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# GENERALIZED HYERS-ULAM STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN GENERALIZED 2-NORMED SPACES

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**Abstract**

In this paper, we investigate the generalized Hyers-Ulam stability of the quadratic functional equation

$$f(3x+2y+z)+f(3x+2y+z)+f(3x+2y+z)+f(3x-2y-z)$$

$$=8[f(x+y)+f(x-y)]+2[f(x+z)+f(x-z)]+16f(x)$$

in generalized 2-normed spaces.

## 1. Introduction

In 1940, S. M. Ulam [15] had raised the following question: Under what condition does there exist a group homomorphism near an approximate group homomorphism. In next year, D. H. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. In 1950, T. Aoki [3]

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generalized Hyers' theorem for additive mappings. In 1978, Th. M. Rassias [14] succeeded in extending Hyers' theorem by weakening the condition for the Cauchy difference controlled by $\| x \|^p + \| y \|^p$, $0 \leq p < 1$, to be unbounded.

The result of Rassias' theorem had been generalized by P. Gavruta [8] who permitted the Cauchy difference to be bounded by a general control function. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2], [7], [11], [12]) and reference cited there in.

K. Balamurugan et al., [6] introduced the following cubic functional equation

$$g(3x + 2y + z) + g(3x + 2y - z) + g(3x - 2y + z) + g(3x - 2y - z)$$
$$= 24[g(x + y) + g(x - y)] + 6[g(x + z) + g(x - z)] + 48g(x) \tag{1.1}$$

and they investigated the generalized Hyers-Ulam stability for Equation (1.1).

In this paper, we investigated the generalized Hyers-Ulam stability of the quadratic functional equation

$$f(3x + 2y + z) + f(3x + 2y - z) + f(3x - 2y + z) + f(3x - 2y - z)$$
$$= 8[f(x + y) + f(x - y)] + 2[f(x + z) + f(x - z)] + 16f(x) \tag{1.2}$$

in generalized 2-normed spaces. The function $f(x) = \alpha x^2$ is a solution for the functional equation (1.2)

## 2. Basic Definitions on Generalized 2-Normed Space

In this section, we demonstrated the generalized Ulam-Hyers stability of the functional equation (1.2) in generalized 2-normed space.

Now, we present some basic definitions related to generalized 2-normed spaces.

**Definition 2.1.** Let $X$ be a linear space. A function $N(.,.) : X \times X \to [0, \infty)$ is called a generalized 2-norm on $X$ if it satisfies the following

(G1) $N(x, y) = 0$ if and only if $x$ and $y$ are linearly dependent.

(G2) $N(x, y) = N(y, x)$ for all $x, y \in X$,

(G3) $N(\lambda x, y) = |\lambda| N(y, x)$ for all $x, y \in X$ and $\lambda \in \varphi$, $\varphi$ is a real or complex field,

(G4) $N(x + y, z) \leq N(x, z) + N(y, z)$ for all $x, y, z \in X$.

Then $(X, N(., .))$ is called generalized 2-normed space.

**Definition 2.2.** A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(., .))$ is called convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, y) = 0$, that is, $\lim_{n\to\infty} N(x_n, y) = N(x, y)$ for all $y \in X$.

**Definition 2.3.** A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(., .))$ is said to be a Cauchy sequence if there exist two linearly independent elements $y$ and $z$ in $X$ such that $\{N(x_n, y)\}$ and $\{N(x_n, y)\}$ are real Cauchy sequences.

**Definition 2.4.** A generalized 2-normed space $(X, N(., .))$ is said to be generalized 2-Banach space if every Cauchy sequence in $X$ is convergent in $X$.

## 3. Generalized Hyers-Ulam Stability of (1.2)

In this section, we investigate the generalized Hyers-Ulam stability of the functional equation (1.2) in generalized 2-normed spaces. Let $X$ be a generalized 2-normed space and $Y$ be generalized 2-Banach space. Define a mapping $f : X^3 \to Y$ by

$$Df(x, y, z) = f(3x + 2y + z) + f(3x + 2y - z) + f(3x - 2y + z)$$

$$+ f(3x - 2y - z) - 8[f(x + y) + f(x - y)] - 2[f(x + z) + f(x - z)] - 16f(x)$$

for all $x, y, z \in X$. Also, throughout this paper, we adopt the following convention $\psi(x, y, z) = \psi((x, s), (y, s), (z, s))$ and $\| x \| = \| x, s \|$ for all $x, y, z, s \in X$.

**Theorem 3.1.** *Let* $j = \pm 1$ *and* $\psi : X^3 \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{\psi(3^{nj}x, 3^{nj}y, 3^{nj}z)}{9^{nj}} = 0 \tag{3.1}$$

*for all* $\xi, \psi, z \in X$. *Let* $g : X \to Y$ *be a mapping satisfying the inequality*

$$N(Dg(x, y, z), s) \le \psi(x, y, z) \ \forall\, x, y, z, s \in X. \tag{3.2}$$

*Then there exists a unique quadratic mapping* $Q : X \to Y$ *which satisfies* (1.2) *and*

$$N(g(x) - Q(x), s) \le \frac{1}{36} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\psi(3^{kj}x, 0, 0)}{9^{kj}} \tag{3.3}$$

*where* $N(Q(x), s)$ *is defined by*

$$N(Q(x), s) = \lim_{n\to\infty} N\left(\frac{g(3^{nj}x)}{9^{nj}}, s\right) \forall\, x, s \in X. \tag{3.4}$$

**Proof.** Replacing $(x, y, z)$ by $(x, 0, 0)$ in (3.2), we get

$$N(g(3x) - 9g(x), s) \le \frac{\psi(x, 0, 0)}{4} \tag{3.5}$$

for all $x \in X$. Dividing the above inequality by 9, we obtain

$$N\left(\frac{g(3x)}{9} - g(x), s\right) \le \frac{\psi(x, 0, 0)}{36} \tag{3.6}$$

for all $x \in X$. Now replacing $x$ by $3x$ and dividing by 9 in (3.6), we get

$$N\left(\frac{g(3^2x)}{9^2} - \frac{g(3x)}{9}, s\right) \le \frac{\psi(x, 0, 0)}{36 \cdot 9} \tag{3.7}$$

for all $x \in X$. From (3.6) and (3.7), we obtain

$$N\left(\frac{g(3^2x)}{9^2} - g(x), s\right) \le N\left(\frac{g(3x)}{9} - g(x), s\right) + N\left(\frac{g(3^2x)}{9^2} - \frac{g(3x)}{9}, s\right)$$

$$\leq \frac{1}{36}\left[\psi(x, 0, 0) + \frac{\psi(3x, 0, 0)}{9}\right] \tag{3.8}$$

for all $x \in X$. Proceeding further and using induction on a positive integer $n$, we get

$$N\left(\frac{g(3^n x)}{9^n} - g(x), s\right) \leq \frac{1}{36}\sum_{k=0}^{n-1}\frac{\psi(3^k x)}{9^k}$$

$$\leq \frac{1}{36}\sum_{k=0}^{\infty}\frac{\psi(3^k x)}{9^k} \tag{3.9}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{g(3^n x)}{9^n}\right\}$, replace $x$ by $3^m x$ and dividing by $9^m$ in (3.9), for any $m, n > 0$, we deduce

$$N\left(\frac{g(3^{n+m}x)}{9^{(n+m)}} - \frac{9(3^m x)}{9^m}, s\right) = \frac{1}{9^m} N\left(\frac{9(3^n \cdot 3^m x)}{9^n} - g(3^m x), s\right)$$

$$\leq \frac{1}{36}\sum_{k=0}^{n-1}\frac{\psi(3^{k+m}x, 0, 0)}{9^{(k+m)}}$$

$$\leq \frac{1}{36}\sum_{k=0}^{\infty}\frac{\psi(3^{k+m}x, 0, 0)}{9^{(k+m)}}$$

for all $x \in X$ and all $s \in X$. Also

$$N\left(\frac{g(3^{n+m}x)}{9^{(n+m)}} - \frac{9(3^m x)}{9^m}, t\right) = \frac{1}{9^m} N\left(\frac{9(3^n \cdot 3^m x)}{9^n} - g(3^m x), t\right)$$

$$\leq \frac{1}{36}\sum_{k=0}^{n-1}\frac{\psi(3^{k+m}x, 0, 0)}{9^{(k+m)}}$$

$$\leq \frac{1}{36}\sum_{k=0}^{\infty}\frac{\psi(3^{k+m}x, 0, 0)}{9^{(k+m)}}$$

$$\to 0 \text{ as } m \to \infty$$

for all $x \in X$ and all $t \in X$. Hence there exist two linearly independent elements $s$ and $t$ in $X$ such that $\left\{N\left(\frac{g(3^n x)}{9^n}, s\right)\right\}$ and $\left\{N\left(\frac{g(3^n x)}{9^n}, t\right)\right\}$ are real Cauchy sequences. Thus the sequence $\left\{\frac{g(3^n x)}{9^n}\right\}$ is a Cauchy sequence. Since $Y$ is complete, there exists a mapping $Q : X \to Y$ such that

$$N(C(x), s) = \lim_{n\to\infty} N\left(\frac{g(3^n x)}{9^n}, s\right), \forall\ x, s \in X.$$

Letting $n \to \infty$ in (3.9), we see that (3.3) holds for all $x \in X$. To prove that $Q$ satisfies (1.2), replacing $(x, y, z)$ by $(3^n x, 3^n y, 3^n z)$ and dividing by $9^n$ in (3.2)), we obtain

$$\frac{1}{9^n} N(g(3^n(3x + 2y + z))) + g(3^n(3x + 2y - z)) + g(3^n(3x - 2y + z))$$

$$+ g(3^n(3x - 2y - z)) - 8[g(3^n(x + y)) + g(3^n(x - y))]$$

$$- 2[g(3^n(x + z)) + g(3^n(x - z))] - 16g(3^n x), s) \leq \frac{1}{9^n} \psi(3^n x, 3^n y, 3^n z)$$

for all $x, y, z \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that $Q$ satisfies (1.2) for all $x, y, z \in X$. To prove that $Q$ is unique, let $R(x)$ be another quadratic mapping satisfying (1.2) and (3.3), then

$$N(Q(x) - R(x), s) = \frac{1}{9^n} N(Q(3^n x), s)$$

$$\leq \frac{1}{9^n} \{N(Q(3^n x) - g(3^n x), s) + N(g(3^n x) - R(3^n x), s\}$$

$$\leq \frac{2}{36} \sum_{k=0}^{\infty} \frac{\psi(3^{k+n} x, 0, 0)}{9^{(k+n)}}$$

$$\to 0 \text{ as } n \to \infty$$

for all $x \in X$. Thus $Q$ is unique. Hence for $j = 1$ the theorem holds.

Now, replacing $x$ by $\frac{x}{3}$ in (3.5), we reach

$$N\left(g(x) - 9g\left(\frac{x}{3}\right), s\right) \leq \frac{1}{4}\psi\left(\frac{x}{3}, 0, 0\right) \tag{3.10}$$

for all $x \in X$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem holds. This completes the proof of the theorem. □

**Corollary 3.2.** *Let* ρ *and* $s$ *be nonnegative real numbers. Let* $g : X \to Y$ *be a mapping satisfying the inequality*

$$N(Dg(x, y, z), s) \leq \begin{cases} \rho, \\ \rho\{\| x \|^{\mu} + \| y \|^{\mu} + \| z \|^{\mu}\}, & \mu \neq 2; \\ \rho\{\| x \|^{\mu}\| y \|^{\mu}\| z \|^{\mu} \\ + \{\| x \|^{3\mu} + \| y \|^{3\mu} + \| z \|^{3\mu}\}\}, & 3\mu \neq 2; \end{cases} \tag{3.11}$$

*for all* $x, y, z \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$N(g(x) - C(x), s) \leq \begin{cases} \dfrac{\rho}{4| 3^2 - 1 |}, \\ \dfrac{\rho\| x \|^{\mu}}{4| 3^2 - 3^{\mu} |}, \\ \dfrac{\rho\| x \|^{3\mu}}{4| 3^2 - 3^{3\mu} |}, \end{cases} \tag{3.12}$$

*for all* $x \in X$.

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