fundamentally managed through recruitment and promotion. Human Resource systems at the highest level can be considered as a system with two groups with certain characteristics. This paper analyses a two-grade system and proposes an approach to determine grade size characteristics when the system allows for training of personnel at a higher grade by the movement to the previous grade and assumes recruitment of a certain proportion of voluntary attritions into the system. The transient behaviour of the system and the steady state characteristics are derived.

Introduction

The policies on hierarchical grade sizes are motivated by the need to control the operations of an organization competently. In certain industries, the policies on grade sizes are determined by trade union agreements. Further, good financial planning demands a control on grade sizes. Additionally, employee engagement and motivation are accomplished through policies on grade structures. In this context, grade sizes in hierarchical human resource systems assume importance.

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A human resource system can be viewed as a system with two compartments or grades. The groups can be a set of temporary workers and a set of permanent employees or technical and support staff or technical and managerial staff and so on and so forth. Analysing a two-grade model is an attempt to understand the characteristics of the workplace. Observations made on a two-grade human resource model can be extended to systems that encompass a homogeneous classification of employees.

(Bartholomew, [1]) has presented structures to examine attainability and maintainability in a k-grade university environment where the total system size is fixed but the individual grade sizes vary. Attainability is concerned with reaching a desired state in a manpower system while maintainability is concerned with remaining in that desired state. (Bartholomew et al., [2]) have studied systems that expand at a constant rate and have observed that it is the relative size of the stock in the grades that is of importance. They have developed manpower models influenced by recruitment, promotion and wastage processes. (Nilakantan and Raghavendra, [6]) have analysed the control aspects of hierarchical manpower organisations under the influence of proportionality policies. The proportionality in this context is the ratio of recruitment to internal promotion in a grade. (B. Thilaka et al., [8]) have proposed an approach to determine time dependent grade sizes when there is a pattern between intermediate grades that suit the real-time non-fixed grade size scenarios.

(Srinivasa Rao et al., [7]) have analysed a two-grade manpower planning model with direct recruitment into both the grades assuming Poisson processes for recruitment, promotion and attrition. The expected number of employees in the two grades, the variances of the grade sizes and the covariance between the two grade sizes are derived through cumulant generating function of the joint grade size distribution.

(Govinda Rao and Srinivasa Rao, [3]) have discussed a methodology to derive the mean number of employees in a two-grade manpower system that considers temporary and permanent employees, and assumes Poisson processes for recruitment, promotion and wastage. They have analysed a two-grade manpower system assuming three-level recruitment at the initial grade and Poisson processes for recruitment, promotion, and wastage (Govinda Rao and Srinivasa Rao, [4]). (Konda Babu and Srinivasa Rao, [5]) have proposed a two-grade model with bulk recruitments in both the grades.

In this paper, a two-grade model is considered and an approach to compute the mean, variance and covariance of the sizes of the two grades is proposed under the assumptions of (a) movement of personnel to the lower grade for training and (b) recruitment of a proportion of personnel who had moved out of the system.

Two-Grade model description and analysis

In a two-grade human resource system, it is assumed that the recruitment is of two types: (a) recruitment of persons who had been in the system earlier and (b) recruitment of persons who are from the external environment. It is further assumed that recruitment, promotion and attrition each follow Poisson process where,

i. λ is the rate of transition from grade 1 to 2.

ii. Training of persons from grade 2 and placed at grade 1 is at rate μ.

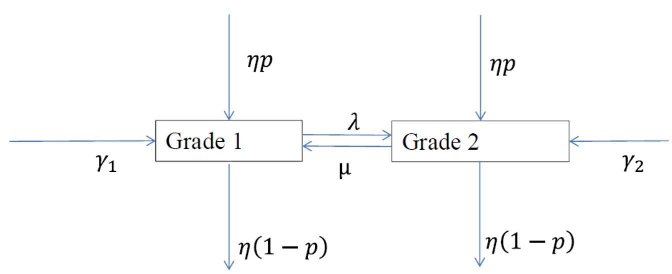
iii. γ_i is the rate of recruitment of persons from the external environment to grade i, $i = 1, 2$.

iv. Recruitment of persons who had been in the system is at the rate ηp, $0 < p < 1$.

v. Wastage rate is $\eta(1 - p)$.

General Case $(\mu > 0)$

It is assumed that $\mu > 0$. A graphical representation of the system is shown below:



Let $N_1(t)$ denote the number of persons in grade 1 at time t. Let $N_2(t)$ denote the number of persons in grade 2 at time t. The governing equations of the model are given by:

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1,\ N_2(t) = n_2]$$

$$= 1 - \gamma_1 dt - n_1 \lambda dt - \eta p dt - n_1 \eta (1-p) dt - \gamma_2 dt - \eta p dt$$

$$- n_2 \mu dt - n_2 \eta (1-p) dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1 - 1,\ N_2(t) = n_2]$$

$$= \gamma_1 dt + \eta p dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1 - 1,\ N_2(t) = n_2 + 1]$$

$$= (n_2 + 1) \mu dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1 + 1,\ N_2(t) = n_2 - 1]$$

$$= (n_1 + 1) \lambda dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1 + 1,\ N_2(t) = n_2]$$

$$= (n_1 + 1) \eta (1-p) dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1,\ N_2(t) = n_2 - 1]$$

$$= \gamma_2 dt + \eta p dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1,\ N_2(t) = n_2 + 1]$$

$$= (n_2 + 1) * \eta (1-p) dt + o(dt). \tag{1}$$

Let $\pi(n_1, n_2, t)$ denote the joint probability mass function of the system size at time t. i.e.,

$$\pi(n_1, n_2, t) = P[N_1(t) = n_1,\ N_2(t) = n_2]. \tag{2}$$

Let $G(s_1, s_2, t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \pi(n_1, n_2, t) s_1^{n_1} s_2^{n_2}$ denote the probability generating function of

$$\pi(n_1, n_2, t). \tag{3}$$

It is assumed that the initial size of the system is $(l, m)\, l, m \geq 0$ i.e.,

$$N_1(0) = l \geq 0 \;\; N_2(0) = m \geq 0. \tag{4}$$

Employing probability arguments for the various mutually exclusive and collectively exhaustive events, the following differential equation is obtained:

$$[\eta(1-p)(1-s_1) + \lambda(s_2 - s_1)]\frac{\partial G}{\partial s_1} + [\eta(1-p)(1-s_2) + \mu(s_1 - s_2)]\frac{\partial G}{\partial s_2}$$

$$-\frac{\partial G}{\partial t} = [(\gamma_1 + \eta p)(1-s_1) + (\gamma_2 + \eta p)(1-s_2)]G. \tag{5}$$

Solving the above Lagrange's equation, the joint probability generating function is given by:

$$G(s_1,\, s_2,\, t) = e^{\frac{\gamma_1 + \eta p}{\eta(1-p)\mu}(\mu s_1 + \lambda s_2)} e^{-\left(\frac{\gamma_1+\eta p}{\eta(1-p)\mu}\right)(\lambda+\mu)}$$

$$* \left\{1 - \left[1 - \frac{\mu s_1 + \lambda s_2}{\lambda + \mu}\right] e^{-\eta(1-p)t}\right\}\left(\frac{1}{\lambda + \mu}\right)^{(l+m)}$$

$$* [(\lambda + \mu) * \left\{1 - \left[1 - \frac{\mu s_1 + \lambda s_2}{\lambda + \mu}\right] e^{-\eta(1-p)t}\right\}$$

$$+ \lambda(s_1 - s_2)e^{-(\lambda+\mu+\eta(1-p))t}]^l [(\lambda + \mu)$$

$$* \left\{1 - \left[1 - \frac{\mu s_1 + \lambda s_2}{\lambda + \mu}\right] * e^{-\eta(1-p)t}\right\} - \mu(s_1 - s_2)e^{-(\lambda+\mu+\eta(1-p))t}]^m.$$

where,

$$\frac{\lambda}{\mu} = \frac{\gamma_1 + \eta p}{\gamma_2 + \eta p}. \tag{6}$$

Using the properties of probability generating functions, the expected size of the two grades is obtained as:

$$E[X_1(t)] = \frac{\gamma_1 + \eta p}{\eta(1-p)}[1 - e^{-\eta(1-p)t}] + \frac{(l+m)}{\lambda + \mu}e^{-\eta(1-p)t}$$

$$+ \frac{e^{-(\lambda+\mu+\eta(1-p))t}}{\lambda + \mu}(\lambda l - m\mu). \tag{7}$$

and

$$E[X_2(t)] = \frac{\gamma_2 + \eta p}{\eta(1-p)}[1 - e^{-\eta(1-p)t}] + \frac{(l+m)\lambda}{\lambda+\mu} e^{-\eta(1-p)t}$$

$$+ \frac{e^{-(\lambda+\mu+\eta(1-p))t}}{\lambda+\mu}(m\mu - \lambda l). \tag{8}$$

The matrix of the means is given by:

$$\overline{X(t)} = \begin{bmatrix} E[X_1(t)] \\ E[X_2(t)] \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{\gamma_1 + \eta p}{\eta(1-p)}[1 - e^{-\eta(1-p)t}] + \frac{(l+m)\mu}{\lambda+\mu} e^{-\eta(1-p)t} + \frac{e^{-(\lambda+\mu+\eta(1-p))t}}{\lambda+\mu}(\lambda l - m\mu) \\ \frac{\gamma_2 + \eta p}{\eta(1-p)}[1 - e^{-\eta(1-p)t}] + \frac{(l+m)\mu}{\lambda+\mu} e^{-\eta(1-p)t} + \frac{e^{-(\lambda+\mu+\eta(1-p))t}}{\lambda+\mu}(m\mu - \lambda l) \end{bmatrix}^T. \tag{9}$$

The covariance matrix is given by:

$$C[X(t)] = \begin{bmatrix} Var[X_1(t)] & Cov[X_1(t), X_2(t)] \\ Cov[X_2(t), X_1(t)] & Var[X_2(t)] \end{bmatrix}. \tag{10}$$

where,

$$Var[X_1(t)] = \frac{\gamma_1 + \eta p}{\eta(1-p)}[1 - e^{-\eta(1-p)t}] + \frac{l}{\lambda+\mu}\{\mu e^{-\eta(1-p)t} + \lambda e^{-(\lambda+\mu+\eta(1-p))t}\}$$

$$* \left[1 - \frac{l}{\lambda+\mu}\{\mu e^{-\eta(1-p)t} + \lambda e^{-(\lambda+\mu+\eta(1-p))t}\}\right]$$

$$+ \frac{l}{\lambda+\mu}\{\mu e^{-\eta(1-p)t} + \mu e^{-(\lambda+\mu+\eta(1-p))t}\}$$

$$* \left[1 - \frac{l}{\lambda+\mu}\{\mu e^{-\eta(1-p)t} + \lambda e^{-(\lambda+\mu+\eta(1-p))t}\}\right]. \tag{11}$$

$$Var[X_2(t)] = \frac{\gamma_2 + \eta p}{\eta(1-p)}[1 - e^{-\eta(1-p)t}] + \frac{l}{\lambda+\mu}[\mu e^{-\eta(1-p)t} + \lambda e^{-(\lambda+\mu+\eta(1-p))t}]$$

$$* \left[1 - \frac{l}{\lambda+\mu}\{\lambda e^{-\eta(1-p)t} + \lambda e^{-(\lambda+\mu+\eta(1-p))t}\}\right]$$

$$+ \frac{l}{\lambda+\mu}[\mu e^{-\eta(1-p)t} + \mu e^{-(\lambda+\mu+\eta(1-p))t}]$$

$$* \left[1 - \frac{l}{\lambda + \mu} \{ \lambda e^{-\eta(1-p)t} + \mu e^{-(\lambda+\mu+\eta(1-p))t} \} \right]. \quad (12)$$

$$Cov[X_1(t), X_2(t)] = Cov[X_2(t), X_1(t)]$$

$$= -\frac{e^{-2\eta(1-p)t}}{(\lambda + \mu)^2} [1 - e^{-(\lambda+\mu)t}] * [\lambda\mu(l + m) + (\lambda^2 l + \mu^2 m) e^{-(\lambda+\mu)t}]. \quad (13)$$

The covariance between grade size 1 and grade size 2 is negative. This implies that the system size is maintained.

At steady state.

In the special case, when $t \to \infty$

$$E[X_1] = \frac{\gamma_1 + \eta p}{\eta(1-p)} \; E[X_2] = \frac{\gamma_2 + \eta p}{\eta(1-p)}. \quad (14)$$

$$Var[X_1] = \frac{\gamma_1 + \eta p}{\eta(1-p)} \; Var[X_2] = \frac{\gamma_2 + \eta p}{\eta(1-p)}. \quad (15)$$

$$Cov[X_1, X_2] = 0. \quad (16)$$

Numerical illustration

The parameter values for $\gamma_1, \gamma_2, \lambda, \eta, l$ and m from (Srinivasa Rao et al., [7]) have been used and the expected sizes of grade 1 and grade 2, the standard deviation of the grade sizes and the covariance between the sizes of grade 1 and grade 2 have been computed. Figure 1 to Figure 12 illustrate the behaviour of the expected size and standard deviation over time when each of the parameters is varied holding all other parameters constant. The grade sizes are $l = 1000$ and $m = 100$. $\mu = 0.005$, $p = 0.0025$ and $\lambda = 0.8$. The initial values of the parameters that are varied are: $\gamma_1 = 0.9$, $\gamma_2 = 0.175$ and $\eta = 0.25$.

Varying γ_1 :

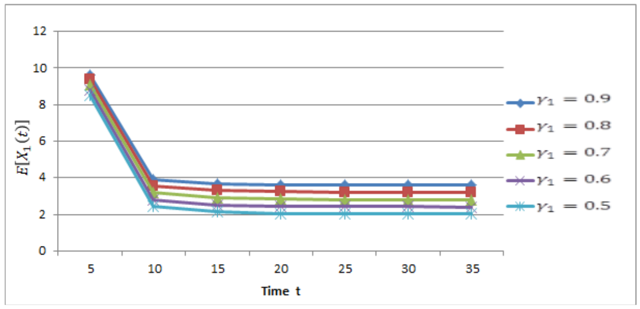


Figure 1. $E[X_1(t)]$ varying γ_1.

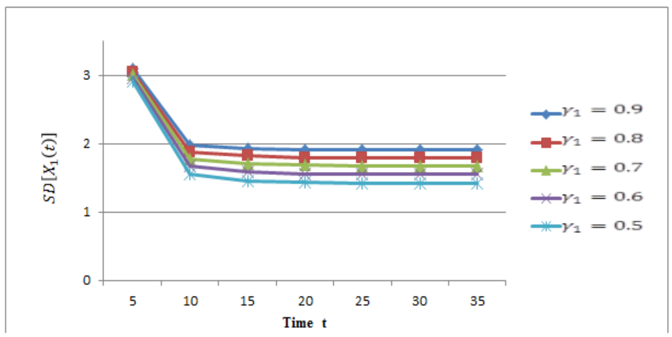


Figure 2. $SD[X_1(t)]$ varying γ_1.

$E[X_1(t)]$ and $SD[X_1(t)]$ decrease as γ_1 decreases. $E[X_1(t)]$ and $SD[X_1(t)]$ decrease as time increases and attain steady state.

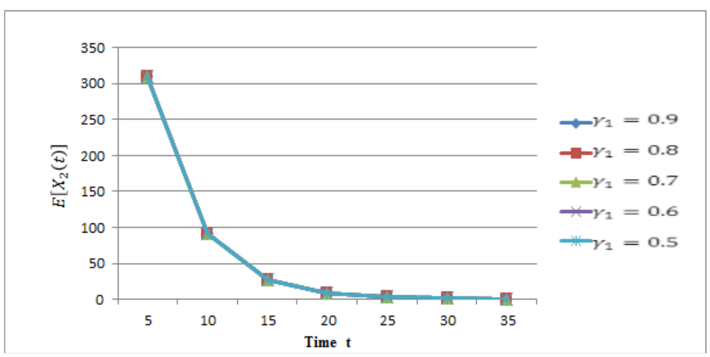


Figure 3. $E[X_2(t)]$ varying γ_1.

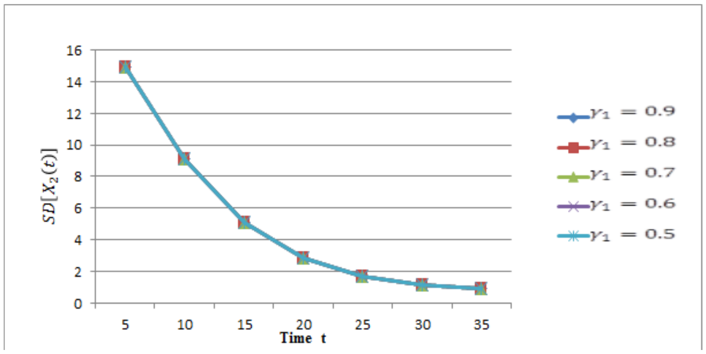


Figure 4. $SD[X_2(t)]$ varying γ_1.

Varying γ_1 does not impact $E[X_2(t)]$ and $SD[X_2(t)]$. The recruitment rate at grade 1 does not impact the size of grade 2 directly. Both $E[X_2(t)]$ and $SD[X_2(t)]$ decrease as time increases and attain steady state in the long term.

Varying γ_2 :

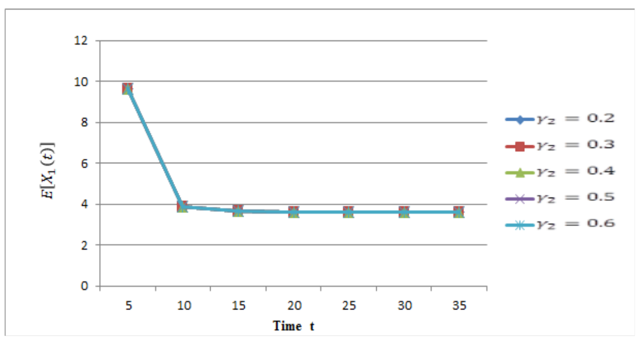


Figure 5. $E[X_1(t)]$ varying γ_2

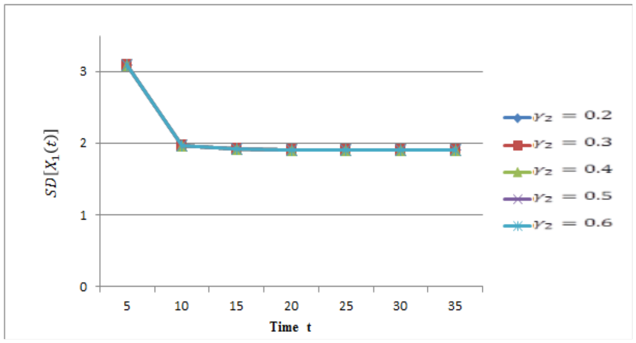


Figure 6. $SD[X_1(t)]$ varying γ_2

Varying γ_2 does not impact $E[X_1(t)]$ and $SD[X_1(t)]$. The recruitment rate at grade 2 does not impact the size of grade 1. Both $E[X_1(t)]$ and $SD[X_1(t)]$ decrease as time increases and attain steady state in the long term.

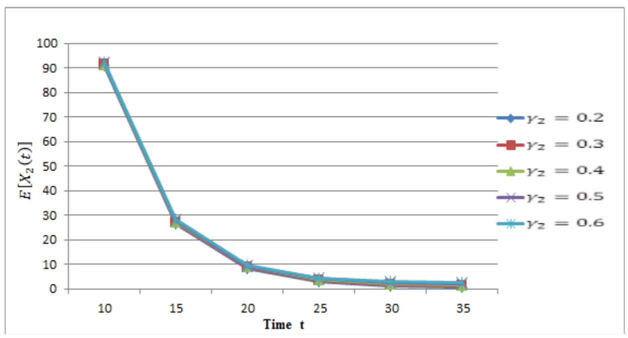


Figure 7. $E[X_2(t)]$ varying γ_2.

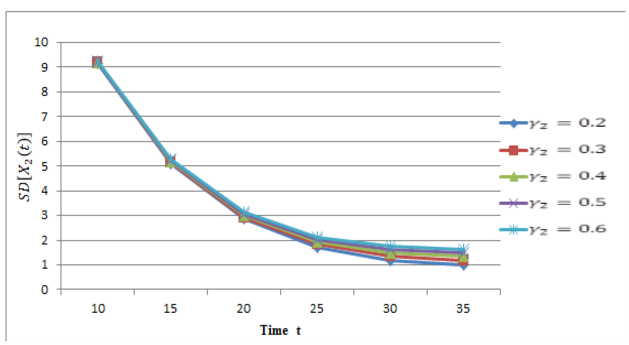


Figure 8. $SD[X_2(t)]$ varying γ_2.

It is observed that varying γ_2 has a marginal effect on $E[X_2(t)]$ and $SD[X_2(t)]$. This is due to the contributions from γ_2 dominated by $\eta(1-p)$ in the denominator. Both $E[X_2(t)]$ and $SD[X_2(t)]$ decrease as time increases and attain steady state in the long term.

Varying η:

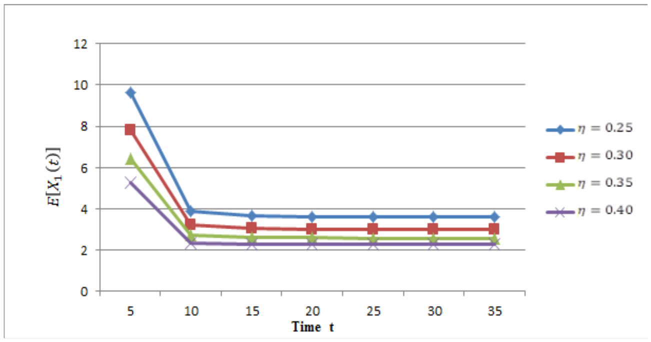


Figure 9. $E[X_1(t)]$ varying η.

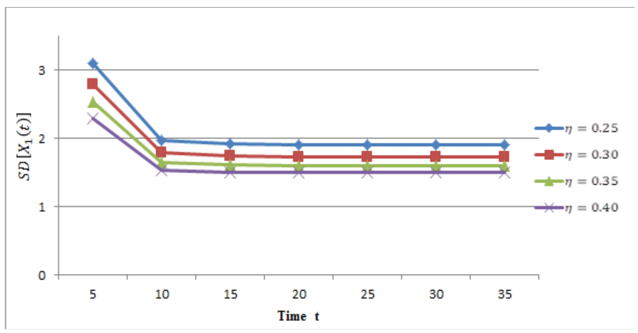


Figure 10. $SD[X_1(t)]$ varying η.

$E[X_1(t)]$ and $SD[X_1(t)]$ decrease as η increases. $E[X_1(t)]$ and $SD[X_1(t)]$ decrease as time increases and attain steady state in a certain period of time when η is varied.

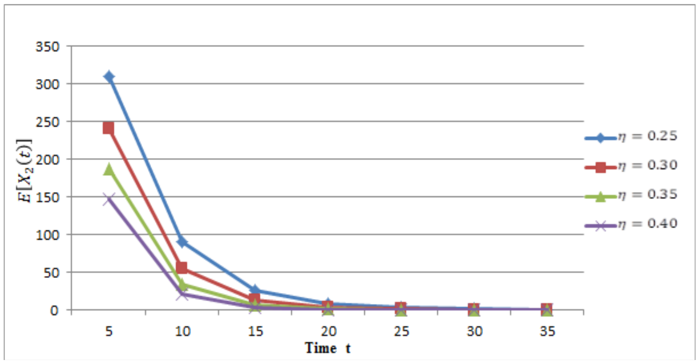


Figure 11. $E[X_2(t)]$ varying η.

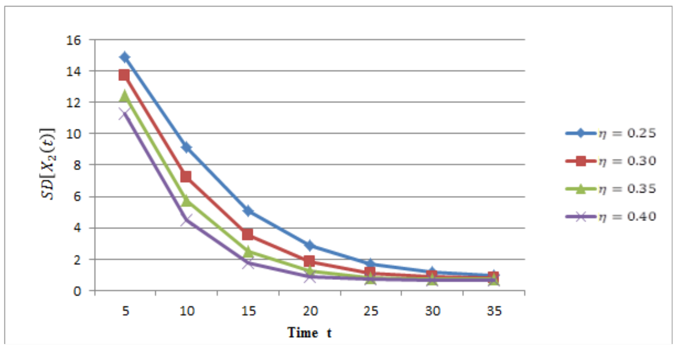
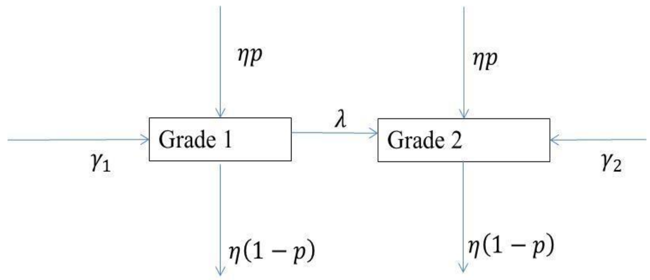


Figure 12. $SD[X_2(t)]$ varying η.

$E[X_2(t)]$ and $SD[X_2(t)]$ decrease as η increases. It is observed that $E[X_2(t)]$ and $SD[X_2(t)]$ decrease as time increases and attain steady state after a long time period depending on the value of η.

Special Case $(\mu = 0)$

When $\mu = 0$, there is no transition from Grade 2 to Grade 1 for training. The system is represented by:



The governing equations of the model are given by:

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1,\ N_2(t) = n_2]$$

$$= 1 - \gamma_1 dt - n_1 \lambda dt - \eta p dt - n_1 \eta (1-p) dt - \gamma_2 dt - \eta p dt$$

$$- n_2 \eta (1-p) dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1 - 1,\ N_2(t) = n_2]$$

$$= \gamma_1 dt + \eta p dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1 + 1,\ N_2(t) = n_2 - 1]$$

$$= (n_1 + 1) \lambda dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1 + 1,\ N_2(t) = n_2]$$

$$= (n_1 + 1) \eta (1-p) dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1,\ N_2(t) = n_2 - 1]$$

$$= \gamma_2 dt + \eta p dt + o(dt).$$

$$P[N_1(t+dt) = n_1,\ N_2(t+dt) = n_2 / N_1(t) = n_1,\ N_2(t) = n_2 + 1]$$

$$= (n_2 + 1) * \eta (1-p) dt + o(dt). \tag{17}$$

Employing probability arguments for the various mutually exclusive and collectively exhaustive events, the following differential equation is obtained:

$$[\eta(1-p)(1-s_1) + \lambda(s_1 - s_2)] \frac{\partial G}{\partial s_1} + [\eta(1-p)(1-s_2)] \frac{\partial G}{\partial s_2} - \frac{\partial G}{\partial t}$$

$$= [(\gamma_1 + \eta p)(1-s_1) + (\gamma_2 + \eta p)(1-s_2)] G. \tag{18}$$

Solving the above Lagrange's equation the joint probability generating function is obtained as:

$$G(s_1, s_2, t)$$

$$= \{1 - (1 - s_2)e^{-\eta(1-p)t}\}^m$$

$$* \{1 - e^{-\eta(1-p)t} + s_2 e^{-\eta(1-p)t}(1 - e^{-\lambda t}) + s_1 e^{-\eta(1-p)t} e^{-\lambda t}\}^l$$

$$* e^{\frac{\gamma_1 + \eta p}{\eta(1-p)} s_1} e^{-\left(\frac{\gamma_1+\eta p}{\eta(1-p)}\right)\frac{\lambda}{\lambda+\eta(1-p)} s_1} e^{\frac{\gamma_2+\eta p}{\eta(1-p)} s_2} e^{\left(\frac{\gamma_2+\eta p}{\eta(1-p)}\right)\frac{\lambda}{\lambda+\eta(1-p)} s_2}$$

$$* e^{-\left(\frac{\gamma_1+\eta p}{\eta(1-p)}\right)[\{1-e^{-\eta(1-p)t} + s_1 e^{-\eta(1-p)t} e^{-\lambda t} + s_2 e^{-\eta(1-p)t}(1-e^{-\lambda t})\}]}$$

$$* e^{-\left(\frac{\gamma_2+\eta p}{\eta(1-p)}\right)[\{1-e^{-\eta(1-p)t} + s_2 e^{-\eta(1-p)t}\}]} e^{-\left(\frac{\gamma_1+\eta p}{\eta(1-p)}\right)\frac{\lambda}{\lambda+\eta(1-p)}(s_1 - s_2)} e^{-(\lambda+\eta(1-p))t}. \quad (19)$$

Equation (19) is different from equation (6) because it is assumed that $\mu > 0$ in deriving equation (6).

The expected sizes of the two grades are:

$$E[X_1(t)] = \frac{\gamma_1 + \eta p}{\eta(1-p)} - \frac{\gamma_1 + \eta p}{\eta(1-p)}\frac{\lambda}{\lambda + \eta(1-p)} + le^{-(\lambda+\eta(1-p))t}$$

$$- \frac{\gamma_1 + \eta p}{\eta(1-p)} e^{-(\lambda+\eta(1-p))t} + \frac{\gamma_1 + \eta p}{\eta(1-p)}\frac{\lambda}{\lambda + \eta(1-p)} e^{-(\lambda+\eta(1-p))t}. \quad (20)$$

and

$$E[X_2(t)] = \frac{\gamma_2 + \eta p}{\eta(1-p)}[1 - e^{-(\eta(1-p))t}] + \frac{\gamma_1 + \eta p}{\eta(1-p)}\frac{\lambda}{\lambda + \eta(1-p)}[1 - e^{-(\lambda+\eta(1-p))t}]$$

$$- \frac{\gamma_1 + \eta p}{\eta(1-p)} e^{-(\eta(1-p))t}[1 - e^{-\lambda t}] + me^{-(\eta(1-p))t}$$

$$+ le^{-(\eta(1-p))t}[1 - e^{-\lambda t}]. \quad (21)$$

The matrix of the means is given by:

$$\overline{X(t)} = \begin{bmatrix} E[X_1(t)] \\ E[X_2(t)] \end{bmatrix}^T. \quad (22)$$

The covariance matrix is given by:

$$C[X(t)] = \begin{bmatrix} Var[X_1(t)] & Cov[X_1(t), X_2(t)] \\ Cov[X_2(t), X_1(t)] & Var[X_2(t)] \end{bmatrix}. \tag{23}$$

where,

$$Var[X_1(t)] = \frac{\gamma_1 + \eta p}{\lambda + \eta(1-p)}[1 - e^{-(\lambda + \eta(1-p))t}]$$

$$+ le^{-(\lambda+\eta(1-p))t}[1 - e^{-(\lambda+\eta(1-p))t}].$$

$$Var[X_2(t)] = \frac{\gamma_2 + \eta p}{\eta(1-p)}[1 - e^{-\eta(1-p)t}]$$

$$+ \frac{\gamma_1 + \eta p}{\eta(1-p)} \frac{\lambda}{(\lambda + \eta(1-p))}[1 - e^{-(\lambda+\eta(1-p))t}]$$

$$- \frac{\gamma_1 + \eta p}{\eta(1-p)} e^{-\eta(1-p)t}[1 - e^{-\lambda t}] + me^{-\eta(1-p)t}[1 - e^{-\eta(1-p)t}]$$

$$+ le^{-\eta(1-p)t}[1 - e^{-\eta(1-p)t}[1 - e^{-\lambda t}]]. \tag{25}$$

$$Cov[X_1(t), X_2(t)] = Cov[X_2(t), X_1(t)] = -le^{-2\eta(1-p)t}e^{-\lambda t}[1 - e^{-\lambda t}]. \tag{26}$$

It is observed that the covariance is negative. This implies that the system size is maintained.

At steady state.

$$E[X_1] = \frac{\gamma_1 + \eta p}{\lambda + \eta(1-p)} \; E[X_2] = \frac{\gamma_2 + \eta p}{\eta(1-p)} + \frac{\gamma_1 + \eta p}{\eta(1-p)} \frac{\lambda}{(\lambda + \eta(1-p))}. \tag{27}$$

$$Var[X_1] = \frac{\gamma_1 + \eta p}{\lambda + \eta(1-p)}.$$

$$Var[X_2] = \frac{\gamma_2 + \eta p}{\lambda(1-p)} + \frac{\gamma_1 + \eta p}{\eta(1-p)} \frac{\lambda}{(\lambda + \eta(1-p))}. \tag{28}$$

$$Cov[X_1, X_2] = 0. \tag{29}$$

Numerical illustration

For various values of $\gamma_1, \gamma_2, \lambda, \eta, p, l$ and m, the expected size of grade 1 and 2, the variance of the grade size and the covariance between the size of grade 1 and 2 have been computed. The values of γ_1, γ_2, λ, η, p, l and m are the same as those considered in the case when $\mu > 0$.

The following Figure 13 to Figure 24 illustrate the pattern of the expected size and the standard deviation over time when each of the parameters is varied holding all other parameters constant.

Varying γ_1 :

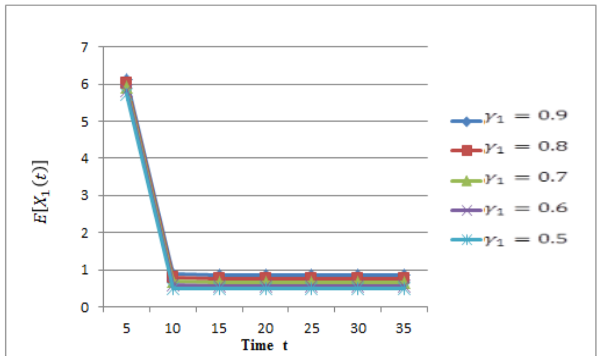


Figure 13. $E[X_1(t)]$ varying γ_1.

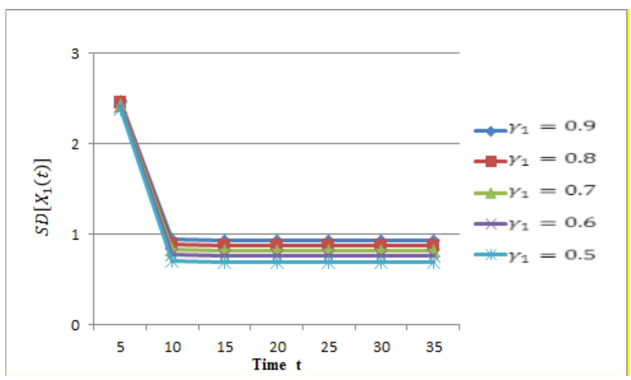


Figure 14. $SD[X_1(t)]$ varying γ_1

$E[X_1(t)]$ and $SD[X_1(t)]$ decrease as γ_1 decreases. $E[X_1(t)]$ and $SD[X_1(t)]$ decrease as time increases and then attain steady state.

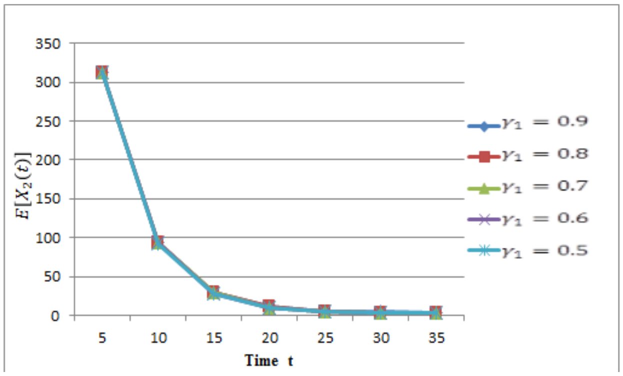


Figure 15. $E[X_2(t)]$ varying γ_1.

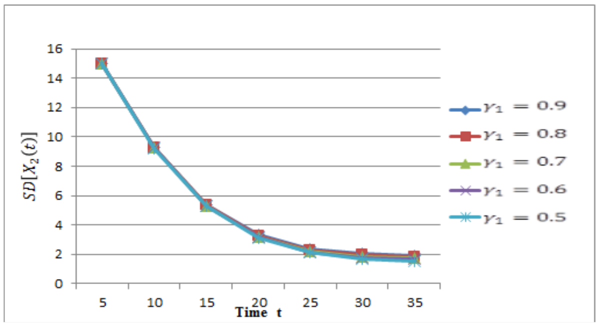


Figure 16. $SD[X_2(t)]$ varying γ_1.

Varying γ_1 does not impact $E[X_2(t)]$ and $SD[X_2(t)]$. The recruitment rate at grade 1 does not impact the size of grade 2 directly. Both $E[X_2(t)]$ and $SD[X_2(t)]$ decrease as time increases and attain steady state in the long term.

Varying γ_2 :

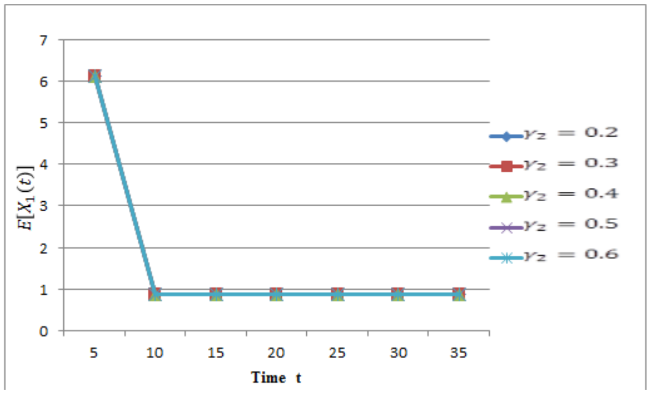


Figure 17. $E[X_1(t)]$ varying γ_2

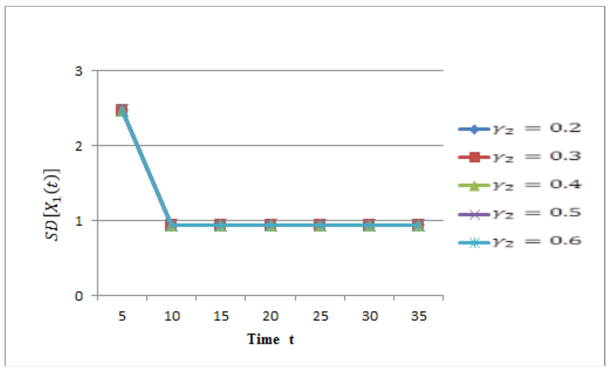


Figure 18. $SD[X_1(t)]$ varying γ_2.

Varying γ_2 does not impact $E[X_1(t)]$ and $SD[X_1(t)]$. The recruitment rate at grade 2 does not impact the size of grade 1. Both $E[X_1(t)]$ and $SD[X_1(t)]$ decrease as time increases and then attain steady state.

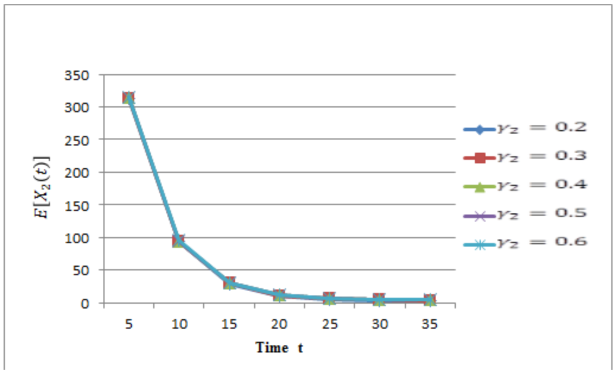


Figure 19. $E[X_2(t)]$ varying γ_2.

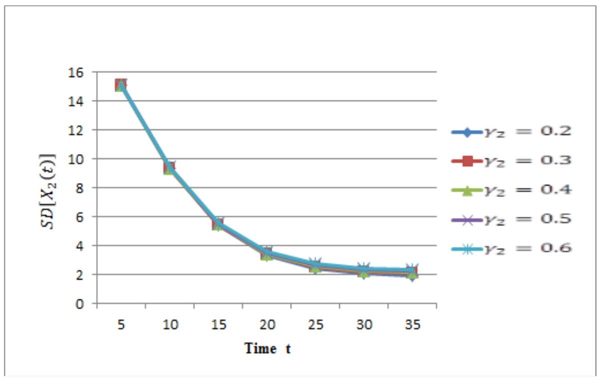


Figure 20. $SD[X_2(t)]$ varying γ_2.

It is observed that varying γ_2 does not impact $E[X_2(t)]$ and $SD[X_2(t)]$. As observed earlier, this is due to the contributions from γ_2 dominated by $\eta(1-p)$ in the denominator. Both $E[X_2(t)]$ and $SD[X_2(t)]$ decrease as time increases and then attain steady state.

Varying η:

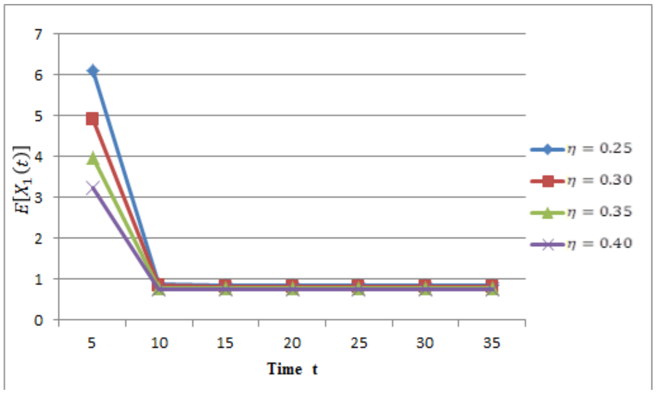


Figure 21. $E[X_1(t)]$ varying η.

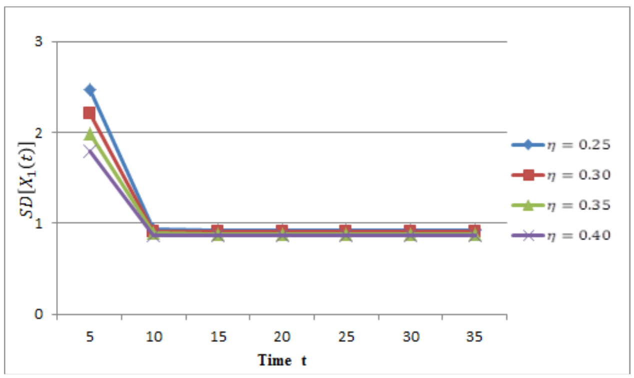


Figure 22. $SD[X_1(t)]$ varying η

$E[X_1(t)]$ and $SD[X_1(t)]$ decrease as η increases. $E[X_1(t)]$ and $SD[X_1(t)]$ decrease as time increases and then attain steady state.

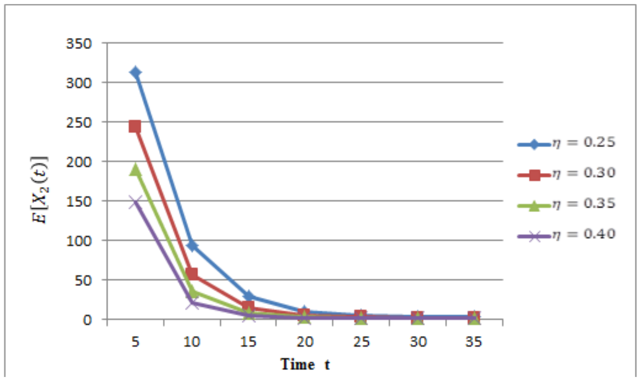


Figure 23. $E[X_2(t)]$ varying η.

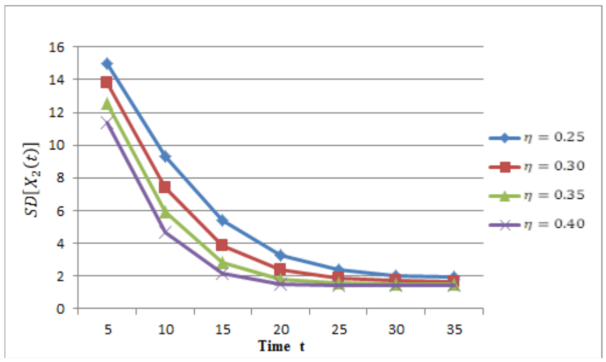


Figure 24. $SD[X_2(t)]$ varying η.

$E[X_2(t)]$ and $SD[X_2(t)]$ decrease as η increases. It is observed that $E[X_2(t)]$ and $SD[X_2(t)]$ decrease as time increases and attain steady state after a certain time period depending on the value of η.

A comparison of the expected grade sizes and the standard deviation of grade sizes when $\mu > 0$ and when $\mu = 0$ are shown in Table 1.

Table 1. A comparison when $\mu > 0$ and when $\mu = 0$.

t	$E[X_1(t)]$		$E[X_2(t)]$		$SD[X_1(t)]$		$SD[X_2(t)]$	
	$\mu > 0$	$\mu = 0$	$\mu > 0$	$\mu = 0$	$\mu > 0$	$\mu = 0$	$\mu > 0$	$\mu = 0$
5	9.6361	6.1177	309.5810	313.0994	3.0965	2.4678	14.9242	15.0079
10	3.9037	0.8859	90.9152	93.9330	1.9757	0.9412	9.1382	9.2969

15	3.6881	0.8584	26.6384	29.4681	1.9204	0.9265	5.1016	5.3711
20	3.6335	0.8582	8.1578	10.9331	1.9062	0.9264	2.8473	3.2988
25	3.6178	0.8582	2.8464	5.6060	1.9021	0.9264	1.6859	2.3668
30	3.6133	0.8582	1.3199	4.0750	1.9009	0.9264	1.1487	2.0186
35	3.6121	0.8582	0.8812	3.6350	1.9005	0.9264	0.9387	1.9066
40	3.6117	0.8582	0.7551	3.5085	1.9004	0.9264	0.8690	1.8731
45	3.6116	0.8582	0.7189	3.4722	1.9004	0.9264	0.8479	1.8634
50	3.6115	0.8582	0.7085	3.4618	1.9004	0.9264	0.8417	1.8606
55	3.6115	0.8582	0.7055	3.4588	1.9004	0.9264	0.8399	1.8598
60	3.6115	0.8582	0.7046	3.4579	1.9004	0.9264	0.8394	1.8595
65	3.6115	0.8582	0.7044	3.4576	1.9004	0.9264	0.8393	1.8595
70	3.6115	0.8582	0.7043	3.4576	1.9004	0.9264	0.8392	1.8595

It is observed that as t increases, the expected grade size and the standard deviation approach a constant value when $\mu > 0$ and when $\mu = 0$.

A typical covariance pattern is shown in Figure 25.

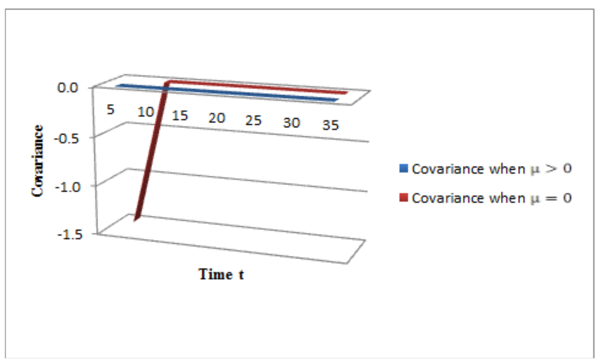


Figure 25. Covariance in grade sizes when $\mu > 0$ and when $\mu = 0$.

The covariance between the grade sizes when $\mu > 0$ approaches zero faster than the covariance between grade sizes when $\mu = 0$.

Conclusion

In this paper, an approach to determine the characteristics of a two-grade human resource system is proposed when (a) transition from a higher to a lower grade for training and skill upgrade and (b) recruitment of personnel who had earlier left the system in both the grades are allowed. The transient and steady state behaviours are discussed. In the discussions, assumptions have been made with respect to the movement of personnel to a lower grade for training and the recruitment of personnel who had moved out of the grade to the external environment.

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