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THIRD HANKEL DETERMINANT FOR A CLASS OF FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS ASSOCIATED WITH q-DERIVATIVE OF EXPONENTIAL FUNCTIONS

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Abstract

The objective of this paper is to investigate the possible upper bound of third order Hankel determinant for q-starlike and q-convex functions with respect to symmetric points associated with exponential functions.

1. Introduction

Let $\mathcal{B}$ denote the class of functions in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. The class has the following Taylor's series expansion

$$f(z) = z + b_2z^2 + b_2z^3 + \dots \tag{1}$$

Let P denote the class of functions defined by

$$p(z) = 1 + c_1z + c_2z^2 + c_2z^3 + \dots \tag{2}$$

which are univalent and analytic in $\mathcal{U}$ is denoted by $\mathbb{S}$ and maps $\mathcal{U}$ onto the

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right half plane.

Let $g, h \in \mathcal{U}$ such that $g \prec h$, if there exist a function ω analytic in $\mathcal{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ and such that $g(z) = h(\omega(z))$. If the function h is univalent in $\mathcal{U}$ then if and only $g \prec h$ if $g(0) = h(0)$ and $g(\mathcal{U}) = h(\mathcal{U})$. The q^{th} Hankel determinant for $q \geq 1$ and $n \in N$ is in [20] given as:

$$H_q(n) = \begin{vmatrix} k_n & k_{n-1} & k_{n+2} & \cdots & k_{n+q-1} \\ k_{n+1} & k_{n-2} & k_{n+3} & \cdots & k_{n+q} \\ k_{n+2} & k_{n-3} & k_{n+4} & \cdots & k_{n+q+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n+q-1} & k_{n-q} & k_{n+q+1} & \cdots & k_{n+2q-2} \end{vmatrix} \tag{3}$$

Where $k_1 = 1$. In [18] Inayath determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f(z)$ given in equation (1) in $\mathbb{S}$ with bounded boundary. The Hankel determinant for exponential polynomials for $q = 2$ and $n = 1$ was studied by R. Ehrenborg [7]. From the relation (3) we get the Fekete-Szego functional for $\mu = 1$ i.e., $H_2(1) = k_3 - k_2^2$.

$$H_2(1) = \begin{vmatrix} k_1 & k_2 \\ k_2 & k_3 \end{vmatrix}$$

$$= | k_1 k_3 - k_2^2 | \, [k_1 = 1]$$

$$= | k_3 - k_2^2 | \tag{4}$$

from (3) we obtain

$$H_2(2) = \begin{vmatrix} k_2 & k_3 \\ k_3 & k_4 \end{vmatrix}$$

$$= | k_2 k_4 - k_3^2 | \tag{5}$$

and

$$H_2(3) = \begin{vmatrix} k_3 & k_4 \\ k_4 & k_5 \end{vmatrix}$$

$$= | k_3 k_5 - k_4^2 | \quad (6)$$

$H_2(1)$ is called the third order Hankel determinant and is given by

$$H_3(1) = \begin{vmatrix} k_1 & k_2 & k_3 \\ k_2 & k_3 & k_4 \\ k_3 & k_4 & k_5 \end{vmatrix}$$

$$= k_3(k_2 k_4 - k_3^2) - k_4(k_4 - k_2 k_3) + k_5(k_3 - k_2^2) \quad (7)$$

We note that $H_2(1)$ is the well-known Fekete-szego functional [7, 8, 9, 10, 11, and 12]. In recent years many authors studied the second order Hankel determinant $H_2(2)$ and the third Hankel determinant $H_3(1)$ for various classes of function and they can be unified by considering an univalent functions with a positive real part symmetric about the real axis and with respect to starlike functions. Mendiratta et al., [16] introduced and studied the class of q-starlike functions $\mathbb{S}_q^* = \mathbb{S}_q^*(e^z)$ defined by

$$\frac{zf'(z)}{f(z)} \prec e^z (z \in \mathcal{U})$$

The upper bound of the third Hankel determinant for the function class $\mathbb{S}_q^*$ is associated with an exponential functions which is given by Hai-Yan Zhang et al., [8] in 2018.

In this paper we investigate the non-sharp upper bounds of $H_3(1)$ for q-starlike and q-convex functions with respect to symmetric points subordinate to exponential function.

The subclasses are defined as.

Definition 1.1. The q-difference operator introduced by Jackson [3] is defined as

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} \; z \in \mathcal{U}. \quad (8)$$

In addition, the q-derivative at zero is $D_q f(0) = D_{q^{-1}} f(0)$ for $|q| > 1$.

The q-derivative at zero is defined as $f^{r}(0)$ if it exists. Equivalently (8) can be written as

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},\ z \neq 0$$

where,

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases} \tag{9}$$

Definition 1.2. A function $f' \in \mathbb{S}_q^*(e^z)$ if and only if

$$\frac{2[zD_q f(z)]}{f(z) - f(-z)} \prec e^z, \text{ for all } z \in \mathcal{U} \tag{10}$$

Definition 1.3. A function $f \in C_q(e^z)$ if and only if

$$\frac{2[D_q z D_q^2 f(z)]}{D_q f(z) - D_q f(-z)} \prec e^z, \text{ for all } z \in \mathcal{U} \tag{11}$$

2. Preliminaries

The lemmas listed below are needed to prove the desired results.

Lemma 2.1 [10]. *If* $p \in \mathcal{P}$ *then* $|p_n| \leq 2, \forall n \in \mathbb{N}$.

Lemma 2.2 [9]. *If* $p(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \ldots$ *is such that* $\operatorname{Req}(z) \geq 0$ *in* $\mathcal{U}$ *then for some* x, z *with* $|x| \leq 1$ *we have*

$$2d_2 = d_1^2 + x(4 - d_1^2) \text{ for some } x, |x| \leq 1$$

$$2d_3 = d_1^3 + 2d_1(4 - d_1^2)x - d_1(4 - d_1^2)x^2 + 2(4 - d_1^2)(1 - |x|^2)z \tag{12}$$

Lemma 2.3 [12]. *If* $p \in \mathcal{P}$ *then* $|d_2 - v d_1^2| \leq \text{maximum}\{1, |2v - 1|\}$ *for any* $v \in \mathbb{C}$.

3. Main Results

Theorem 3.1. *If* $f \in \mathbb{S}_q^*(e^z)$ *then*

$$| k_2 | \le \frac{1}{[2]_q}, \; | k_3 | \le \frac{1}{[3]_q - 1}, \; | k_4 | \le \frac{6(1-[3]_q) + 6(2-[3]_q) + ([3]_q - q)}{6[4]_q(1-[3]_q)}$$

$$| k_5 | \le \frac{-3(1-[3]_q) - 3(1-[3]_q) + 1}{3(1-[5]_q)(1-[3]_q)}.$$

Proof. As $f \in \mathbb{S}_q^*(e^z)$ from (10) and using the principle of subordination we have

$$\frac{2[zD_q f(z)]}{f(z) - f(-z)} = e^{\omega(z)} \tag{13}$$

Define $p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \ldots$ analytic in $\mathcal{U}$ with $p(0) = 1$ and maps $\mathcal{U}$ onto the right half of the ω-plane. Computing $\omega(z)$ in terms of $p(z)$ we get

$$\omega(z) = \frac{d_1}{2} z + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right) z^2 + \left(\frac{d_3}{2} - \frac{d_1 d_2}{2} + \frac{d_1^2}{8}\right) z^3$$

$$+ \left(\frac{d_4}{2} - \frac{d_1 d_4}{2} - \frac{3d_1^2 d_2}{8} - \frac{d_1^4}{16}\right) z^4 + \ldots$$

Substituting for $\omega(z)$ in the exponential series $e^{\omega(z)}$ we get

$$e^{\omega(z)} = 1 + \omega(z) + \frac{(\omega(z))^2}{2!} + \frac{(\omega(z))^3}{3!} + \frac{(\omega(z))^4}{4!} + \ldots \tag{15}$$

$$e^{\omega(z)} = 1 + \frac{d_1}{2} z + \left(\frac{d_2}{2} - \frac{d_1^2}{8}\right) z^2 + \left(\frac{d_3}{2} - \frac{d_1 d_2}{4} + \frac{d_1^3}{48}\right) z^2$$

$$+ \left(\frac{d_4}{2} - \frac{d_1 d_4}{4} - \frac{d_2^2}{8} + \frac{d_1^2 d_2}{16} - \frac{d_1^4}{384}\right) z^4 + \ldots \tag{16}$$

Substituting for $f(z)$, $D_q(f(z))$ and $e^{\omega(z)}$ in (13) we get,

$$k_2 = \frac{d_1}{2[2]_q}$$

$$k_3 = \frac{1}{[3]_q - 1}\left(\frac{d_2}{2} - \frac{d_1^2}{8}\right)$$

$$k_4 = \frac{d_3}{2[2]_q} - \frac{d_1 d_2 (2 - [3]_q)}{4[4]_q (1 - [3]_q)} - \frac{d_1^3 (4 - [3]_q)}{48[4]_q (1 - [3]_q)}$$

$$k_5 = \frac{1}{(1 - [5]_q)4} - \frac{d_4}{(1 - [5]_q)2} + \frac{d_1^4}{48[4]_q (1 - [3]_q)} \tag{17}$$

Taking modulus on both side of each expression in (17) and applying Lemma 2.1 and Lemma 2.2 we get

$$|k_2| \le \frac{1}{[2]_q},\ |k_3| \le \frac{1}{[3]_q - 1},\ |k_4| \le \frac{6(1 - [3]_q) + 6(2 - [3]_q) + ([3]_q - q)}{6[4]_q (1 - [3]_q)}$$

$$|k_5| \le \frac{-3(1 - [3]_q) - 3(1 - [3]_q) + 1}{3(1 - [5]_q)(1 - [3]_q)}. \tag{18}$$

Theorem 3.2. *If* $f \in \mathbb{S}_q^*(e^z)$ *then* $|k_2 - k_2^2| \le \frac{1}{[3]_q - 1}$.

Proof. From theorem 3.1 and using the equation (18) we have

$$k_2 = \frac{d_1}{2[2]_q} \text{ and } k_3 = \frac{1}{[3]_q - 1}\left(\frac{d_2}{2} - \frac{d_1^2}{8}\right) \tag{19}$$

Consider

$$|k_2 - k_2^2| = \left|\frac{d_2}{2([3]q - 1)} - \frac{([2]_q - 2([3]_q - 1))d_1^2}{8[2]_q^2([3]_q - 1)}\right|$$

Using lemma 2.3, we get

$$|k_2 - k_2^2| \le \frac{1}{[3]_q - 1} \tag{20}$$

Theorem 3.3. *If* $f \in \mathbb{S}_q^*(e^z)$ *then* $|k_2k_3 - k_4|$ $\le \frac{-1808447 + 204851\sqrt{118}}{58956}$

Proof. From equation (18) of theorem 3.1 we have

$$|k_2k_3 - k_4| = \left|\frac{d_1d_2(2-[2]_q-[3]_q)}{4[4]_q(1-[3]_q)} - \frac{d_1}{2[4]_q} + \frac{d_1^3(3[2]_q + ([3]_q - 4))}{48[4]_q(1-[3]_q)}\right| \tag{21}$$

Now applying Lemma 2.2, we get

$|k_2k_3 - k_4| =$

$$\left| \frac{d_1(4-d_2^2)x^2}{8[4]_q} - \frac{(4-d_2^2)(1-|x^2|)z}{4[4]_q} - \frac{(1-[3]_q)2-(2-[2]_q-[3]_q)(4-d_2^2)d_1x}{8[4]_q(1-[3]_q)}\right.$$

$$\left. + \frac{(6(2-[2]_q-[3]_q) - 6(1-[3]_q) + 3[2]_q + ([3]_q - 4))d_1^3}{48[4]_q(1-[3]_q)}\right| \tag{22}$$

Denote $|x| = y \in [0, 1]$, $d_1 = p \in [0, 2]$ then using triangle inequality, equation (22) gives

$$|k_2k_3 - k_4| \le \frac{(4-p^2)py^2}{8[4]q} + \frac{(4-p^2)}{4[4]_q} + \frac{(1-[3]_q)2-(2-[2]_q-[3]_q)(4-p^2)py}{8[4]_q(1-[3]_q)}$$

$$+ \frac{(6(2-[2]_q-[3]_q) - 6(1-[3]_q) + (3[2]_q + ([3]_q - 4))p^2}{48[4]_q(1-[3]_q)}$$

Suppose

$$F(p, 1) = \frac{p(4-p^2)y^2}{8[4]q} + \frac{(4-p^2)}{4[4]_q} + \frac{(1-[3]_q)2-(2-[2]_q-[3]_q)(4-p^2)py}{8[4]_q(1-[3]_q)}$$

$$+ \frac{(6(2-[2]_q-[3]_q) - 6(1-[3]_q) + (3[2]_q + ([3]_q - 4))p^2}{48[4]_q(1-[3]_q)}$$

Thus, we get

$$\frac{\partial F}{\partial y} = \frac{(4-p^2)py}{4[4]_q} \frac{(1-[3]_q)2-(2-[2]_q-[3]_q)(4-p^2)p}{8[4]_q(1-[3]_q)}$$

The function $F(p, y)$ is non-decreasing for any value of $y \in [0, 1]$. Hence $F(p, y)$ has maximum value at

$$\operatorname{Max} F(p, y) = F(p, 1) = \frac{p(4-p^2)p}{8[4]q} + \frac{(4-p^2)}{4[4]_q}$$

$$+ \frac{(1-[3]_q)2-(2-[2]_q-[3]_q)(4-p^2)p}{8[4]_q(1-[3]_q)}$$

$$+ \frac{(6(2-[2]_q-[3]_q)-6(1-[3]_q)+(3[2]_q+([3]_q-4))p^3}{48[4]_q(1-[3]_q)}$$

Let us define

$$M(p) := \frac{(4-p^2)p}{8[4]q} + \frac{(4-p^2)}{4[4]_q} + \frac{(1-[3]_q)2-(2-[2]_q-[3]_q)(4-p^2)p}{8[4]_q(1-[3]_q)}$$

$$+ \frac{(6(2-[2]_q-[3]_q)-6(1-[3]_q)+(3[2]_q+([3]_q-4))p^2}{48[4]_q(1-[3]_q)}$$

Then,

$$M(p) = \frac{4-3p^2}{8[4]_q} + \frac{(4-3p^2)(1-[3]_q)2-(2-[2]_q-[3]_q)}{8[4]_q(1-[3]_q)} + \frac{p}{2[4]_q}$$

$$+ \frac{(6(2-[2]_q-[3]_q)-6(1-[3]_q)+(3[2]_q+([3]_q-4))p^2}{16\,[4]_q(1-[3]_q)}$$

$M'(p)$ vanishes at $p = r^* = [M(p)]_{\max} = \frac{-3+2\sqrt{118}}{17}$. A simple yields that $M''(p) < 0$ which implies $[M(p)]_{\max}$ at $r^* = [M(p)]_{\max} = \frac{-3+2\sqrt{118}}{17}$.

Hence we have

$$|k_2k_3 - k_4| \le M(r^*) = \frac{-1808447 + 204851\sqrt{118}}{58956} \tag{23}$$

Hence the theorem is roved.

Theorem 3.4. *If* $f \in \mathbb{S}_q^*(e^z)$ *then* $|k_2k_3 - k_3^2| \le \frac{[4]_q[2]_q + ([3]_q - 1)^2}{[4]_q[2]_q([3]_q - 1)^2}$.

Proof. From equation (17) of theorem 3.1, we have

$$|k_2k_3 - k_3^2| = \left| \frac{d_1}{2[2]_q}\left(\frac{d_3}{2[4]_q} - \frac{d_1d_2(2-[3]_q)}{4[4]_q(1-[3]_q)} - \frac{d_1^2([3]_q - 4)}{48[4]_q([3]_q - 1)}\right) - \left(\frac{-1}{1-[3]_q}\left(\frac{d_2}{2} - \frac{d_1^2}{8}\right)\right)^2 \right|$$

$$= \left| \frac{d_1d_3}{4[4]_q[2]_q} - \frac{d_1^2d_2(2-[3]_q)}{8[2]_q[4]_q(1-[3]_q)} - \frac{d_1^4([3]_q - 4)}{96[2]_q[4]_q(1-[3]_q)} - \frac{d_2^2}{4(1-[3]_q)^2} - \frac{d_1^4}{64(1-[3]_q)^2} + \frac{2d_1^2d_2}{16(1-[3]_q)^2} \right|$$

According to lemma 2.2, we get

$$|k_2k_4 - k_3^2| =$$

$$\left| \frac{(4-d_1^2)d_1(1-|x|^2)z}{8[4]_q[2]_q} - \frac{(4-d_1^2)d_1^2x^2}{16[4]_q[2]_q} - \frac{d_1^2(4-d_1^2)x(1-[3]_q)^2 - (2-[3]_q)(1-[3]_q)[2]_q[4]_q)}{16[4]_q[2]_q(1-[3]_q)^2} - \frac{(4-d_1^2)^2x^2}{16(1-[3]_q)^2} - \frac{(6(1-[3]_q) - 6(2-[3]_q)(1-[3]_q) - (1-[3]_q) - 6[2]_q)d_1^4}{96[2]_q[4]_q(1-[3]_q)^2} \right|$$

Denote $|x| = y \in [0, 1]$ and $d_1 = p \in [0, 2]$ then by using triangle inequality we get,

$$| k_2k_4 - k_3^2 | \le \frac{(4-p^2)}{4[4]_q[2]_q} - \frac{(4-p^2)p^2y^2}{16[4]_q[2]_q}$$

$$+ \frac{p^2(4-p^2)y(2([3]_q - 1) + ([3]_q - 2) - 2[4]_q + [4]_q)}{16[4]_q[2]_q([3]_q - 1)^2} + \frac{(4-p^2)^2y^2}{16([3]_q - 1)^2}$$

$$+ \frac{(6([3]_q - 1) - 6([3]_q - 2) - (4 - [3]_q) - 6[4]_q - 3[2]_q + 3[4]_q)p^4}{96[2]_q[4]_q([3]_q - 1)^2}$$

Now we consider

$$F(p, t) = \frac{(4-p^2)}{4[4]_q[2]_q} - \frac{(4-p^2)p^2y^2}{16[4]_q[2]_q}$$

$$+ \frac{p^2(4-p^2)y(2([3]_q - 1) + ([3]_q - 2) - 2[4]_q + [4]_q)}{16[4]_q[2]_q([3]_q - 1)^2} + \frac{(4-p^2)^2y^2}{16([3]_q - 1)^2}$$

$$+ \frac{(6(1 - [3]_q) - 6(2 - [3]_q) - (1 - [3]_q) - ([3]_q - 4)(1 - [3]_q) - 6[2]_q)p^4}{96[2]_q[4]_q(1 - [3]_q)^2}.$$

Thus we get,

$$\frac{\partial F}{\partial y} = \frac{(4-p^2)p^2y}{8[4]_q[2]_q} + \frac{p^2(4-p^2)((1-[3]_q)^2 - (2-[3]_q)(1-[3]_q) - [2]_q[4]_q)}{16[2]_q[4]_q(1-[3]_q)^2}$$

$$+ \frac{(4-p^2)^2y}{8(1-[3]_q)^2} \ge 0$$

Which gives that $F(p, y)$ is increasing for any value of y in $[0, 1]$. This shows that $F(p, y)$ has maximum value at $y = 1$.

$$\text{Max } F(p, y) = F(p, 1) = \frac{p^2(4-p^2)[(1-[3]_q)^2 - (2-[3]_q)(1-[3]_q) - [2]_q[4]_q]}{16[2]_q[4]_q(1-[3]_q)^2}$$

$$+ \frac{(4-p^2)p^2}{16[4]_q[2]_q} \frac{(4-p^2)p^2}{16(1-[3]_q)^2} + \frac{(4-p^2)}{4[4]_q[2]_q}$$

$$+\frac{[6(1-[3]_q)-6(2-[3]_q)(1-[3]_q)-([3]_q-4)(1-[3]_q)-6[2]_q]p^4}{96[2]_q[4]_q(1-[3]_q)^2}$$

we define

$$M(p) := \frac{(4-p^2)}{4[4]q[2]_q} + \frac{p^2(4-p^2)((1-[3]_q)^2-(2-[3]_q)(1-[3]_q)-[2]_q[4]_q)}{16[2]_q[4]_q(1-[3]_q)^2}$$

$$+\frac{(4-p^2)}{16(1-[3]_q)^2} + \frac{(6(1-[3]_q)-6(2-[3]_q)(1-[3]_q)-([3]_q-4)(1-[3]_q)-5[2]_q)p^4}{96[2]_q[4]_q(1-[3]_q)^2}$$

then

$$M(p) := \frac{(-2p^2)}{4[4]q[2]_q} + \frac{(2p(4-p^2)-2p^2)((1-[3]_q)^2-(2-[3]_q)(1-[3]_q)-[2]_q[4]_q))}{16[2]_q[4]_q(1-[3]_q)^2}$$

$$+\frac{2(4-p^2)(-2p)}{16(1-[3]_q)^2} + \frac{(6(1-[3]_q)-6(2-[3]_q)(1-[3]_q)-([3]_q-4)(1-[3]_q)-6[2]_q)4p^2}{96[2]_q[4]_q(1-[3]_q)^2}$$

If $M'(p)$ vanishes at $p = 0$. A simple computation yields that $M''(p) < 0$ which implies at $[M(p)]_{\max}$ at $p = 0$. Hence we have

$$|k_2k_3 - k_3^2| \le M(0) = \frac{[4]_q[2]_q + (1-[3]_q)^2}{[4]_q[2]_q(1-[3]_q)^2} \tag{24}$$

Theorem 3.5. *If* $f \in \mathbb{S}_q^*(e^z)$ *then* $|H_3(1)| \le \frac{-33071381 + 3892169\sqrt{118}}{2812608}$ $= 3.275$.

Proof. Since $H_3(1) = k_3(k_2k_4 - k_3^2) - k_4(k_4 - k_2k_3) + k_5(k_3 - k_2^2)$.

By applying triangle inequality we get

$$|H_3(1)| = |k_3||k_2k_4 - k_3^2| + |k_4||k_4 - k_2k_3| + |k_5||k_3 - k_2^2|. \tag{25}$$

Now substituting the equations (20), (23) and (24) in (25) we get

$$|H_3(1)| \le 3.275.$$

Now, we establish some results related to the class defined in Definition 1.3.

Theorem 3.6. *If* $f \in C_q^*(e^z)$ *then*

$$|k_2| \leq \frac{1}{2[2]_q}, \; |k_3| \leq \frac{-1}{3(1-[3]_q)}$$

$$|k_4| \leq \frac{6(1-[3]_q)+6(2-[3]_q)+([3]_q-4)}{48[4]_q(1-[3]_q)},$$

$$|k_5| \leq \frac{-3(1-[3]_q)-3(1-[3]_q)+1}{3(1-[5]_q)(1-[3]_q)[5]_q}$$

Theorem 3.7. *If* $f \in C_q^*(e^z)$ *then* $|k_2 - k_2^2| \leq \frac{1}{2[2]_q}$.

Theorem 3.8. *If* $f \in C_q^*(e^z)$ *then*

$$|k_2k_3 - k_4| \leq \frac{-1039504 + 698425\sqrt{(118)}}{63200832} = 0.10359.$$

Theorem 3.9. *If* $f \in C_q^*(e^z)$ *then* $|k_2k_4 - k_3^2| \leq \frac{-1}{3[3]_q(1-[3]_q)}$.

Theorem 3.10. *If* $f \in C_q^*(e^z)$ *then* $|H_3(1)| \leq 0.03756$.

Remark 3.1. As $q \to 1$ we get the results of Ganesh K., Bharavi Sharma and Rajya Laxmi K.

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