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2-ODD LABELING OF SOME GRAPHS AND ITS JOINS

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Abstract

A graph G with p vertices and q edges is called a 2-Odd graph if the vertices of G can be labelled with integers (necessarily distinct) such that any two vertices that are adjacent will have their modulus difference of their labels as either exactly 2 or an odd integer. In this paper we study on comb graph $P_n \odot K_1$ and its M-joins, comb graph $P_n \odot K_2$ and its M-Joins, armed crown graph $C_3 \Theta P_n$ and its M-joins, human chain graph $HC_{n,3}(p, q)$ and proved that they are 2-odd graph and in further we study some characteristics of these graphs and their labelling schema.

1. Introduction

The study of labelling of vertices and edges of a finite graph plays a significant role and development of the subject graph theory. A dynamic study on different labelling techniques is discussed by J. A. Gallian [1]. In the study of labelling process prime distance graphs introduced by Eggleton, Erdos and Skilton in [2, 3] looks more significant which states that for any set D of positive integers the distance graph $Z(D)$ as the graph with vertex set Z and an edge between integers x and y if an only if $|x - y| \in D$. The prime distance graph $Z(P)$ is the distance graph with $D = P$, the set of all

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primes. Further study on prime distance graph 2-odd graph was carried out in various papers [4] [5] [6] [7] and we have taken those papers as our literature survey and have studied on Comb graph $P_n \odot K_1$ and its M-Joins, Comb graph $P_n \odot K_2$ and its M-Joins, $P_n \odot C_3$ graph and its M-Joins, Armed Crown graph $C_3 \Theta P_n$ and its M-Joins, Human Chain graph $HC_{n,3}(p, q)$ and proved that they are 2-Odd graph and further study some characteristics of these graphs and their labelling schema.

2. Preliminaries

Definition 2.1. A graph G is called a prime distance graph if there exists a one to one labelling of its vertices $f : V(G) \to Z$ such that for every two adjacent vertices in G which induces an edge labelling with the condition that for all $x, y \in V(G)$, adjacent vertices we have the induced edge labeling $f^*(xy) = |f(x) - f(y)| =$ prime integer.

Definition 2.2. A graph is 2-Odd if for any two adjacent vertices of G with the condition that the induced edge labelling is $f^*(xy) = |f(x) - f(y)|$ which is either exactly 2 or an Odd integer.

Definition 2.3. Comb graph $P_n \odot K_1$ has $2n$ vertices and $2n - 1$ edges which is obtained by joining single pendant edge to each vertex of a path. Below is the example of Comb graph $P_5 \odot K_1$

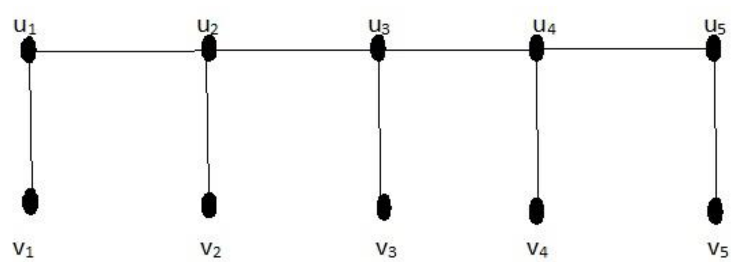

Figure 1. Comb graph $P_5 \odot K_1$.

Definition 2.4. Comb graph $P_n \odot K_2$ has $3n$ vertices and $3n - 1$ edges which is obtained by joining two pendant edges to each vertex of a path.

Below is the example of Comb graph $P_3 \odot K_2$.

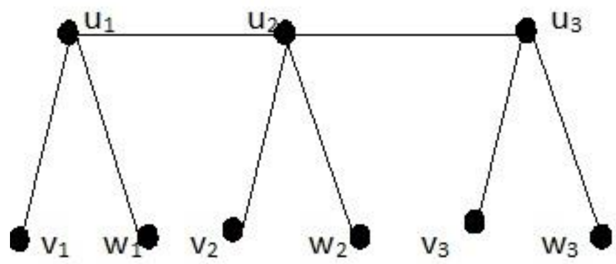

Figure 2. Comb graph $P_3 \odot K_2$.

Definition 2.5. $P_n \odot C_3$ graph is the graph in which every vertex of a path graph is attached with a cycle C_3. Below is the example of $P_3 \odot C_3$.

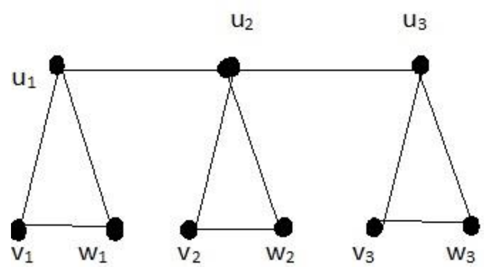

Figure 3. $P_3 \odot C_3$ Graph.

Definition 2.6. An armed crown graph $C_n \Theta P_n$ is a graph in which path P_n is attached at each vertex of the cycle C_n. Below is the example of $C_3 \Theta P_2$ graph

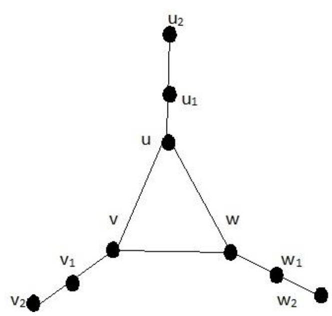

Figure 4. Armed Crown graph $C_3 \Theta P_2$.

Definition 2.7. A human chain graph $HC_{n,m}(p, q)$ is obtained by a path $u_1, u_2, \ldots, u_{2n+1}, n \in N$ joining a cycle C_m of length m and Y-tree Y_{m+1} for $m \geq 3$ to each u_{2i} where $1 \leq i \leq n$. The vertices of C_m and Y-tree Y_{m+1} are $v_1, v_2, \ldots, v_{(m-1)m}$ and $w_1, w_2, \ldots, w_{nm}$ respectively. Below is the example of human chain graph $HC_{2,3}$.

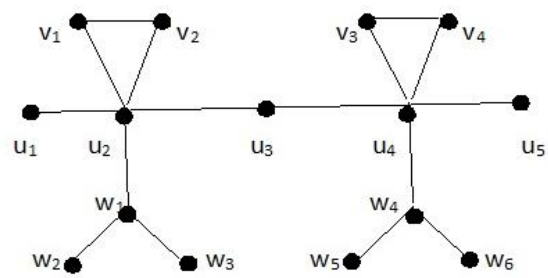

Figure 5. Human Chain graph $HC_{2,3}$.

3. Main Results

Theorem 3.1. *The Comb graph* $P_n \odot K_1$ *is* 2-*Odd graph*.

Proof. Consider the graph G to be a Comb graph $P_n \odot K_1$. Let the vertices of $P_n \odot K_1$ be $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ where the vertices of the path graph are $\{u_1, u_2, \ldots, u_n\}$ and the vertices of K_1 attached to each of the path vertices are $\{v_1, v_2, \ldots, v_n\}$. Now consider the edges of $P_n \odot K_1$ graph are $E(G) = \{u_i u_{i+1}, 1 \leq i \leq n-1\} \cup \{u_i v_i, 1 \leq i \leq n\}$. Now let us label the vertices as follows

$f(u_i) = 2i - 1$ for $1 \leq i \leq n$;

$f(v_i) = 2i$ for $1 \leq i \leq n$.

Hence the induced edge labelling are computed as

$f^*(u_i u_{i+1}) = | f(u_i) - f(u_{i+1}) | = 2;$

$f^*(u_i v_i) = | f(u_i) - f(v_i) | = 1.$

Therefore the induced edge labelling satisfies the condition and hence the Comb graph $P_n \odot K_1$ is 2-Odd graph.

To illustrate with example we have consider the Comb graph $P_5 \odot K_1$ and verified that it is 2-Odd gr

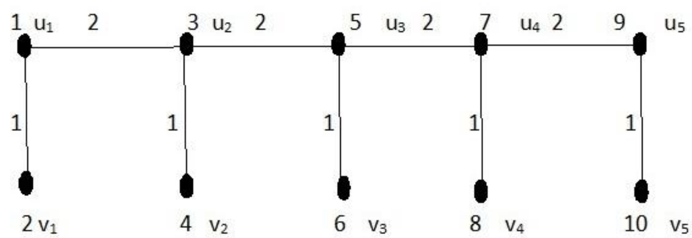

Figure 6. Comb graph $P_5 \odot K_1$ 2-Odd graph.

Theorem 3.2. *The* 1*-Join of Comb graph* $P_n \odot K_1$ *is* 2*-Odd graph*.

Proof. Consider the graph G to be a 1-Join of Comb graph $P_n \odot K_1$. The vertex set is $V(G) = \{u_1, u_2, \ldots, u_n, u_1^1, u_2^1, \ldots, u_n^1, v_1, v_2, \ldots, v_n, v_1^1, v_2^1, \ldots, v_n^1\}$ where the vertices of the path graph are $\{u_1, u_2, \ldots, u_n, u_1^1, u_2^1, \ldots, u_n^1\}$ and the vertices of K_1 attached to each of the path vertices are $\{v_1, v_2, \ldots, v_n, v_1^1, v_2^1, \ldots, v_n^1\}$. The edges of 1-Join of Comb graph $P_n \odot K_1$ $E(G) = \{u_i u_{i+1}, 1 \leq i \leq n_i - 1\} \cup \{u_i v_i, 1 \leq i \leq n_i\} \cup \{u_i^1 u_{i+1}^1, 1 \leq i \leq n_2 - 1\} \cup \{u_i^1 v_i^1, 1 \leq i \leq n_2\} \cup (v_{n_1} u_1^1)$. Now let us label the vertices as follows.

$f(u_i) = 2i - 1$ for $1 \leq i \leq n_1$;

$f(v_i) = 2i$ for $1 \leq i \leq n_1$

$f(u_i^1) = 2n_1 + (2i - 1)$ for $1 \leq i \leq n_2$

$f(v_i^1) = 2n_1 + 2i$ for $1 \leq i \leq n_2$.

Now computing the induced edges we have

$f^*(u_i u_{i+1}) = |\, f(u_i) - f(u_{i+1}) \,| = 2;$

$f^*(u_i v_i) = | f(u_i) - f(v_i) | = 1$

$f^*(u_i^1 u_{i+1}^1) = | f(u_i^1) - f(u_{i+1}^1) | = 2$

$f^*(u_i^1 v_i^1) = | f(u_i^1) - f(v_i^1) | = 1$

$f^*(v_{n_1} u_1^1) = | f(v_{n_1}) - f(u_1^1) | = 1.$

Therefore the induced edge labelling satisfies the condition and hence the 1-Join Comb graph $P_n \odot K_1$ is 2-Odd graph.

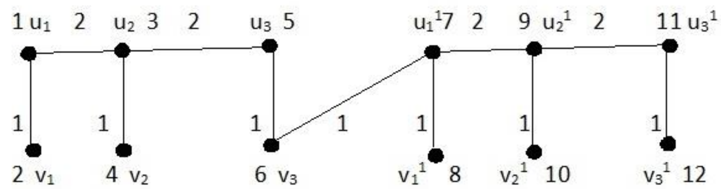

Figure 7. 1-Join Comb graph $P_3 \odot K_1$ 2-Odd graph.

Theorem 3.3. *The M-Join of Comb graph* $P_n \odot K_1$ *is* 2-*Odd graph.*

Proof. Consider the M-Join of Comb graph $P_n \odot K_1$. To prove that the graph is 2-Odd graph. Let us prove by applying the method of mathematical induction. We have proved in the previous theorem that 1-Join of Comb graph $P_n \odot K_1$ is 2-Odd graph. Let us now assume that the theorem is true for $n = k$ i.e K-join of Comb graph $P_n \odot K_1$ is 2-Odd graph. Now let us prove for $n = k + 1$ i.e $k + 1$ Join of Comb graph $P_n \odot K_1$ is 2-Odd graph. This is done by adding 1-Join of Comb graph $P_n \odot K_1$ with k-Join of Comb graph $P_n \odot K_1$ to form $k + 1$ Join of Comb graph $P_n \odot K_1$. We know that 1-Join of Comb graph $P_n \odot K_1$ and k-Join of Comb graph $P_n \odot K_1$ are 2-Odd graph and hence $k + 1$ Join of Comb graph $P_n \odot K_1$ is 2-Odd graph. Hence the proof by induction.

Theorem 3.4. *The Comb graph* $P_n \odot K_2$ *is* 2-*Odd graph.*

Proof. Consider the graph G to be a comb graph $P_n \odot K_2$. The vertex set is $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ where the vertices of the path graph are $\{u_1, u_2, \ldots, u_n\}$ and the vertices of K_2 attached to each of the path vertices are $\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$. Now let us label the vertices. We find the corresponding edges of the Comb graph $P_n \odot K_2$ as follows $\{u_i u_{i+1}, 1 \le i \le n-1\} \cup \{u_i v_i, 1 \le i \le n\} \cup \{u_i w_i, 1 \le i \le n\}$. Now let us label the vertices as follows

$f(u_i) = 3i - 2$ for $1 \le i \le n$

$f(v_i) = 3i - 1$ for $1 \le i \le n$

$f(w_i) = 3i$ for $1 \le i \le n$.

Then the induced edges are computed as follows

$f^*(u_i u_{i+1}) = |\, f(u_i) - f(u_{i+1}) \,| = 3$

$f^*(u_i v_i) = |\, f(u_i) - f(v_i) \,| = 1$

$f^*(u_i w_i) = |\, f(u_i) - f(w_i) \,| = 2.$

Then the induced edge labelling satisfies the condition and hence the Comb graph $P_n \odot K_2$ is 2-Odd graph.

To illustrate we have the following example of Comb graph $P_3 \odot K_2$ which is 2-Odd graph

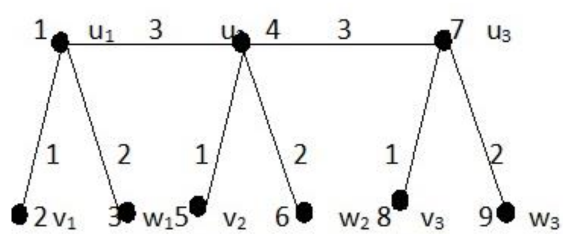

Figure 8. Comb Graph $P_3 \odot K_2$ 2-Odd graph.

Theorem 3.5. *The* 1-*Join Comb graph* $P_n \odot K_2$ *is* 2-*Odd graph.*

Proof. Consider the graph G to be 1-Join Comb graph $P_n \odot K_2$. The vertex set is $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n, u_1^1, u_2^1, \ldots, u_n^1, , v_1^1, v_2^1, \ldots, v_n^1, w_1^1, w_2^1, \ldots, w_n^1\}$ where the vertices of the path graph are $\{u_1, u_2, \ldots, u_n\}$ and $\{u_1^1, u_2^1, \ldots, u_n^1\}$, the vertices of the corresponding K_2 are $\{v_1, v_2, \ldots, v_n\}$, $\{w_1, w_2, \ldots, w_n\}$, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{w_1^1, w_2^1, \ldots, w_n^1\}$. Now the edges comprises of $\{u_i u_{i+1}, 1 \le i \le n_1 - 1\} \cup \{u_i v_i, 1 \le i \le n_1\} \cup \{u_i w_i, 1 \le i \le n_1\} \cup (w_{n_1} u_1^1) \cup \{u_i^1 u_{i+1}^1, 1 \le i \le n_2 - 1\} \cup \{u_i^1 v_i^1, 1 \le i \le n_2\} \cup \{u_i^1 w_i^1, 1 \le i \le n_2\}$. Now let us label the vertices as follows

$f(u_i) = 3i - 2$ for $1 \le i \le n_1$

$f(v_i) = 3i - 1$ for $1 \le i \le n_1$

$f(w_i) = 3i$ for $1 \le i \le n_1$

$f(u_i^1) = 3n_1 + (3i - 2)$ for $1 \le i \le n_2$

$f(v_i^1) = 3n_1 + (3i - 1)$ for $1 \le i \le n_2$

$f(w_i^1) = 3n_1 + 3i$ for $1 \le i \le n_2$.

Now the induced edges are computed as follows

$f^*(u_i u_{i+1}) = | f(u_i) - f(u_{i+1}) | = 3$

$f^*(u_i v_i) = | f(u_i) - f(v_i) | = 1$

$f^*(u_i w_i) = | f(u_i) - f(w_i) | = 2$

$f^*(u_i^1 u_{i+1}^1) = | f(u_i^1) - f(u_{i+1}^1) | = 3$

$f^*(u_i^1 v_i^1) = | f(u_i^1) - f(v_i^1) | = 1$

$f^*(u_i^1 w_i^1) = | f(u_i^1) - f(w_i^1) | = 2$

$f^*(w_{n_1} u_1^1) = | f(w_{n_1}) - f(u_1^1) | = 1.$

We find that the induced edge labelling satisfies the condition and therefore 1-Join Comb graph $P_n \odot K_2$ is 2-Odd graph.

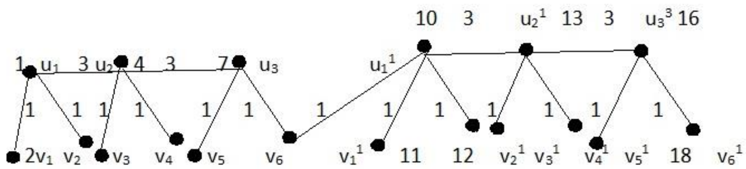

Figure 9. 1-Join Comb graph $P_3 \odot K_2$ is 2-Odd graph.

Theorem 3.6. *The M-Join Comb graph* $P_n \odot K_2$ *is* 2-*Odd graph*.

Proof. Consider the M-Join Comb graph $P_n \odot K_2$ is 2-Odd graph. We can prove the theorem by applying mathematical induction or by successively adding 1-join of Comb graph $P_n \odot K_2$ to the Comb graph $P_n \odot K_2$ M times and we can obtain M-Join of Comb graph $P_n \odot K_2$ which we can prove that it is 2-Odd graph as each of the joins which we add to is 2-Odd graph. In a similar way the same proof can be given in the form of applying principle of Mathematical induction. We know that 1-Join of Comb graph $P_n \odot K_2$ is 2-Odd graph from theorem 3.5. Now if we assume that for some k-Join of Comb graph $P_n \odot K_2$ is 2-Odd graph then we can prove that for $k+1$- Join of Comb graph $P_n \odot K_2$ is 2-Odd graph. Hence the theorem.

Theorem 3.7. *The graph* $P_n \odot C_3$ *is* 2-*Odd graph*.

Proof. Consider the graph $P_n \odot C_3$ with the vertices as $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ where the vertices of the path graph are $\{u_1, u_2, \ldots, u_n\}$ and the vertices of K_2 attached to each of the path vertices are $\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$. Now let us label the vertices of $P_n \odot C_3$. We find the corresponding edges of the Comb graph $P_n \odot C_3$ as follows $E(G) = \{u_i u_{i+1}, 1 \le i \le n-1\} \cup \{u_i v_i, 1 \le i \le n\}$ $\cup \{u_i w_i, 1 \le i \le n\} \cup \{v_i w_i, 1 \le i \le n\}$. Now let us label the vertices as follows

$f(u_i) = 3i - 2$ for $1 \le i \le n$

$f(v_i) = 3i - 1$ for $1 \le i \le n$

$f(w_i) = 3i$ for $1 \le i \le n$.

Then the induced edges are computed as follows

$f^*(u_iu_{i+1}) = |\, f(u_i) - f(u_{i+1})\,| = 3$

$f^*(u_iv_i) = |\, f(u_i) - f(v_i)\,| = 1$

$f^*(u_iw_i) = |\, f(u_i) - f(w_i)\,| = 2$

$f^*(v_iw_i) = |\, f(v_i) - f(w_i)\,| = 1.$

Then the induced edge labelling satisfies the condition and hence the Comb graph $P_n \odot C_3$ is 2-Odd graph.

We illustrate by example that $P_n \odot C_3$ is 2-Odd graph

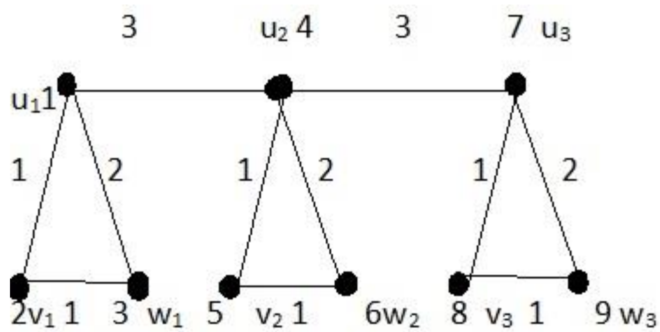

Figure 10. $P_n \odot C_3$ is 2-Odd Graph.

Theorem 3.8. *The* 1-*Join of graph* $P_n \odot C_3$ *is* 2-*Odd graph*.

Proof. Consider the 1-Join of $P_n \odot C_3$ graph with the vertices as $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n, u_1^1, u_2^1, \ldots, u_n^1, , v_1^1, v_2^1, \ldots, v_n^1, w_1^1, w_2^1, \ldots, w_n^1\}$ with the vertices of path graph are $\{u_1, u_2, \ldots, u_n\}$ and $\{u_1^1, u_2^1, \ldots, u_n^1\}$, the vertices of the corresponding attachment to form a C_3 are $\{v_1, v_2, \ldots, v_n\}$, $\{w_1, w_2, \ldots, w_n\}$, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{w_1^1, w_2^1, \ldots, w_n^1\}$. Now the edges comprises of $\{u_iu_{i+1}, 1 \le i \le n_1 - 1\} \cup \{u_iv_i, 1 \le i \le n_1\} \cup \{u_iw_i, 1 \le i \le n_1\} \cup \{v_iw_i, 1 \le i \le n_1\} \cup (w_{n_1}u_1^1) \cup \{u_i^1u_{i+1}^1, 1 \le i \le n_2 - 1\} \cup$

$\{u_i^1 v_i^1, 1 \leq i \leq n_2\} \cup \{u_i^1 w_i^1, 1 \leq i \leq n_2\} \cup \{v_i^1 w_i^1, 1 \leq i \leq n_2\}$. Now let us label the vertices as follows

$f(u_i) = 3i - 2$ for $1 \leq i \leq n_1$

$f(v_i) = 3i - 1$ for $1 \leq i \leq n_1$

$f(w_i) = 3i$ for $1 \leq i \leq n_1$

$f(u_i^1) = 3n_1 + (3i - 2)$ for $1 \leq i \leq n_2$

$f(v_i^1) = 3n_1 + (3i - 1)$ for $1 \leq i \leq n_2$

$f(w_i^1) = 3n_1 + 3i$ for $1 \leq i \leq n_2$.

Now the induced edges are computed as follows

$f^*(u_i u_{i+1}) = | f(u_i) - f(u_{i+1}) | = 3$

$f^*(u_i v_i) = | f(u_i) - f(v_i) | = 1$

$f^*(u_i w_i) = | f(u_i) - f(w_i) | = 2$

$f^*(v_i w_i) = | f(v_i) - f(w_i) | = 1$

$f^*(u_i^1 u_{u+1}^1) = | f(u_i^1) - f(u_{i+1}^1) | = 3$

$f^*(u_i^1 v_i^1) = | f(u_i^1) - f(v_i^1) | = 1$

$f^*(u_i^1 w_i^1) = | f(u_i^1) - f(w_i^1) | = 2$

$f^*(v_i^1 w_i^1) = | f(v_i^1) - f(w_i^1) | = 1$

$f^*(w_{n_1} u_1^1) = | f(w_{n_1}) - f(u_1^1) | = 1.$

We find that the induced edge labelling satisfies the condition and therefore 1-Join Comb graph $P_n \odot C_3$ is 2-Odd graph.

Theorem 3.9. *The M-Join of graph* $P_n \odot C_3$ *is* 2-*Odd graph.*

Proof. Consider M-Join of $P_n \odot C_3$. To prove that it is 2-Odd graph. We can consider the basic $P_n \odot C_3$ graph which is proved as 2-Odd graph. By introducing Joins to the $P_n \odot C_3$ graph we obtain 1-Join of $P_n \odot C_3$ graph which is also proved as 2-Odd graph and in succession by adding joins to $P_n \odot C_3$ graph we can conclude that M-Join $P_n \odot C_3$ graph is 2-Odd graph. Hence the theorem.

Theorem 3.10. *The Armed Crown graph* $C_3 \Theta P_n$ *is* 2-*Odd graph*.

Proof. Consider the Armed Crown graph $C_3 \Theta P_n$. We have the Vertex set as $V(G) = \{u_1, v_1, w_1, u_1^i, v_1^i, w_1^i\}$ for $1 \le i \le n$. Where u_1, v_1, w_1 are the vertices of cycle graph C_3 and u_1^i, v_1^i, w_1^i for $1 \le i \le n$ are the corresponding vertices attached to the cycle graph C_3 vertices u_1, v_1, w_1 as path vertices connecting them. Now the edge set are $E(G) = \{u_1v_1, v_1w_1, w_1u_1, u_1u_1^1, u_1^iu_1^{i+1}, v_1v_1^1, v_1^iv_1^{i+1}, w_1w_1^1, w_1^iw_1^{i+1}\}$ for $1 \le i \le n-1$. Now let us label the vertices as follows

$f(u_1) = 1$

$f(v_1) = 2$

$f(w_1) = 3$

$f(u_1^i) = 3i + 1$ for $1 \le i \le n$

$f(v_1^i) = 3i + 2$ for $1 \le i \le n$

$f(w_1^i) = 3i + 3$ for $1 \le i \le n$.

Now computing the induced edge labels we have

$f^*(u_1v_1) = |f(u_1) - f(v_1)| = 1$

$f^*(v_1w_1) = |f(v_1) - f(w_1)| = 1$

$f^*(u_1w_1) = |f(u_1) - f(w_1)| = 2$

$f^*(u_1u_1^1) = | f(u_1) - f(u_1^1) | = 3$

$f^*(u_1^iu_1^{i+1}) = | f(u_1^i) - f(u_1^{i+1}) | = 3$ for $1 \leq i \leq n-1$

$f^*(v_1v_1^1) = | f(v_1) - f(v_1^1) | = 3$

$f^*(v_1^iv_1^{i+1}) = | f(v_1^i) - f(v_1^{i+1}) | = 3$ for $1 \leq i \leq n-1$

$f^*(w_1w_1^1) = | f(w_1) - f(w_1^1) | = 3$

$f^*(w_1^iw_1^{i+1}) = | f(w_1^i) - f(w_1^{i+1}) | = 3$ for $1 \leq i \leq n-1$.

Hence we find that the induced edge labelling satisfy the condition and therefore the Armed Crown graph $C_3 \Theta P_n$ is 2-Odd graph.

To illustrate we consider armed crown graph $C_3 \Theta P_n$ which is 2-Odd graph

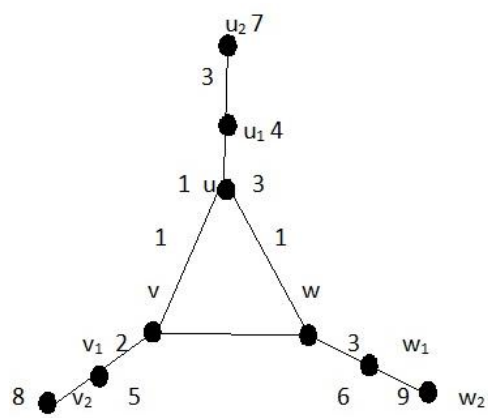

Figure 11. Armed Crown Graph $C_3 \Theta P_2$ 2-Odd graph.

Theorem 3.11. *The* 1-*Join of Armed Crown graph* $C_3 \Theta P_n$ *is* 2-*Odd graph*.

Proof. Consider 1-Join Armed Crown graph $C_3 \Theta P_n$. The vertex set is $V(G) = \{u_1, v_1, w_1, u_1^i, v_1^i, w_1^i\} \cup \{u_1^1, v_1^1, w_1^1, u_1^{i1}, v_1^{i1}, w_1^{i1}\}$ where the vertices $u_1, v_1, w_1, u_1^1, v_1^1, w_1^1$ are the vertices of C_3 which are attached by corresponding paths whose vertices are u_1^i, v_1^i, w_1^i, u_1^{i1}, v_1^{i1}, w_1^{i1} for $1 \leq i \leq n$. Now we have the edge set as $E(G) = \{u_1v_1, v_1w_1, w_1u_1\} \cup \{u_1^1v_1^1, v_1^1w_1^1, w_1^1u_1^1\} \cup \{u_1u_1^1, u_1^iu_1^{i+1}\} \cup \{u_1^1u_1^{11}, u_1^{i1}u_1^{(i+1)1}\} \cup \{v_1v_1^1, v_1^iv_1^{i+1}\} \cup \{v_1^1v_1^{11}, v_1^{i1}v_1^{(i+1)1}\} \cup \{w_1w_1^1, w_1^iw_1^{i+1}\} \cup \{w_1^1w_1^{11}, w_1^{i1}w_1^{(i+1)1}\} \cup \{w_1^{i+1}u_1^1\}$ for $1 \leq i \leq n-1$. Now let us label the vertices as follows

$f(u_1) = 1$

$f(v_1) = 2$

$f(w_1) = 3$

$f(u_1^i) = 3i + 1$ for $1 \leq i \leq n_1$

$f(v_1^i) = 3i + 2$ for $1 \leq i \leq n_1$

$f(w_1^i) = 3i + 3$ for $1 \leq i \leq n_1$

$f(u_1^1) = (3n_1 + 4)$

$f(v_1^1) = (3n_1 + 5)$

$f(w_1^1) = (3n_1 + 6)$

$f(u_1^{i1}) = 3(n_1 + 2) + i$ for $1 \leq i \leq n_2$

$f(v_1^{i1}) = 3(n_1 + 2) + (i + 1)$ for $1 \leq i \leq n_2$

$f(w_1^{i1}) = 3(n_1 + 2) + (i + 2)$ for $1 \leq i \leq n_2$.

Where n_1, n_2 represents the size of path graph P_n and its join respectively. Now computing the induced edge labels we have

$f^*(u_1v_1) = | f(u_1) - f(v_1) | = 1$

$f^*(v_1w_1) = | f(v_1) - f(w_1) | = 1$

$f^*(u_1w_1) = | f(u_1) - f(w_1) | = 2$

$f^*(u_1u_1^1) = |\ f(u_1) - f(u_1^1)\ | = 3$

$f^*(u_1^iu_1^{i+1}) = |\ f(u_1^i) - f(u_1^{i+1})\ | = 3$ for $1 \le i \le n_1 - 1$

$f^*(v_1v_1^1) = |\ f(v_1) - f(v_1^1)\ | = 3$

$f^*(v_1^iv_1^{i+11}) = |\ f(v_1^i) - f(v_1^{i+1})\ | = 3$ for $1 \le i \le n_1 - 1$

$f^*(w_1w_1^1) = |\ f(w_1) - f(w_1^1)\ | = 3$

$f^*(w_1^iw_1^{i+1}) = |\ f(w_1^i) - f(w_1^{i+1})\ | = 3$ for $1 \le i \le n_1 - 1$

$f^*(u_1^1v_1^1) = |\ f(u_1^1) - f(v_1^1)\ | = 1$

$f^*(v_1^1w_1^1) = |\ f(v_1^1) - f(w_1^1)\ | = 1$

$f^*(u_1^1w_1^1) = |\ f(u_1^1) - f(w_1^1)\ | = 2$

$f^*(u_1^1u_1^{11}) = |\ f(u_1^1) - f(u_1^{11})\ | = 3$

$f^*(u_1^{i1}u_1^{(i+1)1}) = |\ f(u_1^{i1}) - f(u_1^{(i+1)1})\ | = 3$ for$1 \le i \le n_2 - 1$

$f^*(v_1^1v_1^{11}) = |\ f(v_1^1) - f(v_1^{11})\ | = 3$

$f^*(v_1^{i1}v_1^{(i+1)1}) = |\ f(v_1^{i1}) - f(v_1^{(i+1)1})\ | = 3$ for $1 \le i \le n_2 - 1$

$f^*(w_1^1w_1^{11}) = |\ f(w_1^1) - f(w_1^{11})\ | = 3$

$f^*(w_1^{i1}w_1^{(i+1)1}) = |\ f(w_1^{i1}) - f(w_1^{(i+1)1})\ | = 3$ for $1 \le i \le n_2 - 1$

$f^*(w_1^{i+1}u_1^1) = |\ f(w_1^{i+1}) - f(u_1^1)\ | = 1$ for $1 \le i \le n_1 - 1$.

Hence we find that the induced edge labelling satisfy the condition and therefore the 1-Join Armed Crown graph $C_3 \Theta P_n$ is 2-Odd graph.

Theorem 3.12. *The M-Join of Armed Crown graph* $C_3 \Theta P_n$ *is 2-Odd graph.*

Proof. The proof is analogous to the proof given in Theorem 3.9.

Theorem 3.13. *The human chain graph* $HC_{n,3}$ *is* 2-*odd graph.*

Proof. Consider the Human Chain graph $HC_{n,3}$ which is a combination of cycle graph and Y-tree attached to the path graph. Now the vertex set is given as $V(G) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, v_3, \ldots, v_m\} \cup \{w_1, w_2, w_3, \ldots, w_n\}$ where the vertices $\{u_1, u_2, \ldots, u_m\}$ represents the path graph, vertices $\{v_1, v_2, v_3, \ldots, u_n\}$ represents the Cycle graph C_3 and the vertices $\{w_1, w_2, w_3 \ldots, w_n\}$ represents the Y-tree. Now the edges of the Human Chain graph $HC_{n,3}$ is $E(G) = \{u_1u_2, u_2u_3, \ldots, u_{ii+1} : 1 \leq i \leq m-1\} \cup \{u_1v_1, u_1v_2, v_1v_2 \ldots\} \cup \{u_2w_1, w_1w_2, w_1w_3, \ldots,\}$. Now let us label the vertices as follows

$f(u_1) = 6$; $f(u_2) = 3$;

$f(u_3) = 8$; $f(v_1) = 1$; $f(v_2) = 2$; $f(w_1) = 4$; $f(w_2) = 5$; $f(w_3) = 7$

$f(v_{2i+1}) = 7i + 2, 1 \leq i \leq m$; $f(v_{2i}) = 7i + 3, 1 \leq i \leq m$, $f(u_{2i}) = 7i + 3,$

$2 \leq i \leq n$; $f(u_{2i+1}) = 7i, 2 \leq i \leq n$; $f(w_{3i-2}) = 7i - 2, 2 \leq i \leq n$; $f(w_{3i+2}) = 7i + 6, 1 \leq i \leq n$; $f(w_{3i+3}) = 7i + 8, 1 \leq i \leq n$.

Now we can find that the induced edge labelling satisfies the condition and hence the Human chain graph $HC_{n,3}$ is 2-Odd graph.

To Illustrate let us explain through the following example

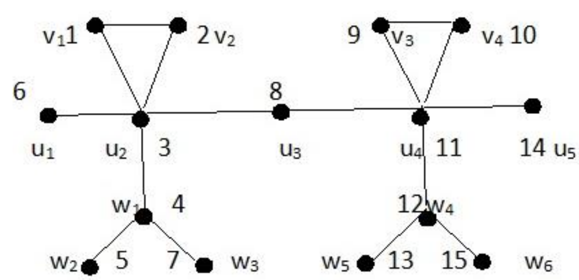

Figure 12. Human Chain graph $HC_{2,3}$.

Theorem 3.14. *The M-join human chain graph* $HC_{2,3}$ *is* 2-*odd graph.*

Proof. Consider *M*-Join Human Chain graph $HC_{2,3}$ by constructing joins from the given Human Chain graph successively and it is possible to label the vertices as given in the Theorem 3.13 and can find that the *M*-Join Human Chain graph $HC_{n,3}$ is 2-Odd graph as we can find that the induced edges are either 2 or Odd integer. Hence the proof.

Observation 1. We have observed from the literature study of paper [4] "Finite prime distance graphs and 2-odd graphs" that for proposition. 8 given showing figure 4 is prime distance graph but has a 2-odd coloring that is not a prime distance graph. Motivated towards that we tried with the all the graphs for which we have labelled the vertices and proved to be 2-Odd graph has the same phenomena or not. In the discussion we found that the 2-coloring does not hold good. Hence we conclude that even though the induced edges are labelled with prime according to coloring schema provided in [4] we understand that they are not prime distance labelling.

Observation 2. We also infer on studying the importance of Human chain graph $HC_{n,3}$ from [7] and found that instead of vertex coloring if we adopt to edge coloring we find that the edge coloring is not possible for Human Chain graph $HC_{n,3}$ as the induced edge labelling are repeated (which is possible according to [4], as the vertex are distinct in case of finite prime distance labelling and 2-Odd graphs whereas edges need not be distinct).

Conclusion

We in this paper have studied on 2-Odd graphs and have further analyzed whether it can be considered to be Finite Prime distance labelling graphs. We further like to investigate on this and like to identify the minimum number of edge coloring that can be used to color the graphs that we have studied.

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