Advances and Applications in Mathematical Sciences
Volume 20, Issue 9, July 2021, Pages 1995-2013

LIGHTLIKE HYPERSURFACES OF AN INDEFINITE STATISTICAL MANIFOLD

VANDANA RANI and JASLEEN KAUR

Department of Mathematics
Punjabi University Patiala
Punjab 147001, India
E-mail: vgupta.87@rediffmail.com
jasleen_2311@yahoo.co.in

Abstract

In this paper, various properties of the killing and geodesic lightlike hypersurfaces of an indefinite statistical manifold have been developed. The characterizations of parallel vector fields and the screen distributions with respect to the dual connections in lightlike hypersurfaces have been obtained. The Lie-derivatives of the induced metric and second fundamental form have been established. Also, some formulae for the curvature tensors of the lightlike hypersurfaces of the indefinite statistical manifold have been developed and further used to prove assertions related to parallelism of lightlike transversal vector bundle and geodesicity of the lightlike hypersurface.

1. Introduction

Lightlike hypersurfaces, being models of certain horizons like killing, dynamical and conformal ones studied in general relativity, have gained considerable importance nowadays. Also, the classical theory of hypersurfaces is not applicable, in the usual way, to define any induced object on a lightlike hypersurface. So, the theory of degenerate submain folds of semi-Riemannian manifolds and subsequently the lightlike geometry of hypersurfaces has become an important area of research in differential geometry having applications in various branches of mathematics and physics.

2010 Mathematics Subject Classification: 53C15, 53C25, 53C40.
Keywords: Indefinite statistical manifold, Lightlike hypersurface, Hypersurface, Killing distribution, Dual connection, Totally geodesic hypersurface.
Received July 30, 2020; Accepted April 23, 2021

Lightlike hypersurfaces in different spaces endowed with degenerate (indefinite) metric have been investigated by many geometers such as [6], [7], [8], [13], [17], [21], [22], and [15]. For the statistical manifolds, which have developed from the investigation of geometric structures on sets of certain probability distributions, the theory of hypersurfces was introduced by [9] and later developed in [10]. These manifolds have been investigated extensively by [19], Amari et al. [1], [2], [10], [11], [16], [20] and are currently being explored in the realm of degenerate metric.

Recently, the lightlike geometry in the sphere of statistical manifolds has been developed by [12], [14], [5], [18]. The concept of lightlike hypersurfaces of statistical manifolds has been introduced by [4] et al. and various conditions related to dual connections and characterizations of the screen distribution have been derived. [3] has also introduced significant results on hypersurfaces of an indefinite Sasakian statistical manifold. Since the literature related to the lightlike hypersurfaces of an indefinite statistical manifold available so far is very limited, it has motivated us to continue the study of geometry of these hypersurfaces.

In this paper, the characterizations of parallel vector fields and the screen distributions with respect to the dual connections in a lightlike hypersurface of an indefinite statistical manifold have been derived. Certain properties related to the killing and geodesic lightlike hypersurfaces and Lie-derivatives of the second fundamental form have been obtained. Various formulae for the curvature tensors of the lightlike hypersurfaces of the indefinite statistical manifold instrumental in developing results concerning parallel second fundamental form of lightlike transversal vector bundle have also been derived.

2. Lightlike Hypersurface

Let $(\overline{M}, \overline{g})$ be an $(m+2)$ dimensional semi-Riemannian manifold of constant index $q \geq 1$. Let (M, g) be a lightlike hypersurface of $(\overline{M}, \overline{g})$ with $g = \overline{g} \mid M$. If the induced metric g on M is degenerate, then M is called a lightlike or degenerate hypersurface of $\overline{M}$. There exists a vector field $\xi \neq 0$ on M such that $g_x(\xi, X) = 0$ for all $X \in \Gamma(TM)$.

The null space or radical space of $T_x(M)$ at each point $x \in M$ is a subspace Rad $T_x(M)$ defined as

Rad $T_x(M) = \{\xi \in T_x(M) : g_x(\xi, X) = 0$ for all $X \in \Gamma(TM)\}$ whose dimension is called the nullity degree of g.

Since g is degenerate and any null vector is perpendicular to itself, therefore $T_x M^{\perp}$ is also null and

$$Rad\, T_x M = T_x M \cap T_x M^{\perp}.$$

For a hypersurface M, dimension of $T_x M^{\perp}$ equals 1 which implies that the dimension of $Rad\, T_x M$ is also 1 and $Rad\, T_x M = T_x M^{\perp}$. Here $Rad\, TM$ is called a radical distribution of M.

Now consider $S(TM)$, known as screen distribution, as a complementary vector bundle of $Rad\,(TM)$ in TM. i.e.

$$TM = Rad\, TM \perp S(TM). \tag{2.1}$$

It follows that $S(TM)$ is a non-degenerate distribution. Thus, we have

$$TM_{|M} = S(TM) \perp S(TM)^{\perp},$$

where $S(TM)^{\perp}$, known as screen transversal vector bundle, is the orthogonal complement to $S(TM)$ in $TM_{|M}$.

Theorem 2.1 [4]. *Let (M, g) be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr\,(TM)$ known as lightlike transversal vector bundle of rank 1 over M, such that for any non-zero local normal section ξ of $Rad\,(TM)$ there exist a unique section N of $tr(TM)$ satisfying*

$$\overline{g}(N, \xi) = 1 \tag{2.2}$$

$$\overline{g}(N, N) = 0,\ \overline{g}(N, W) = 0 \forall W \in \Gamma(S(TM)).$$

Then the tangent bundle $T\overline{M}$ of $\overline{M}$ is decomposed as follows:

$$T\overline{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM).$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, from the above theory, we have the Gauss and Weingarten formulae as follows:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^t_X N \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y, A_N X \in \Gamma(TM)$ and $h(X, Y), \nabla^t_X N \in \Gamma(tr(TM))$; ∇ is a torsion free linear connection on M; h is a symmetric bilinear form on $\Gamma(TM)$; ∇ and ∇^t are the induced connections on M and $tr(TM)$ respectively.

Now, define

$$B(X, Y) = \overline{g}(h(X, Y), \xi), \ \tau(X) = \overline{g}(\nabla^t_X N, \xi).$$

It follows that

$$h(X, Y) = B(X, Y)N, \ \nabla^t_X N = \tau(X)N.$$

Then (2.4) and (2.5) become

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N$$

respectively. Here, we call B and A the second fundamental form and the shape operator respectively of the lightlike hypersurface M. Let P be the projection morphism of TM on $S(TM)$, then from equation (2.1), the Gauss and Weingarten equations for the screen distribution are

$$\nabla_X PY = \nabla'_X PY + h'(X, PY) \ \forall X, Y \in \Gamma(TM) \tag{2.6}$$

and

$$\nabla_X \xi = -A'_\xi X + \nabla'^t_X \xi \ \forall X \in \Gamma(TM), \ \xi \in \Gamma(TM^{\perp}) \tag{2.7}$$

where $\nabla'_X PY$, $A'_\xi X \in \Gamma(S(TM))$ and $h'(X, PY)$, $\nabla'^t_X \xi \in \Gamma(TM^\perp)$; ∇' and ∇'^t are linear connections on $S(TM)$ and $TM^\perp$ respectively. h' and A' are respectively the second fundamental form and the shape operator of $S(TM)$. From equations (2.3), (2.4), (2.6) and (2.7), we have

$$g(A_N Y, PW) = \bar{g}(N, h'(Y, PW)), \bar{g}(A_N Y, N) = 0 \tag{2.8}$$

$$g(A'_\xi X, PY) = \bar{g}(\xi, h(X, PY)), \bar{g}(A'_\xi X, N) = 0 \tag{2.9}$$

for any $X, Y, W \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$. Then, we define

$$C(X, PY) = \bar{g}(h'(X, PY), N), \epsilon(X) = \bar{g}(\nabla'^t_X \xi, N).$$

Hence,

$$h'(X, PY) = C(X, PY)\xi, \Delta'^t_X \xi = \epsilon(X)\xi$$

$$\nabla_X PY = \nabla'_X PY + C(X, PY)\xi$$

$$\nabla_X \xi = -A'_\xi X - \tau(X)\xi,$$

where $\epsilon(X) = -\tau(X)$. Now from equations (2.8), (2.9) we get

$$g(A_N Y, PW) = C(Y, PW), \bar{g}(A_N Y, N) = 0$$

$$g(A'_\xi X, PY) = B(X, PY), \bar{g}(A'_\xi X, N) = 0.$$

We also have

$$B(X, \xi) = 0, A^*_\xi \xi = 0.$$

Let P be the projection of $S(TM)$ on M. Then for any $X \in \Gamma(TM)$, then we have

$$X = PX + \eta(X)\xi,$$

where η is a 1-form given by

$$\eta(X) = \bar{g}(X, N).$$

As $\overline{\nabla}$ is a metric connection and using equation (2.6), we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y)$$

for all $X, Y, Z \in \Gamma(TM)$, where the induced connection ∇ is a non-metric connection on M.

3. Lightlike Hypersurfaces in the Context of an Indefinite Statistical Manifold

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold, $\overline{D}$ be an affine connection on $\overline{M}$ and $\overline{g}$ be a semi-Riemannian metric on $\overline{M}$. A semi-Riemannian manifold $(\overline{M}, \overline{g})$ which admits an affine connection $\overline{D}$ such that for all $X, Y, Z \in \Gamma(T, \overline{M})$

(i) $\overline{D}_X Y - \overline{D}_Y X = [X, Z]$;

(ii) $(\overline{D}_X \overline{g})(Y, Z) = (\overline{D}_Y \overline{g})(X, Z)$ hold,

is said to be indefinite statistical manifold and is denoted by $(\overline{M}, \overline{g}, \overline{D})$.

Moreover, there exists $\overline{D}^*$ which is a dual connection of $\overline{D}$ with respect to $\overline{g}$, satisfying

$$X\overline{g}(Y, Z) = \overline{g}(\overline{D}_X Y, Z) + \overline{g}(Y, \overline{D}^*_X Z)\ X, Y, Z \in \Gamma(T\overline{M}). \tag{3.1}$$

If $(\overline{M}, \overline{g}, \overline{D})$ is an indefinite statistical manifold, then so is $(\overline{M}, \overline{g}, \overline{D}^*)$. Hence the indefinite statistical manifold is denoted by $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$.

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \overline{g})$. Then Gauss and Weingarten formulae with respect to dual connections as given by [20], [9], [4] are as follows:

$$\overline{D}_X Y = D_X Y + h(X, Y),\ \overline{D}_X N = -A^*_N X + D^{\perp}_X N,$$

$$\overline{D}^*_X Y = D^*_X Y + h^*(X, Y),\ \overline{D}^*_X N = -A_N X + D^{\perp *}_X N,$$

for $X, Y \in \Gamma(TM)$, $N \in \Gamma(T^{\perp}M)$, where $D_X Y, D^*_X Y, A_N X, A^*_N X \in \Gamma(TM)$ and $h(X, Y), h^*(X, Y), D^{\perp}_X N, D^{\perp *}_X N \in \Gamma(tr(TM))$. Here D, D^* are called induced connections on M and A_N, A_N^* are called shape operator with respect to $\overline{D}$ and $\overline{D}^*$ respectively. Also, we denote by B and B^*, the second fundamental forms with respect to $\overline{D}$ and $\overline{D}^*$.

Now the following holds:

$$B(X, Y) = \overline{g}(\overline{D}_X Y, \xi), \tau^*(X) = \overline{g}(\overline{D}_X N, \xi)$$

$$B^*(X, Y) = \overline{g}(\overline{D}_X^* Y, \xi), \tau(X) = \overline{g}(\overline{D}_X^* N, \xi).$$

It follows that

$$h(X, Y) = B(X, Y)N, h^*(X, Y) = B^*(X, Y)N$$

$$D_X^{\perp} N = \tau^*(X,)N, D_X^{\perp *} N = \tau(X)N.$$

Hence,

$$\overline{D}_X Y = D_X Y + B(X, Y)N, \overline{D}_X N = -A_N^* X + \tau^*(X)N \tag{3.2}$$

$$\overline{D}_X^* Y = D_X^* Y + B^*(X, Y)N, \overline{D}_X^* = -A_N X + \tau(X)N. \tag{3.3}$$

Using Gauss formula and the relation between dual connections, we have [3]

$$Xg(Y, Z) = g(\overline{D}_X Y, Z) + g(Y, \overline{D}_X^* Z)$$

$$= g(D_X Y, Z) + g(Y, D_X^* Z) + B(X, Y)\eta(Z) + B^*(X, Z)\eta(Y).$$

From the above equation, we obtain that induced connections D and D^* are not dual connections and a lightlike hypersurface of a statistical manifold need not a statistical manifold with respect to the dual connections. Also, the induced connections D and D^* and the second fundamental forms B and B^* are symmetric.

Further, using Gauss and Weingarten formulae, we have

$$(D_X g)(Y, Z) + (D_X^* g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y)$$

$$+ B^*(X, Y)\eta(Z) + B^*(X, Z)\eta(Y). \tag{3.4}$$

Let P denote the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). Then we have

$$D_X PY = \nabla_X PY + h'(X, PY),\ D^*_X PY = \nabla^*_X PY + h^{*'}(X, PY)$$

$$D_X \xi = -A'_\xi X + \nabla'^t_X \xi,\ D^*_X \xi = -A^{*'}_\xi X + \nabla^{*'t}_X \xi$$

for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$ where $\nabla_X PY, \nabla^*_X PY, A'_\xi X$ and $A^{*'}_\xi X \in \Gamma(S(TM)), \nabla, \nabla^*$ and $\nabla'^t, \nabla^{*'t}$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. Here $h', h^{*'}$ and $A', A^{*'}$ are called screen second fundamental form and screen shape operator of $S(TM)$. We define the local second fundamental forms of $S(TM)$ as

$$C(X, PY) = \bar{g}(h'(X, PY), N),\ C^*(X, PY) = \bar{g}(h^{*'}(X, PY), N)$$

$$\epsilon(X) = g(\nabla'^t_X \xi, N),\ \epsilon^*(X) = g(\nabla'^t_X \xi, N)\ \forall X, Y \in \Gamma(TM)$$

Therefore, we have

$$h'(X, PY) = C(X, PY)\xi,\ h^{*'}(X, PY) = C^*(X, PY)\xi$$

$$\nabla'^t_X \xi = -\tau(X)\xi,\ \nabla^{*'t}_X \xi = -\tau^*(X)\xi$$

$$D_X PY = \nabla_X PY + C(X, PY)\xi\ D^*_X PY = \nabla^*_X PY + C^*(X, PY)\xi \tag{3.5}$$

$$D_X \xi = -A'_\xi X - \tau(X)\xi,\ D^*_X \xi = -A^{*'}_\xi X - \tau^*(X)\xi\ \forall X, Y \in \Gamma(TM) \tag{3.6}$$

where $\epsilon(X) = -\tau(X)$

Using above equation, the relationship between induced objects are as follows:

$$B(X, \xi) + B^*(X, \xi) = 0,\ g(A_N X + A^*_N X, N) = 0, \tag{3.7}$$

$$C(X, PY) = g(A_N X, PY),\ C^*(X, PY) = g(A^*_N X, PY)$$

From the equations (2.2), (3.1), (3.2), (3.3) and (3.6), we have the following proposition.

Proposition 3.1. *Let* (M, g) *be a lightlike hypersurface of a statistical manifold* $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$. *Then the second fundamental forms B and* B^* *are related to the shape operators* $A'_\xi X$ *and* $A^{*'}_\xi X$ *of* $S(TM)$ *as follows*:

$$g(A'_\xi X, PY) = B^*(X, PY), \; g(A^{*'}_\xi X, PY) = B(X, PY). \tag{3.8}$$

Hence from equation (3.8), we obtain

$$B(A^{*'}_\xi X, Y) = B(X, A^{*'}_\xi, Y), \; B^*(A'_\xi X, Y) = B^*(X, A'_\xi Y). \tag{3.9}$$

Proposition 3.2. *Let* (M, g) *be a lightlike hypersurface of a statistical manifold* $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$. *Then the screen distribution* $(S(TM), g, \nabla, \nabla^*)$ *has a statistical structure*.

Proposition 3.3 [4]**.** *Let* (M, g) *be a lightlike hypersurface of an indefinite statistical manifold* $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$. *Then the following are equivalent*:

(i) *The screen distribution is parallel*.

(ii) $C(X, Y) = 0$ *for all* $X, Y \in \Gamma(S(TM))$.

(iii) $C^*(X, Y) = 0$ *for all* $X, Y \in \Gamma(S(TM))$.

Definition 3.1 [11]**.** Let (M, g) be a lightlike hypersurface of an indefinite statistical manifold $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$. Then

(1) M is called totally tangentially umbilical with respect to $\overline{D}$ if $B(X, Y) = kg(X, Y)$ for all $X, Y \in \Gamma(TM)$, where k is smooth function.

(2) M is called totally tangentially umbilical with respect to D^* if $B^*(X, Y) = k^*g(X, Y)$ for all $X, Y \in \Gamma(TM)$, where k^* is smooth function.

(3) M is called totally normally umbilical with respect to $\overline{D}$ if $A^*_N X = kX$ for any $X, Y \in \Gamma(TM)$, where k is smooth function.

(4) M is called totally normally umbilical with respect to $\overline{D}^*$ if $A_N X = k^* X$ for any $X, Y \in \Gamma(TM)$, where k^* is smooth function.

4. Killing and Geodesic Lightlike Hypersurfaces of an Indefinite Statistical Manifold

Definition 4.1 [11]**.** Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$. Then

(1) M is called totally geodesic with respect to $\overline{D}$ if $B = 0$.

(2) M is called totally geodesic with respect to $\overline{D}^*$ if $B^* = 0$.

Definition 4.2. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$. Then

(1) A Distribution E on M is called a Killing distribution if $(L_X g)(Y, Z) = 0$ for any $X \in \Gamma(E)$ and $Y, Z \in \Gamma(TM)$.

If $Y, Z \in \Gamma(S(TM))$, then E is called an $S(TM)$- killing distribution and X is called an $S(TM)$- killing vector field.

(2) A vector field W is parallel with respect to the induced connection D if $D_X W = 0$ for any $X \in \Gamma(TM)$.

Theorem 4.3. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \overline{g}, \overline{D}, \overline{D}^*)$. If the second fundamental form B and B^* vanishes identically on M, then $TM^{\perp}$ is a killing distribution on M.*

Proof. Let $X, Z \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$. From equations (2.2), (3.4) and (3.7) we have,

$$(D_X g)(\xi, Z) + (D_X^* g)(\xi, Z) = B(X, \xi)\eta(Z) + B(X, Z)\eta(\xi)$$

$$+ B^*(X, \xi)\eta(Z) + B^*(X, Z)\eta(\xi) = B(X, Z) + B^*(X, Z).$$

Also,

$$(D_X g)(\xi, Z) + (D_X^* g)(\xi, Z) = Xg(\xi, Z) - g(D_X \xi, Z) - g(\xi, D_X Z) + Xg(\xi, Z)$$

$$- g(D_X^* \xi, Z) - g(\xi, D_X^* Z)$$

$$= -g(D_X \xi, Z) - g(D_X^* \xi, Z).$$

Therefore we obtain

$$g(D_X \xi, Z) + g(D_X^* \xi, Z) = -B(X, Z) - B^*(X, Z).$$

Now,

$$(L_{\xi g})(X, Z) = \xi g(X, Z) - g([\xi, X], Z) - g(X, [\xi, Z])$$

$$= \xi g(X, Z) - g(D_\xi X, Z) + g(D_X \xi, Z) - g(X, D_\xi Z) + g(X, D_Z \xi)$$

$$(L_\xi g)(X, Z) = (D_\xi g)(X, Z) + g(D_X \xi, Z) + g(X, D_Z \xi). \tag{4.1}$$

Similarly, we derive

$$(L_{\xi g})(X, Z) = (D_\xi^* g)(X, Z) + g(D_X^* \xi, Z) + g(X, D_Z^* \xi). \tag{4.2}$$

Using equation (4.1), (4.2), (3.4) and (3.7), we get

$$2(L_{\xi g})(X, Z) = (D_\xi g)(X, Z) + (D_\xi^* g)(X, Z) + g(D_X \xi, Z) + g(X, D_Z \xi)$$

$$+ g(D_X^* \xi, Z) + g(X, D_Z^* \xi) = -B(X, Z) - B^*(X, Z) - B(Z, X) - B^*(Z, X)$$

$$= -2B(X, Z) - 2B^*(X, Z)$$

which proves the result.

Theorem 4.4. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \overline{g})$. Then the Lie derivative of g with respect to $N \in \Gamma(tr(TM))$ is given by*

$$(L_{Ng})(X, Y) = X\eta(Y) + Y\eta(X) + \eta([X, Y]) - \eta(D_X^* Y) - \eta(D_X Y). \tag{4.3}$$

Proof. Let $X, Y \in \Gamma(TM)$. Then

$$\eta(D_X Y) = \overline{g}(D_X Y, N) = \overline{g}(\overline{D}_X Y, N).$$

Now using the relation between the dual connections in a statistical manifold, we obtain from (3.1)

$$\eta(D_X Y) = X\overline{g}(Y, N) - \overline{g}(Y, D_X^* N)$$

$$= X\eta(Y) - \bar{g}(Y, [X, N]) - \bar{g}(Y, \bar{D}_N^* X)$$

$$= X\eta(Y) + \bar{g}(Y[N, X]) - N\bar{g}(Y, X) + \bar{g}(\bar{D}_N Y, X)$$

$$= X\eta(Y) + \bar{g}(Y[N, X]) - N\bar{g}(Y, X) + \bar{g}(X, [N, Y]) + \bar{g}(\bar{D}_N Y, X)$$

$$= X\eta(Y) + g(Y[N, X]) - Ng(Y, X) + g(X, [N, Y]) + \bar{g}(\bar{D}_N Y, X)$$

$$= X\eta(Y) - (L_N g)(X, Y) + Y\bar{g}(N, X) - \bar{g}(\bar{D}_Y^* X, N)$$

$$= X\eta(Y) + Y\eta(X) - (L_N g)(X, Y) - \bar{g}(\bar{D}_X^* Y, N) + \bar{g}([X, Y], N)$$

$$= X\eta(Y) + Y\eta(X) - (L_N g)(X, Y) - \eta(\bar{D}_X^* Y) + \eta([X, Y]).$$

This gives the desired result.

Definition 4.5. The screen distribution $S(TM)$ is said to be totally umbilical with respect to D (resp. D^*) if on any coordinate neighbourhood $\mathcal{U}$ of M, there exist smooth functions λ (resp. λ^*) such that

$$C(X, PY) = \lambda g(X, PY)(resp. C^*(X, PY) = \lambda^* g(X, PY)) \; \forall X, Y \in \Gamma(TM). \quad (4.4)$$

Definition 4.6. The screen distribution $S(TM)$ is said to be totally geodesic with respect to D (resp. D^*) if

$$C(X, PY) = 0 \; (resp. C^*(X, PY) = 0) \; \forall X, Y \in \Gamma(TM). \quad (4.5)$$

Theorem 4.7. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g})$. Then, the screen distribution $S(TM)$ is totally geodesic with respect to D if and only if the vector Y is parallel with respect to the induced connection D.*

Proof. Let the screen distribution $S(TM)$ be a totally geodesic with respect to D. Then, for any $X, Y \in \Gamma(S(TM))$, we have $C(X, Y) = 0$. Now using (3.5), we have

$$\bar{g}(D_X Y, N) = \bar{g}(\nabla_X Y + C(X, Y)\xi, N) = 0$$

which implies that $D_X Y = 0$.

Conversely, if the vector field Y is parallel with respect to induced connection D, then $\bar{g}(D_X Y, N) = 0$. Therefore

$$\bar{g}(\nabla_X Y + C(X, Y)\xi, N) = 0$$

which implies $C(X, Y) = 0$. Hence the proof.

Theorem 4.8. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \bar{g})$. Suppose that the vector field Y is parallel with respect to the induced connection D (resp. D^*). If a vector field N is parallel with respect to connection $\overline{D}$, then for any $X \in \Gamma(TM)$, $Y \in \Gamma(S(TM))$,*

$$(L_{Ng})(X, Y) = 0.$$

Proof. From (4.1), we have for any $X \in \Gamma(TM)$, $Y \in \Gamma(S(TM))$, $(L_{Ng})(X, Y) = Y\eta(X) - \eta([Y, X])$ since $\eta(Y) = 0$ for $Y \in \Gamma(S(TM))$ and $\eta(D_X Y) = 0 = \eta(D_X^* Y)$.

Now, from the definition of statistical manifold, we have

$$Y\bar{g}(N, X) = \bar{g}(\overline{D}_Y N, X) + \bar{g}(N, \overline{D}_Y^* X).$$

If a vector N is parallel with respect to connection $\overline{D}$, then $\overline{D}_Y N = 0$. Using equation (3.1), (3.3) we have

$$\begin{aligned} 0 &= \bar{g}(\overline{D}_Y N, X) = Y\bar{g}(N, X) - \bar{g}(N, D_Y^* X) \\ &= Y\bar{g}(N, X) - \bar{g}(N, [Y, X]) - \bar{g}(N, \overline{D}_X^* Y) \\ &= Y\bar{g}(N, X) - \bar{g}(N, [Y, X]) - \bar{g}(N, D_X^* Y + B^*(X, Y)N) \\ &= Y\eta(X) - \eta([Y, X]). \end{aligned}$$

Hence the required result.

Theorem 4.9. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \bar{g})$ If the screen distribution $S(TM)$ is totally geodesic with respect to D* (resp. D^*), *then for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(tr(TM))$,*

(i) $(L_N g)(X, Y) = \eta[X, Y]$

(ii) $(L_N g)(X, Y) = -\eta([Y, X]) = -(L_N g)(Y, X)$

Moreover, $N \in \Gamma(tr(TM))$, *is an* $S(TM)$ *-Killing vector field.*

Proof. Let the screen distribution $S(TM)$ be totally geodesic with respect to D (resp. D^*). Then for any $X, Y \in \Gamma(S(TM))$, we have

$$C(X, Y) = 0 \ (resp.\ C^*(X, Y) = 0).$$

Now using equation (3.5), we have

$$\eta(D_X Y) = \bar{g}(D_X Y, N) = \bar{g}(\nabla_X Y + C(X, Y)\xi, N) = 0,$$

$$\eta(D^*_X Y) = \bar{g}(D^*_X Y, N) = \bar{g}(\nabla^*_X Y + C^*(X, Y)\xi, N) = 0,$$

Therefore

$$(L_{Ng})(X, Y) = X\eta(Y) + Y\eta(X) + \eta([X, Y]) - \eta(D^*_X Y) - \eta(D_X Y) \qquad (4.6)$$

As $\eta(X) = \bar{g}(X, N)$, it is clear that $X \in \Gamma(S(TM))$ if and only if $\eta(X) = 0$, so we have

$$(L_N g)(X, Y) = \eta([X, Y]).$$

Hence

$$(L_N g)(X, Y) = -\eta([Y, X]) = -(L_N g)(Y, X).$$

Now, using the statistical character of the manifold and equation (4.6), we have

$$(L_N g)(X,Y) - (L_N g)(Y,X) = X\eta(Y) + Y\eta(X) + \eta([X,Y]) - \eta(D^*_X Y) - \eta(D_X Y)$$

$$- Y\eta(X) - X\eta(Y) - \eta([Y, X]) + \eta(D^*_Y X) + \eta(D_Y X)$$

$$= 2\eta([X,Y]) - \eta([X,Y]) - \eta([X,Y]) = 0.$$

Therefore

$$(L_N g)(X, Y) = (L_N g)(Y, X) = -(L_N g)(X, Y).$$

Thus, the result and the further assertion follow from the hypothesis.

Theorem 4.10. *Let* (M, g) *be a lightlike hypersurface of a statistical manifold. If the local second fundamental form B is parallel with respect to D, then*

$$(L_{\xi}B)(X, Y) = -B(A'_{\xi}X, Y) - B(X, A'_{\xi}Y) - \tau(X)B(\xi, Y) - \tau(Y)B(X, \xi)$$

where $(L_{\xi}B)(X, Y)$ *stands for the Lie-derivative of B with respect to* ξ.

Proof. Let $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Then

$$\begin{aligned}(L_{\xi}B)(X, Y) &= \xi B(X, Y) - B([\xi, X], Y) - B(X, [\xi, Y])\\ &= \xi B(X, Y) - B(D_{\xi}X, Y) + B(D_X\xi, Y) - B(X, D_{\xi}Y) + B(X, D_Y\xi)\\ &= (D_{\xi}B)(X, Y) + B(D_X\xi, Y) + B(X, D_Y\xi).\end{aligned}$$

Now, from equations (3.6) and (3.9), we obtain

$$\begin{aligned}(L_{\xi}B)(X, Y) &= (D_{\xi}B)(X, Y) + B(-A'_{\xi}X - \tau(X)\xi, Y) + B(X, A'_{\xi}Y - \tau(Y)\xi)\\ &= (D_{\xi}B)(X, Y) + B(A'_{\xi}X, Y) - B(X, A'_{\xi}Y) - \tau(X)B(\xi, Y) - \tau(Y)B(X, \xi)\end{aligned}$$

$$(L_{\xi}B)(X, Y) = (D_{\xi}B)(X, Y) - B(A'_{\xi}X, Y) - B(X, A'_{\xi}Y) - \tau(X)B(\xi, Y) - \tau(Y)B(X, \xi). \quad (4.7)$$

Now from the given hypothesis, we get the desired result.

5. Some Formulae on the Curvature Tensors

Let $\overline{R}$, R and R' be the curvature tensors of the connection $\overline{D}$ on the indefinite statistical manifold $\overline{M}$, of the induced connections D on the lightlike hypersurface M and ∇ on the screen distribution $S(TM)$ respectively. Let $\overline{R}^*$, R^* and $R^{*'}$ be the curvature tensors of the corresponding dual connection $\overline{D}^*$ and of the induced connections D^* and ∇^* respectively. Using Gauss and Weingarten formulae, we have derived the following equations:

$$\overline{R}(X, Y)N = -D_X(A^*_N Y) + D_Y(A^*_N X) - B(X, A^*_N Y)N + B(Y, A^*_N X)N$$

$$- A_N^* X\tau^*(Y) + A_N^* Y\tau^*(X) + A_N^*[X, Y] + 2d\tau^*(X, Y)N,$$

$$\overline{R}^*(X, Y)N = -D_X^*(A_N Y) + D_Y^*(A_N X) - B^*(X, A_N Y)N + B^*(Y, A_N X)N$$
$$-A_N X\tau(Y) + A_N Y\tau(X) + A_N[X, Y] + 2d\tau(X, Y)N,$$

$$g(\overline{R}(X, Y)PZ, N) = (D_X C)(Y, PZ) - (D_Y C)(X, PZ) - \tau(X)C(Y, PZ)$$
$$+ \tau(Y)C(X, PZ) + B(X, PZ)g(A_N^* Y, N)$$
$$- B(Y, PZ)g(A_N^* X, N),$$

$$g(\overline{R}^*(X, Y)PZ, N) = (D_X^* C^*)(Y, PZ) - (D_Y^* C^*)(X, PZ) - \tau^*(X)C^*(Y, PZ)$$
$$+ \tau^*(Y)C^*(X, PZ) + B^*(X, PZ)g(A_N Y, N)$$
$$- B^*(Y, PZ)g(A_N X, N),$$

where we set

$$(D_X C)(Y, PZ) = XC(Y, PZ) - C(D_X Y, PZ) - C(Y, \nabla_X PZ),$$

$$(D_X^* C^*)(Y, PZ) = XC^*(Y, PZ) - C^*(D_X^* Y, PZ) - C^*(Y, \nabla_X^* PZ).$$

Also

$$g(\overline{R}(X, Y)PZ, PW) = g(R'(X, Y)PZ, PW) - C(Y, PZ)B^*(X, PW)$$
$$- B(Y, PZ)C^*(X, PW) + C(X, PZ)B^*(Y, PW)$$
$$+ B(X, PZ)C^*(Y, PW),$$

$$g(\overline{R}^*(X, Y)PZ, PW) = g(R'^*(X, Y)PZ, PW) - C^*(Y, PZ)B(X, PW)$$
$$- B^*(Y, PZ)C(X, PW) + C^*(X, PZ)B(Y, PW)$$
$$+ B^*(X, PZ)C(Y, PW),$$

$$g(R(X, Y)\xi, N) = C(Y, A'_\xi X) - C(X, A'_\xi Y) - 2dr(X, Y),$$

$$g(R^*(X, Y)\xi, N) = C^*(Y, A_\xi^{*'} X) - C^*(X, A_\xi^{*'} Y) - 2d\tau^*(X, Y),$$

where

$$2d\tau(X, Y) = X\tau(Y) - Y\tau(X) - \tau([X, Y]),$$

$$2d\tau^*(X, Y) = X\tau^*(Y) - Y\tau^*(X) - \tau^*([X, Y]),$$

We define the curvature tensors $R^{\perp}$ and $R^{*\perp}$ with respect to D and D^* of the transversal bundle of M by

$$R^{\perp}(X, Y)N = D_X^{\perp}D_Y^{\perp}N - D_Y^{\perp}D_X^{\perp}N - D_{[X, Y]}^{\perp}N,$$

$$R^{*\perp}(X, Y)N = D_X^{*\perp}D_Y^{*\perp}N - D_Y^{*\perp}D_X^{*\perp}N - D_{[X, Y]}^{*\perp}N,$$

for all $X, Y \in \Gamma(TM)$.

Definition 5.1. *If* $R^{\perp}$ (resp. $R^{*\perp}$) *vanishes identically, then the transversal connection* $D^{\perp}$(resp. $D^{*\perp}$) *is said to be flat.*

Theorem 5.2. *Let* (M, g) *be a lightlike hypersurfaces of a statistical manifold* $(\overline{M}, \overline{g})$. *Then the following assertions are equivalent*:

(i) *The transversal connection* $D^{\perp}$ *on* M *is flat, i.e.,* $R^{\perp} = 0$ (*resp.* $R^{*\perp} = 0$).

(ii) *Each* 1-*form* τ^* *is closed i.e.* $d\tau^* = 0$ (*resp.* τ *is closed i.e.* $d\tau = 0$).

Proof. For all $X, Y \in \Gamma(TM)$, using $\overline{D}_X N = -A_N^* X + D_X^{\perp}N$, we have

$$\overline{R}(X, Y)N = \overline{D}_X\overline{D}_Y N - \overline{D}_Y\overline{D}_X N - \overline{D}_{[X, Y]}N$$

$$\overline{R}(X, Y)N = R^{\perp}(X, Y)N - D_X(A_N^*Y) + D_Y(A_N^*Y) - B(X, A_N^*Y)N$$
$$+ B(Y, A_N^*X)N - A^*_{D_Y^{\perp}N}X + A^*_{D_X^{\perp}N}Y + A_N^*[X, Y]. \quad (5.1)$$

Now using, $\overline{D}_X N = -A_N^*X + \tau^*(X)N$, we derive

$$\overline{R}(X, Y)N = -D_X(A_N^*Y) + D_Y(A_N^*Y) - B(X, A_N^*Y)N - B(Y, A_N^*X)N$$
$$- A_N^*X\tau^*(Y) + A_N^*Y\tau^*(X) + A_N^*[X, Y] - 2d\tau^*(X, Y)N. \quad (5.2)$$

From equations (5.1) and (5.2) we obtain

$$R^{\perp}(X, Y) = 2d\tau^{*}(X, Y)N \quad \forall X, Y \in \Gamma(TM).$$

The required assertions follow using definition 5.1.

Similarly, the corresponding result holds for the dual connection.

Theorem 5.3. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\overline{M}, \overline{g})$. Then the linear connection $D^{\perp}(resp. D^{*\perp})$ on transversal vector bundle is at if and only if the lightlike transversal vector bundle N is parallel and M is totally geodesic with respect to $\overline{D}(resp. D^{*})$.*

Proof. For $\overline{D}$ and $D^{\perp}$ on $T\overline{M}$ and $tr(TM)$, respectively, we have

$$\overline{R}(X, Y)N = D_X^{\perp}D_Y^{\perp}N - D_Y^{\perp}D_X^{\perp}D_{[X, Y]}^{\perp}N - D_X(A_N^{*}Y) + D_Y(A_N^{*}X)$$

$$- B(X, A_N^{*}Y)N + B(Y, A_N^{*}X)N - A^{*}_{D_Y^{\perp}N}X$$

$$+ A^{*}_{D_X^{\perp}N}Y + A_N^{*}[X, Y].$$

The curvature tensor of lightlike transversal vector bundle is given by

$$\overline{R}^{\perp}(X, Y)N = D_X^{\perp}D_Y^{\perp}N - D_Y^{\perp}D_X^{\perp}N - D_{[X, Y]}^{\perp}N - B(X, A_N^{*}Y)N$$

$$+ B(Y, A_N^{*}X)N.$$

Using the given hypothesis, we get the desired result.

6. Conclusion and Future Work

In this paper, the lightlike geometry of an indefinite statistical manifold has been thoroughly studied. Various results of considerable importance related to the geodesicity and parallelism with respect to the dual connections in the lightlike hypersurfaces have been worked upon. This study can be useful for geometers in developing the theory of lightlike hypersurfaces in the corresponding odd and even dimensional manifolds and thus adding to its significant applications in the theory of general relativity and neural networks.

References

[1] S. Amari, Differential geometrical methods in statistics, Lecture notes in Statistics (28) (1985).

[2] S. Amari, Differential geometrical theory of statistics, Differential Geometry in Statistical Inference (10) (1987), 19-94.

[3] O. Bahadur and M. M. Tripathi, Geometry of lightlike hypersurfaces of a statistical manifold, arXiv preprint arXiv:1901.09251, (2019).

[4] K. Duggal and A. Bejancu, Lightlike submain folds of semi-Riemannian manifolds and applications, Mathematics and its Applications, Kluwer Academic.

[5] K. L. Duggal and D. H. Jin, Null curves and hypersurfaces of semi-Riemannian manifolds, World Scientific Publishing Company (2007).

[6] K. L. Duggal and B. Sahin, Differential geometry of lightlike submanifolds, Springer Science and Business Media (2011).

[7] H. Furuhata, Hypersurfaces in statistical manifolds, Differential Geometry and its Applications 27(3) (2009), 420-429.

[8] H. Furuhata, Statistical hypersurfaces in the space of Hessian curvature zero, Differential Geometry and its Applications 29 (2011), 86-S90.

[9] H. Furuhata and I. Hasegawa, Submanifold theory in holomorphic statistical manifolds, In Geometry of Cauchy-Riemann Submanifolds, (2016), pages 179-215.

[10] D. Jin, Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold. Indian Journal of Pure and Applied Mathematics 41(4) (2010), 569-581.

[11] F. Massamba, Killing and geodesic lightlike hypersurfaces of indefinite Sasakian manifolds, Turkish Journal of Mathematics 32(3) (2008), 325-347.

[12] M. Milijevic, Totally real statistical submain folds, Interdisciplinary Information Sciences 21(2) (2015), 87-96.

[13] B. O'Neill, Semi-Riemannian geometry with applications to relativity, volume 103, Academic Press (1983).

[14] C. Rao, Information and the accuracy attainable in the estimation of statistical parameters, Bulletin of Calcutta Mathematical Society 37 (1945), 81-91.

[15] P. W. Vos, Fundamental equations for statistical submain folds with applications to the bartlett correction, Annals of the Institute of Statistical Mathematics 41(3) (1989), 429-450.

[16] Y. Wang and X. Liu, A note on lightlike hypersurfaces of semi-Riemannian space form, Sarajevo Journal of Mathematics 9 (2013), 283-291.

[17] E. Yasar and I. Ayhan, Some properties of lightlike hypersurfaces in semi-Riemannian manifold, International Journal of Pure and Applied Mathematics 46 (2008).