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SOME FIXED POINT THEOREMS ON COMPLETE *G*-METRIC SPACES WITH BINARY OPERATOR

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Abstract

Using the notion of G-metric space and property of new binary operator we prove some fixed point theorem which unifies and improve some fixed point results.

1. Introduction

The concept of standard metric spaces is a fundamental tool in topology, functional analysis, and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric spaces. In recent years, several generalizations of standard metric spaces have appeared.

The notion of 2-metric space introduced by Gahler [5, 6] as a generalization of usual notion of metric space (X, d). In 1992, Dhage in his Ph.D. thesis introduce a new class of generalized metric space called D-metric spaces ([1, 2]). In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces [2-4]. He claimed that D-metrics provide a generalization of ordinary metric functions. In 2003 Brailey Sims et al. demonstrated in [8] that most of the claims concerning the fundamental topological structure of D-metric space are incorrect, so they introduced notion of generalized metric space [10].

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Definition 1.1[9]**.** Let X be a nonempty set, and let $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2[9]**.** Let (X, G) be a G-metric space, and let (x_n) be sequence of points of X, a point $x \in X$ is said to be the limit of the sequence (x_n), if $\lim_{n, m \to \infty} G(x, x_n, x_m) = 0$, and one says that the sequence (x_n) is G-convergent to x.

Thus, that if $x_n \to x$ in a G-metric space (X, G), then for any $c > 0$, there exists $N \in \mathbf{N}$ such that $G(x, x_n, x_m) < c$, for all $n, m \geq N$.

Proposition 1.1 [9]**.** *Let (X, G) be a G-metric space, then the following are equivalent.*

(1) (x_n) *is G-convergent to x.*

(2) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$.

(3) $G(x_n, x, x) \to 0$, *as* $n \to \infty$.

(4) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Definition 1.3[9]**.** Let (X, G) be a G-metric space, a sequence (x_n) is called G-Cauchy if for every $c > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < c$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.2[8]**.** *If* (X, G) *is a G-metric space, then the following are equivalent.*

(1) *The sequence* (x_n) *is G-Cauchy.*

(2) *For every* $c > 0$, *there exists* $N \in \mathbf{N}$ *such that* $G(x_n, x_m, x_m) < c$, *for all* $n, m \geq N$.

Definition 1.4[9]**.** Let (X, G) and (X^{r}, G^{r}) be two G-metric spaces, and let $f : (X, G) \to (X^{r}, G^{r})$ be a function, then f is said to be G-continuous at a point $a \in X$ if and only if, given $c > 0$, there exists $\delta > 0$ such that $x, y \in X$, and $G(a, x, y) < \delta$ implies $G^{r}(f(a), f(x), f(y)) < c$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.3 [9]**.** *Let* $(X, G), (X^{r}, G^{r})$ *be two G-metric spaces. Then a function* $f : X \to X^{r}$ *is G-continuous at a point* $x \in X$ *if and only if it is G sequentially continuous at* x, *that is, whenever* (x_n) *is G-convergent to* x, $(f(x_n))$ *is G-convergent to* $f(x)$.

Definition 1.5[9]**.** A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.4[9]**.** *Let* (X, G) *be a G-metric space, then the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Proposition 1.5[8]**.** *Every G-metric space* (X, G) *will define a metric space* (X, d_G) *by*

$$d_G(x, y) = G(X, y, y) + G(y, x, x), \forall x, y \in X. \tag{1.1}$$

Note that if (X, G) *is a symmetric G-metric space, then*

$$d_G(x, y) = 2G(x, y, y), \forall x, y \in X. \tag{1.2}$$

However, if (X, G) *is not symmetric, then it holds by the G-metric properties that*

$$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \forall x, y \in X, \tag{1.3}$$

and that in general these inequalities cannot be improved.

Definition 1.6[9]**.** A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.6[9]**.** *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Definition 1.7[7]**.** In what follows, $\mathbf{N}$ is the set of all natural numbers and $\mathbf{R}^+$ is the set of all positive real numbers.

Let $\lozenge : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ be a binary operator satisfying the following conditions:

(1) $\lozenge$ is associative and commutative;

(2) $\lozenge$ is continuous.

Example 1.1.

$$a \lozenge b = \max\{a, b\}; a \lozenge b = a + b; a \lozenge b = ab + a + b; a \lozenge b = a/b (b \neq 0);$$

and $a \lozenge b = \dfrac{ab}{\max\{a, b, 1\}}$ for each $a, b \in \mathbf{R}^+$.

Definition 1.8[7]**.** The binary operation is said to satisfy α-property if there exists a positive real number α such that

$$a \lozenge b \leq \alpha \max\{a, b\}.$$

2. Main Results

Theorem 2.1. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions*:

$$G(T(x), T(y), T(z)) \leq k_1 \{G(x, T(x), T(x)) \lozenge G(y, T(y), T(y)) \lozenge G(z, T(z), T(z))\} \tag{2.1}$$

or

$$G(T(x), T(y), T(z)) \leq k_1\{G(x, x, T(x)) \lozenge G(y, y, T(y)) \lozenge G(z, z, T(z))\} \quad (2.2)$$

for all $x, y, z \in X$, *where* $0 \leq k_1 < 1$ *and* a *is a small positive real number such that* $k = ak_1$ *where* $0 \leq k < 1$. *Then* T *has a unique fixed point* (*say* u), *and* T *is* G*-continuous at* u.

Proof. Suppose that T satisfies condition (2.1), then for all $x, y \in X$,

$$G(Tx, Ty, Ty) \leq k_1\{G(x, Tx, Tx) \lozenge G(y, Ty, Ty)\},$$

$$G(Tx, Ty, Ty) \leq k \max\{G(x, Tx, Tx), G(y, Ty, Ty)\},$$

$$G(Ty, Tx, Tx) \leq k_1\{G(y, Ty, Ty) \lozenge G(x, Tx, Tx)\}.$$

$$G(Ty, Tx, Tx) \leq k \max\{G(y, Ty, Ty) \lozenge G(x, Tx, Tx)\}. \quad (2.3)$$

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and definition (1.2) we get

$$d_G(Tx, Ty) \leq k_1\{d_G(x, Tx) \lozenge d_G(y, Ty)\}, \forall x, y \in X.$$

$$d_G(Tx, Ty) \leq k \max\{d_G(x, Tx) \lozenge d_G(y, Ty)\}, \forall x, y \in X. \quad (2.4)$$

Since $k < 1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X, d_G).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and Proposition 1.1, we get

$$d_G(Tx, Ty) \frac{4k}{3} \max\{d_G(x, Tx), d_G(y, Ty)\}, \forall x, y \in X. \quad (2.5)$$

The metric condition gives no information about this map since $4k/3$ need not be less than 1, but we will proof it by G-metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$. By (2.1), we can verify that

$$G(x_n, x_{n+1}, x_{n+1}) \leq k_1\{G(x_{n-1}, x_n, x_n) \lozenge G(x_n, x_{n+1}, x_{n+1})\}$$

$$\leq k \max\{G(x_{n+1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \tag{2.6}$$

$$= k\,G(x_{n-1}, x_n, x_n) \text{ (since } 0 \leq k < 1).$$

Continuing in the same argument, we will find

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1). \tag{2.7}$$

For all $n, m \in \mathbf{N}; n < m$, we have by rectangle inequality that

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})$$

$$+ G(x_{n+2}, x_{n+3}, x_{n+3}) + G(x_{m-1}, x_m, x_m)$$

$$\leq (k^n + k^{n+1} + \ldots + k^{m-1}) G(x_0, x_1, x_1)$$

$$\leq \frac{k^n}{1-k} G(x_0, x_1, x_1).$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, and thus (x_n) is G-Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$ such that $(x_n) \to u$.

Suppose that $T(u) \neq u$, then $G(x_{n+1}, T(u), T(u)) \leq k \max\{G(x_{n+1}, x_{n+2}, x_{n+2}), G(u, T(u), T(u))\}$ and by taking the limit as $n \to \infty$, and using the fact that the function G is continuous, we get that $G(u, T(u), T(u)) \leq k\,G(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that $u \neq v$ such that $T(v) = v$, then $G(u, v, v) \leq k \max\{G(v, v, v), G(u, u, u)\} = 0$ which implies that $u = v$.

To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim (y_n) = u$, then

$$G(u, T(y_n), T(y_n)) \leq k_1\{G(u, T(u), T(u)) \lozenge G(y_n, T(y_n), T(y_n))\}$$

$$= k\,G(y_n, T(y_n), T(y_n)). \tag{2.9}$$

But, $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, the N

$$G(u, T(y_n), T(y_n)) \leq (k/(1-k))G(y_n, u, u).$$

Taking the limit as $n \to \infty$, from which we see that $G(u, T(y_n), T(y_n)) \to 0$, and so by Proposition 1.3, $T(y_n) \to u = Tu$. So, T is G-continuous at u.

Corollary 2.1. *Let* (X, G) *be a complete* G*-metric space and let* $T : X \to X$ *be a mapping satisfying one of the following conditions for some* $m \in N$.

$$G(T^m(x), T^m(y), T^m(z))$$

$$\leq k_1\{G(x, T^m(x), T^m(x)) \lozenge G(y, T^m(y), T^m(y)) \lozenge G(z, T^m(z), T^m(z))\} \quad (2.10)$$

or

$$G(T^m(x), T^m(y), T^m(z)) \leq k_1\{G(x, x, T^m(x)) \lozenge G(y, y, T^m(y)) \lozenge G(z, z, T^m(z))\} \quad (2.11)$$

for all $x, y, z \in X$, *then* T *has a unique fixed point* (*say* u) *and* T^m *is* G*-continuous at* u.

Proof. We can see that T^m has a unique fixed point (say u), that is, $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$.

Theorem 2.2. *Let* (X, G) *be a complete* G*-metric space*, *and let* $T : X \to X$ *be a mapping satisfying one of the following conditions*:

$$G(T(x), T(y), T(y)) \leq k_1\{G(x, T(y), T(y)) \lozenge G(y, T(x), T(x)) \lozenge G(y, T(y), T(y))\} \quad (2.12)$$

or

$$G(T(x), T(y), T(y)) \leq k_1\{G(x, x, T(y)) \lozenge G(y, y, T(x)) \lozenge G(y, y, T(y))\}, \quad (2.13)$$

for all $x, y \in X$, *where* $k \in [0, 1)$. *Then* T *has a unique fixed point* (*say* u), *and* T *is* G*-continuous at* u.

Proof. Suppose that T satisfies condition (2.11), then for all $x, y \in X$,

$$G(Tx, Ty, Ty) \leq k_1\{G(x, Ty, Ty) \Diamond G(y, Tx, Tx) \Diamond G(y, Ty, Ty)\},$$

$$G(Ty, Tx, Tx) \leq k \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(x, Tx, Tx)\}.$$

$$G(Ty, Tx, Tx) \leq k_1\{G(x, Ty, Ty) \Diamond G(y, Tx, Tx) \Diamond G(x, Tx, Tx)\},$$

$$G(Ty, Tx, Tx) \leq k \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(x, Tx, Tx)\}. \tag{2.14}$$

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and definition (1.2), we have

$$d_G(Tx, Ty) \leq \frac{k_1}{2}\{d_G(x, Ty) \Diamond d_G(y, Tx) \Diamond d_G(y, Ty)\}$$

$$+ \frac{k_1}{2}\{d_G(x, Ty) \Diamond d_G(y, Tx) \Diamond d_G(x, Tx)\}$$

$$\leq k \max\{d_G(x, Ty), d_G(y, Tx), d_G(x, Tx), d_G(y, Ty)\}, \forall x, y \in X. \tag{2.15}$$

Since $0 \leq k < 1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X, d_G).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and Proposition 1.1, we have

$$d_G(Tx, Ty) \leq \frac{2k_1}{3}\{d_G(x, Ty) \Diamond d_G(y, Tx) \Diamond d_G(y, Ty)\}$$

$$+ \frac{2k_1}{3}\{d_G(x, Ty) \Diamond d_G(y, Tx) \Diamond d_G(x, Tx)\}$$

$$d_G(Tx, Ty) \leq \frac{2k \max}{3}\{d_G(x, Ty), d_G(y, Tx), d_G(y, Ty)\}$$

$$+ \frac{2k \max}{3}\{d_G(x, Ty), d_G(y, Tx), d_G(x, Tx)\} \tag{2.16}$$

for all $x, y \in X$, then the metric space (X, d_G) gives no information about this map since $4k/3$ need not be less than 1. But we will proof it by G-metric.

Let $x_0 \in X$ be arbitrary point, and define the sequence (x_n) by

$x_n = T_n(x_0)$, then by (2.12) and using $k < 1$, we deduce that

$$G(x_n, x_{n+1}, x_{n+1}) \leq k_1\{G(x_{n-1}, x_{n+1}, x_{n+1}) \lozenge G(x_n, x_{n+1}, x_{n+1})\}$$

$$= k\,G(x_{n-1}, x_{n+1}, x_{n+1}). \tag{2.17}$$

So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq k\,G(x_{n-1}, x_{n+1}, x_{n+1}), \tag{2.18}$$

and using

$$G(x_n, x_{n+1}, x_{n+1})$$

$$\leq k \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n+1}, x_{n+1})\}, \tag{2.19}$$

then,

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^2 \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1})\}. \tag{2.20}$$

Continuing in this procedure, we will have

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n \Gamma n, \tag{2.21}$$

where $\Gamma n = \max\{G(x_i, x_j, x_j);$ for all $i, j \in \{0, 1, \ldots, n+1\}\}$.

For $n, m \in \mathbf{N}; n < m$, let $\Gamma = \max\{\Gamma_i,$ for all $i = n, \ldots, m-1\}$.

Then, for all $n, m \in \mathbf{N}; n < m$, we have by rectangle inequality

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})$$

$$+ G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + G(x_{m-1}, x_m, x_m)$$

$$\leq k^n \Gamma_n + k^{n+1}\Gamma_{n+1} + \ldots + k^{n-1}\Gamma_{m-1}$$

$$\leq (k^n + k^{n+1} + \ldots + k^{m-1})$$

$$\Gamma_{m-1}$$

$$\leq \frac{k^n}{1-k}\Gamma_{m-1} \tag{2.22}$$

This prove that $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, and thus (x_n) is G-

Cauchy sequence. Since (X, G) is G-complete then there exists $u \in X$ such that (x_n) is G-converge to u.

Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \leq k \max\{G(x_{n-1}, T(u), T(u)), G(u, x_{n+1}, x_{n+1}), G(u, T(u), T(u))\}. \tag{2.23}$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous, we get $G(u, T(u), T(u)) \leq k G(u, T(u), T(u))$, this contradiction implies that $u = T(u)$.

To prove the uniqueness, suppose that $u \neq v$ such that $T(v) = v$. So, by (2.12), we have that

$$G(u, v, v) \leq k_1\{G(u, v, v) \Diamond G(v, u, u)\}$$

$$G(u, v, v) \leq k \max\{G(u, v, v), G(v, u, u)\}$$

$$\Rightarrow G(u, v, v) \leq k G(v, u, u). \tag{2.24}$$

Again we will find $G(v, u, u) \leq k G(u, v, v)$, so

$$G(u, v, v) \leq k_2 G(u, v, v), \tag{2.25}$$

since $k < 1$, this implies that $u = v$.

To show that T is G-continuous at u, let (y_n) X be a sequence such that, then

$$G(u, T(y_n), T(y_n)) \leq k_1\{G(u, T(y_n), T(y_n)) \Diamond G(y_n, T(u), T(u)) \Diamond G(y_n, T(y_n), T(y_n))\},$$

$$G(u, T(y_n), T(y_n)) \leq k \max\{G(u, T(y_n), T(y_n)) \Diamond G(y_n, T(u), T(u)) \Diamond G(y_n, T(y_n), T(y_n))\}. \tag{2.26}$$

But, $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, so, $G(u, T(y_n), T(y_n)) \leq (k/(1-k))G(y_n, u, u)$.

Taking the limit as $n \to \infty$, from which we see that $G(u, T(y_n), T(y_n)) \to 0$ and so, by Proposition 1.3, we have $T(y_n) \to u = Tu$ which implies that T is G-continuous at u.

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