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ON THE GAUSSIAN INTEGER SOLUTIONS FOR AN ELLIPTIC DIOPHANTINE EQUATION

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Abstract

The quadratic Diophantine equation with two unknowns represented by an elliptic curve $DE: 65J^2 + 225K^2 - 230JK = 1600$ is analyzed for its non-zero separate solutions in $Z[i]$. We also gain a few formulae and reappearance relations on the Gaussian integer solutions (J_n, K_n) of DE.

1. Introduction

The Diophantine equation $x^4 \pm y^4 = z^2$ where x, y and z being Gaussian integer were examined by Hilbert. It was proved that there exist only inconsequential solutions in $Z[i]$. Elliptic curves have also been used in [3] to prove that the Diophantine equation $x^3 + y^3 = z^3$ has only trivial solutions in Gaussian integers. These outcomes have motivated us to find non-zero distinct Gaussian integer solutions to a homogenous quadratic Diophantine equation in three variables. Convinced Diophantine problems

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move towards beginning problems or immediate arithmetical generalizations and others come from geometry in a variety of ways. Swayed Diophantine problems are neither inconsequential nor complicated to analyze [1, 2, 4, 5, 6].

In this manuscript, we look into Gaussian integer solutions of the Diophantine equation $65J^2 + 225K^2 - 230JK = 1600$ which is malformed into a Pell's equation and is solved by a variety of methods.

2. The Diophantine Equation

Think about the Diophantine equation

$$DE : 65J^2 + 225K^2 - 230JK = 1600 \tag{1}$$

to be solved over $Z[i]$. It is not trouble-free to work out and discover the nature and properties of the solutions of (1). So we concern a linear conversion Trs to (1) to shift to a simpler form for which we can find out the integral solutions.

Let

$$Trs : \begin{cases} J = ha + ib \\ K = a + ikb \end{cases} \tag{2}$$

be the shift where $a, b, h, k \in Z$.

Applying to Trs to DE, we get

$$Trs(DE) = D\widetilde{E} : 65(ha + ib)^2 + 225(a + ikb)^2$$

$$-230((ha + ib)(a + ikb)) = 1600. \tag{3}$$

Equating the imaginary part to zero and the coefficient of a^2 and b^2 to the smallest amount numeral, we get $h = 2$ and $k = 3$.

Hence for $J = 2a + ib$ and $K = a + i3b$, we have the Diophantine equation

$$\widetilde{D}E : a^2 - 56b^2 = 64 \tag{4}$$

which is a Pell equation. Now we try to find all integer solutions (a_n, b_n) of

$\widetilde{D}$ and then we can retransfer all outcome as of $\widetilde{D}$ to by using the converse of *Trs*.

Theorem 2.1. *Let* $\widetilde{D}E$ *be the Diophantine equation in* (4). *Then*

(i) *The continued fraction expansion of* $\sqrt{56}$ *is*

$$\sqrt{56} = [7; \overline{2, 14}]$$

(ii) *The primary result of* $a^2 - 56b^2 = 1$ *is* $(u_1, v_1) = (15, 2)$

(iii) *For* $n \geq 4$,

$$u_n = 31(u_{n-1} - u_{n-2}) + u_{n-3}$$

$$v_n = 31(v_{n-1} - v_{n-2}) + v_{n-3}$$

Proof.

(i) The continued fraction expansion of $\sqrt{56} = 7 + (\sqrt{56} - 7)$

$$= 7 + \frac{1}{\frac{1}{\sqrt{56} - 7}}$$

$$= 7 + \frac{1}{\frac{\sqrt{56} + 7}{7}}$$

$$= 7 + \frac{1}{2 + \frac{\sqrt{56} - 7}{7}}$$

$$= 7 + \frac{1}{2 + \frac{1}{\sqrt{56} + 7}}$$

$$= 7 + \frac{1}{2 + \frac{1}{14 + (\sqrt{56} - 7)}}.$$

Therefore the continued fraction expansion of $\sqrt{56}$ is

$$[7; \overline{2, 14}]$$

(ii) Note that by (3), if $(u_1, v_1) = (15, 2)$ is the primary result of $a^2 - 56b^2 = 1$, then the supplementary solutions (u_n, v_n) of $a^2 - 56b^2 = 1$ can be consequently employing the equalities

$(u_n + v_n\sqrt{56}) = (u_1 + \sqrt{56}v_1)^n$ for $n \geq 2$, in supplementary expressions,

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 56v_1 \\ 2 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $n \geq 2$.

Hence it can be given away by generation on n that

$$u_n = 31(u_{n-1} - u_{n-2}) + u_{n-3}$$

and also

$$v_n = 31(v_{n-1} - v_{n-2}) + v_{n-3}$$

for $n \geq 4$.

Now we think about the problem

$$a^2 - 56b^2 = 64.$$

Make a note of that we stand for the integer solutions of $a^2 - 56b^2 = 64$ by (a_n, b_n), and represent the integer solutions of $a^2 - 56b^2 = 1$ by (u_n, v_n). Then we have the subsequent theorem.

Theorem 2.2. *Describe a progression* $\{(a_n, b_n)\}$ *of positive integers by* $(a_1, b_1) = (120, 16)$ *and*

$$a_n = 120u_{n-1} + 896v_{n-1}$$

$$b_n = 16u_{n-1} + 120v_{n-1}.$$

Somewhere $\{(u_n, v_n)\}$ *is a progression of constructive solutions of* $a^2 - 56b^2 = 1$. *Then*

(1) (a_n, b_n) *is a solution of* $a^2 - 56b^2 = 64$ *for any integer* $n \geq 1$.

(2) *For* $n \geq 2$,

$$a_{n+1} = 15a_n + 112b_n$$

$$b_{n+1} = 2a_n + 15b_n.$$

(3) *For* $n \geq 4$

$$a_n = 31(a_{n-1} - a_{n-2}) + a_{n-3}$$

$$b_n = 31(b_{n-1} - b_{n-2}) + b_{n-3}.$$

Proof.

(1) It is without a doubt seen to

$$(a_1, b_1) = (120, 16)$$

is a way out of $a^2 - 56b^2 = 64$ since

$$a_1^2 - 56b_1^2 = (120)^2 - 56(16)^2$$

$$= 64(15^2 - 56(2^2)) = 64(1)$$

$$= 64.$$

A reminder that by description (u_{n-1}, v_{n-1}) is a method out $a^2 - 56b^2 = 1$, of that is,

$$u_{n-1}^2 - 56v_{n-1}^2 = 1. \tag{5}$$

Moreover, we observe as greater than that (a_1, b_1) is a resolution of $a^2 - 42b^2 = 16$, to is,

$$a_1^2 - 56b_1^2 = 64. \tag{6}$$

Applying (5) and (6), we get

$$a_n^2 - 56b_n^2 = (120u_{n-1} + 896v_{n-1})^2 - 56(16u_{n-1} + 120v_{n-1})^2$$

$$= u_{n-1}^2(2^6) - v_{n-1}^2(2^6(56))$$

$$= 2^6(u_{n-1}^2 - 56v_{n-1}^2)$$

$$= 2^6.$$

Consequently (a_n, b_n) is a way out of $a^2 - 56b^2 = 2^6$.

(2) Bear in mind that

$$a_{n+1} + b_{n+1}\sqrt{d} = (u_1 + v_1\sqrt{d})(a_n + b_n\sqrt{d})$$

Consequently $a_{n+1} = u_1 a_n + v_1 b_n d$ and $b_{n+1} = v_1 a_n + u_1 b_n$.

So

$$a_{n+1} = 15a_n + 112b_n \text{ and } b_{n+1} = 2a_n + 15b_n. \tag{7}$$

Since $u_1 = 15$ and $v_1 = 2$.

(3) Applying the equalities

$$a_n = 2^2(15)u_{n-1} + 2^3(56)v_{n-1} \text{ and } a_{n+1} = 15a_n + 112b_n.$$

We find by generation on n that

$$a_n = 31(a_{n-1} - a_{n-2}) + a_{n-3}$$

for $n \geq 4$. In the same way, it knows how to be given away that

$$b_n = 31(b_{n-1} - b_{n-2}) + b_{n-3}.$$

We saw as on top of that the Diophantine equation DE could be malformed into the $D\widetilde{E}$ via the conversion Trs. Also, we showed that $J = 2a + ib$ and $K = a + i3b$. So we can retransfer all results from $\widetilde{D}E$ to by using the converse of Trs. Thus we can give the subsequent major theorem.

Theorem 2.3. *Let DE be the Diophantine equation in* (1). *Subsequently*

(1) *The primary solution of DE is* $(J_1, K_1) = (240 + 16i, 120 + 48i)$.

(2) *Describe the progression* $\{(J_n, K_n)\}_{n\geq 1} = \{(2a_n + ib_n, a_n + i3b_n)\}$, *wherever* $\{(a_n, b_n)\}$ *defined in* (7).

Then (J_n, K_n) *is a solution of DE.*

(3) *The explanation* (J_n, K_n) *convince*

$$J_n = 2(15a_{n-1} + 112b_{n-1}) + i(2a_{n-1} + 15b_{n-1})$$

$$K_n = (15a_{n-1} + 112b_{n-1}) + i3(2a_{n-1} + 15b_{n-1}).$$

(4) *The solutions* (J_n, K_n) *satisfy the reappearance relations*

$$J_n = (62(a_{n-1} - a_{n-1}) + 2a_{n-3}) + i(31(b_{n-1} - b_{n-2}) + b_{n-3})$$

$$K_n = (31(a_{n-1} - a_{n-1}) + a_{n-3}) + i(93(b_{n-1} - b_{n-2}) + 3b_{n-3}).$$

Conclusion

Diophantine equations are prosperous in diversity. There is no general method for discovering the entire possible Gaussian integer solutions (if it exists) for Diophantine equations one possibly will explore for additional choices of Diophantine equations to find their equivalent Gaussian integer solutions.

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