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SOME PROPERTIES ON BIPOLAR VALUED MULTI FUZZY SUBFIELD OF A FIELD AND ITS (λ, μ)- LEVEL SUBSETS

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Abstract

In this paper, some properties of the bipolar valued multi fuzzy subfield of a field are discussed and studied its lower level subsets and related properties.

Introduction

The fuzzy set theory domain is a wide range and its information is incomplete or inaccurate such as bioinformatics. Initially, the notion of the fuzzy sets and its functions are introduced by Zadeh [16] in 1965. Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of

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research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc. W. R. Zhang [17], Lee [6] introduced the notion of bipolar-valued fuzzy sets. After the introduction of fuzzy subgroups many researchers discussed on the expansion of bipolar-valued fuzzy sets notion. However, they are distinct each other [2, 4, 5]. Sabu Sebastian and T. V. Ramakrishnan [8] discussed multi-fuzzy sets. V. K. Shanthi and G. Shyamala [10] discuss about bipolar-valued fuzzy subgroups of a group. K. Chandrasekar Rao and V. Swaminathan [3] defined the Anti-homomorphism in Fuzzy Ideals. Anitha. M. S et al. [1] defined a homomorphism and antihomomorphism of bipolar-valued fuzzy subgroups of a group. B. Yasodara and K. E. Sathappan [14, 15] defined the bipolar valued multi fuzzy subsemirings of a semiring under homomorpisms, Sivaramakrishna das. *P* [11] studied the Fuzzy groups and level subgroups. This paper confer about the notion of bipolar valued multi fuzzy subfield of a field and established some results.

2. Preliminaries

Definition 2.1 [6]**.** A bipolar valued fuzzy set (BVFS) A in X is defined as an object of the form $A = \{\langle x, A^{+}(x), A^{-}(x)\rangle / x \in X\}$, where $A^{+} : X \to [0, 1]$ and $A^{-} : X \to [-1, 0]$. The positive membership degree $A^{+}(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree $A^{-}(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued fuzzy set A.

Example 2.2. $A = \{\langle a, 0.9, -0.6\rangle, \langle b, 0.8, -0.7\rangle, \langle c, 0.7, -0.5\rangle\}$ is a bipolar valued fuzzy subset of $X = \{a, b, c\}$.

Definition 2.3 [10]**.** A bipolar valued multi fuzzy set (BVMFS) A in X of order n is defined as an object of the form $A = \{\langle x, A_1^{+}(x), A_2^{+}(x), \ldots, A_n^{+}(x), A_1^{-}(x), A_2^{-}(x), \ldots, A_n^{-}(x)\rangle / x \in X\}$, where $A_i^{+} : X \to [0, 1]$ and $A_i^{-} : X \to [-1, 0]$, $i = 1, 2, 3, \ldots, n.$ The positive membership degrees $A_i^{+}(x)$ denote the satisfaction degree of an element x to the property corresponding to a bipolar valued multi fuzzy set A and the negative membership degrees $A_i^-(x)$ denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued multi fuzzy set A.

Note: In this paper, the bipolar valued multi fuzzy subfield of a field A means $A = \langle A^+, A^- \rangle = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^- \rangle$.

Example 2.4. $A = \{\langle a, 0.5, 0.6, 0.3, -0.3, -0.6, -0.5\rangle, \langle b, 0.1, 0.4, 0.7, -0.7, -0.3, -0.6\rangle, \langle c, 0.5, 0.3, 0.8, -0.4, -0.5, -0.3\rangle\}$ is a bipolar-valued multi fuzzy subset of order 3 in $X = \{a, b, c\}$.

Definition 2.5 [13]**.** Let F be a field. A bipolar valued multi fuzzy subset A of F is said to be a bipolar valued multi fuzzy subfield of F if the following conditions are satisfied, for all i,

(i) $A_i^+(x - y) \geq \min \{A_i^+(x), A_i^+(y)\}$ for all x, y in F.

(ii) $A_i^-(x - y) \leq \max \{A_i^-(x), A_i^-(y)\}$ for all x, y in F.

(iii) $A_i^+(xy^{-1}) \geq \min \{A_i^+(x), A_i^+(y)\}$ for all $x, y \neq 0$ in F.

(iv) $A_i^-(xy^{-1}) \leq \max \{A_i^-(x), A_i^-(y)\}$ for all $x, y \neq 0$ in F.

Example 2.6. Let $F = Z_3 = \{0, 1, 2\}$ be a field with respect to the ordinary addition and multiplication. Then $A = \{\langle 0, 0.5, 0.8, 0.6, -0.6, -0.5, -0.7\rangle$ $\langle 1, 0.4, 0.7, 0.5, -0.5, -0.4, -0.6\rangle, \langle 2, 0.4, 0.7, 0.5, -0.5, -0.4, -0.6\rangle\}$ is a bipolar valued multi fuzzy subfield of order 3 in F.

Definition 2.7 [14]**.** Let $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^- \rangle$ be a bipolar valued multi fuzzy subset of X. Then the height $H(A) = \langle H(A_1^+), H(A_2^+), \ldots, H(A_n^+), H(A_1^-), H(A_2^-), \ldots, H(A_n^-)\rangle$ is defined as $H(A_i^+) = \sup A_i^+(x)$ for all x in X and $H(A_i^-) = \inf A_i^-(x)$ for all x in X and for all i.

Definition 2.8 [14]**.** Let $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^- \rangle$ be a bipolar valued multi fuzzy subset of X. Then ${}^0A = \langle {}^0A_1^+, {}^0A_2^+, \ldots,$

${}^{0}A_n^+, {}^{0}A_1^-, {}^{0}A_2^-, \ldots, {}^{0}A_n^-\rangle$ is defined as ${}^{0}A_i^+(x) = A_i^+(x)H(A_i^+)$ for all x in X and ${}^{0}A_i^-(x) = -A_i^-(x)H(A_i^-)$ for all x in X and for all i.

Definition 2.9 [14]. Let $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^-\rangle$ be a bipolar valued multi fuzzy subset of X. Then ${}^{\Delta}A = \langle {}^{\Delta}A_1^+, {}^{\Delta}A_2^+, \ldots, {}^{\Delta}A_n^+, {}^{\Delta}A_1^-, {}^{\Delta}A_2^-, \ldots, {}^{\Delta}A_n^-\rangle$ is defined as ${}^{\Delta}A_i^+(x) = A_i^+(x)/H(A_i^+)$ for all x in X and ${}^{\Delta}A_i^-(x) = -A_i^-(x)/H(A_i^-)$ for all x in X and for all i.

Definition 2.10 [14]. $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^-\rangle$ be a bipolar valued multi fuzzy subset of X. Then ${}^{\oplus}A = \langle {}^{\oplus}A_1^+, {}^{\oplus}A_2^+, \ldots, {}^{\oplus}A_n^+, {}^{\oplus}A_1^-, {}^{\oplus}A_2^-, \ldots, {}^{\oplus}A_n^-\rangle$ is defined as ${}^{\oplus}A_i^+(x) = A_i^+(x) + 1 - H(A_i^+)$ for all x in X and ${}^{\oplus}A_i^-(x) = A_i^-(x) - 1 - H(A_i^-)$ for all x in X and for all i.

Definition 2.11 [14]. Let If $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^-\rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_n^+, B_1^-, B_2^-, \ldots, B_n^-\rangle$ be any two bipolar valued multi fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by $A \times B$, is defined as $A \times B = \{\langle (x, y), (A_1 \times B_1)^+(x, y), (A_2 \times B_2)^+(x, y), \ldots, (A_n \times B_n)^+(x, y), (A_1 \times B_1)^-(x, y), (A_2 \times B_2)^-(x, y), \ldots, (A_n \times B_n)^-(x, y)\rangle$ for all x in G and y in $H\}$ where $(A_i \times B_i)^+(x, y) = \min\{A_i^+(x), B_i^+(y)\}$ and $(A_i \times B_i)^-(x, y) = \max\{A_i^-(x), B_i^-(y)\}$ for all x in G and y in H and for all i.

Definition 2.11 [14]. Let $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^-\rangle$ be a bipolar valued multi fuzzy subset in a set S, the strongest bipolar valued multi fuzzy relation on S, that is a bipolar valued multi fuzzy relation on A is $V = \{\langle (x, y), V_1^+(x, y), V_2^+(x, y), \ldots, V_n^+(x, y), V_1^-(x, y), V_2^-(x, y), \ldots, V_n^-(x, y)\rangle / x$ and y in $S\}$ given by $V_i^+(x, y) = \min\{A_i^+(x), A_i^+(y)\}$ and $V_i^-(x, y) = \max\{A_i^-(x), A_i^-(y)\}$ for all x and y in S and for all i.

3. Properties

Theorem 3.1. *If A and B are any two bipolar valued multi fuzzy subfield of a field* F_1 *and* F_2 *respectively, then* $A \times B$ *is a bipolar valued multi fuzzy subfield of* $F_1 \times F_2$.

Proof. Let x_1 and x_2 be in F_1, y_1 and y_2 be in F_2. Then (x_1, y_1) and (x_2, y_2) are in $F_1 \times F_2$. Now $(A_i \times B_i)^+[(x_1, y_1) - (x_2, y_2)] = (A_i \times B_i)^+$ $[(x_1 - x_2), (y_1 - y_2)] = \min\{A_i^+(x_1 - x_2), B_i^+(y_1 - y_2)\} \geq \min\{\min\{A_i^+(x_1),$ $A_i^+(x_2)\}, \min\{B_i^+(y_1), B_i^+(y_2)\} = \min\{\min\{A_i^+(x_1), B_i^+(y_1)\}, \min\{A_i^+(x_2), B_i^+(y_2)\}\}$ $= \{\min\{(A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2)\}$ for all i. Therefore $(A_i \times B_i)^+[(x_1, y_1) - (x_2, y_2)] \geq \min\{(A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2)\}$ for all (x_1, y_1) and (x_2, y_2) in $F_1 \times F_2$ and for all i. And $(A_i \times B_i)^-[(x_1, y_1) - (x_2, y_2)] = (A_i \times B_i)^-(x_1 - x_2, y_1 - y_2)$ $= \max\{A_i^-(x_1 - x_2), B_i^-(y_1 - y_2)\} \leq \max\{\max\{A_i^-(x_1), A_i^-(x_2)\},$ $\max\{B_i^-(y_1), B_i^-(y_2)\}\} = \max\{\max\{A_i^-(x_1)B_i^-(y_1)\}, \max\{A_i^-(x_2)B_i^-(y_2)\}\}$ $= \max\{(A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2)\}$ for all i. Therefore $(A_i \times B_i)^-[(x_1, y_1) - (x_2, y_2)] \leq \max\{(A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2)\}$ for all (x_1, y_1) and (x_2, y_2) in $F_1 \times F_2$ and for all i. Also $(A_i \times B_i)^+[(x_1, y_1)(x_2, y_2)^{-1}] = (A_i \times B_i)^+(x_1 x_2^{-1}, y_1 y_2^{-1})$ $= \min\{A_i^+(x_1 x_2^{-1}), B_i^+(y_1 y_2^{-1})\} \geq \min\{\min\{A_i^+(x_1), A_i^+(x_2)\},$ $\min\{B_i^+(y_1), B_i^+(y_2)\}\} = \min\{\min\{A_i^+(x_1), B_i^+(y_1)\}, \min\{A_i^+(x_2), B_i^+(y_2)\}\}$ $= \min\{(A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2)\}$ for all i. Therefore $(A_i \times B_i)^+[(x_1, y_1)(x_2, y_2)^{-1}] \geq \min\{(A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2)\}$ for all (x_1, y_1) and (x_2, y_2) in $F_1 \times F_2$ and for all i. And $(A_i \times B_i)^-[(x_1, y_1)(x_2, y_2)^{-1}] = (A_i \times B_i)^-(x_1 x_2^{-1}, y_1 y_2^{-1})$ $= \max\{A_i^-(x_1 x_2^{-1}), B_i^-(y_1 y_2^{-1})\} \leq \max\{\max\{A_i^-(x_1), A_i^-(x_2)\},$ $\max\{B_i^-(y_1), B_i^-(y_2)\}\} = \max\{\max\{A_i^+(x_1), B_i^+(y_1)\}, \max\{A_i^-(x_2), B_i^-(y_2)\}\}$ $= \max\{(A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2)\}$ for all i. Therefore $(A_i \times B_i)^-[(x_1, y_1)(x_2, y_2)^{-1}] \leq \max\{(A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2)\}$ for all (x_1, y_1) and (x_2, y_2) in $F_1 \times F_2$ and for all i. Hence $A \times B$ is a bipolar valued multi fuzzy subfield of $F_1 \times F_2$.

Theorem 3.2. *Let A be a bipolar valued multi fuzzy subset of a sub field F and* $V = \langle V_1^+, V_2^+, \ldots, V_n^+, V_1^-, V_2^-, \ldots, V_n^- \rangle$ *be the strongest bipolar valued multi fuzzy relation of F. If A is a bipolar valued multi fuzzy subfield of F, then V is a bipolar valued multi fuzzy subfield of* $F \times F$.

Proof. Suppose that A is a bipolar valued multi fuzzy subfield of F. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $F \times F$. We have $V_i^+(x - y) = V_i^+[(x_1, x_2) - (y_2, y_2)] = V_i^+(x_1 - y_1, x_2 - y_2) = \min\{A_i^+(x_1 - y_1), A_i^+(x_2 - y_2)\} \geq \min\{\min\{A_i^+(x_1), A_i^+(y_1)\}, \min\{A_i^+(x_2), A_i^+(y_2)\}\} = \min\{\min\{A_i^+(x_1), A_i^+(x_2)\}, \min\{A_i^+(y_1), A_i^+(y_2)\}\} = \min\{V_i^+(x_1, x_2), V_i^+(y_1, y_2)\} = \min\{V_i^+(x), V_i^+(y)\}$ for all i. Therefore $V_i^+(x - y) \geq \min\{V_i^+(x), V_i^+(y)\}$ for all x and y in $F \times F$ and for all i. And $V_i^-(x - y) = V_i^-[(x_1, x_2) - (y_1, y_2)] = V_i^-(x_1 - y_1, x_2 - y_2) = \max\{A_i^-(x_1 - y_1), A_i^-(x_2 - y_2)\} \leq \max\{\max\{A_i^-(x_1), A_i^-(y_1)\}, \max\{A_i^-(x_2), A_i^-(y_2)\} = \max\{\max\{A_i^-(x_1), A_i^-(x_2)\}, \max\{A_i^-(y_1), A_i^-(y_2)\}\} = \max\{V_i^-(x_1, x_2), V_i^-(y_1, y_2)\} = \max\{V_i^-(x), A_i^-(y)\}$ for all i. Therefore $V_i^-(x - y) \leq \max\{A_i^-(x), V_i^-(y)\}$ for all x and y in $F \times F$ and for all i. Also we have $V_i^+(xy^{-1}) = V_i^+[(x_1, x_2)(y_1, y_2)^{-1}] = V_i^+[(x_1 y_1^{-1}, x_2 y_2^{-1}) = \min\{A_i^+(x_1 y_1^{-1}), A_i^+(x_2 y_2^{-1})\} \geq \min\{\min\{A_i^+(x_1), A_i^+(y_1)\}, \min\{A_i^+(x_2), A_i^+(y_2)\} = \min\{\min\{A_i^+(x_1), A_i^+(x_2)\}, \min\{A_i^+(y_1), A_i^+(y_2)\}\} = \min\{V_i^+(x_1, x_2), V_i^+(y_1, y_2)\} = \min\{V_i^+(x), V_i^+(y)\}$ for all i. Therefore $V_i^+(xy^{-1}) \geq \min\{V_i^+(x), V_i^+(y)\}$ for all x and $y \neq 0$ in $F \times F$ and for all i. And $V_i^-(xy^{-1}) = V_i^-[(x_1, x_2)(y_1, y_2)^{-1}] = V_i^-[(x_1 y_1^{-1}, x_2 y_2^{-1}) = \max\{A_i^-(x_1 y_1^{-1}), A_i^-(x_2 y_2^{-1})\} \leq \max\{\max\{A_i^-(x_1), A_i^-(y_1)\}, \max\{A_i^-(x_2), A_i^-(y_2)\}\} = \max\{\max\{A_i^-(x_1), A_i^-(x_2)\}, \max\{A_i^-(y_1), A_i^-(y_1)\}\} = \max\{V_i^-(x_1, x_2), V_i^-(y_1, y_2)\} = \max\{V_i^-(x), V_i^-(y)\}$ for all i. Therefore $V_i^-(xy^{-1}) \leq \max\{V_i^-(x), V_i^-(y)\}$ for all x and $y \neq 0$ in $F \times F$ and for all i. Hence V is a bipolar valued multi fuzzy subfield of $F \times F$.

Theorem 3.3. *If A is a bipolar valued multi fuzzy subfield of a field F, then* $^{\oplus}A$ *is a bipolar valued multi fuzzy subfield of a field F.*

Proof. Let x and y in F. We have ${}^{\oplus}A_i^+(x-y) = A_i^+(x-y) + 1 - H(A_i^+) \geq \min\{A_i^+(x), A_i^+(y)\} + 1 - H(A_i^+) = \min\{A_i^+(x) + 1 - H(A_i^+), A_i^+(y) + 1 - H(A_i^+)\} = \min\{{}^{\oplus}A_i^+(x), {}^{\oplus}A_i^+(y)\}$ which implies ${}^{\oplus}A_i^+(x-y) \geq \min\{{}^{\oplus}A_i^+(x), {}^{\oplus}A_i^+(y)\}$ for all x and y in F and for all i. And ${}^{\oplus}A_i^-(x-y) = A_i^-(x-y) - 1 - H(A_i^-) \leq \max\{A_i^-(x), A_i^-(y)\} - 1 - H(A_i^-) = \max\{A_i^-(x) - 1 - H(A_i^-), A_i^-(y) - 1 - H(A_i^-)\} = \max\{{}^{\oplus}A_i^-(x), {}^{\oplus}A_i^-(y)\}$ which implies ${}^{\oplus}A_i^-(x-y) \leq \max\{{}^{\oplus}A_i^-(x), {}^{\oplus}A_i^-(y)\}$ for all x and y in F and for all i. Also ${}^{\oplus}A_i^+(xy^{-1}) = A_i^+(xy^{-1}) + 1 - H(A_i^+) \geq \min\{A_i^+(x), A_i^+(y)\} + 1 - H(A_i^+) = \min\{A_i^+(x) + 1 - H(A_i^+), A_i^+(y) + 1 - H(A_i^+)\} = \min\{{}^{\oplus}A_i^+(x), {}^{\oplus}A_i^+(y)\}$ which implies ${}^{\oplus}A_i^+(xy^{-1}) \geq \min\{{}^{\oplus}A_i^+(x), {}^{\oplus}A_i^+(y)\}$ for all x and $y \neq 0$ in F and for all i. And ${}^{\oplus}A_i^-(xy^{-1}) = A_i^-(xy^{-1}) - 1 - H(A_i^+) \leq \max\{A_i^-(x), A_i^-(y)\} - 1 - H(A_i^-) = \max\{A_i^-(x) - 1 - H(A_i^-), A_i^-(y) - 1 - H(A_i^-)\} = \max\{{}^{\oplus}A_i^-(x), {}^{\oplus}A_i^-(y)\}$ which implies ${}^{\oplus}A_i^+(xy^{-1}) \leq \max\{{}^{\oplus}A_i^-(x), {}^{\oplus}A_i^-(y)\}$ for all x and $y \neq 0$ in F and for all i. Hence ${}^{\oplus}A$ is a bipolar valued multi fuzzy subfield of a field F.

Theorem 3.4. *Let A be a bipolar valued multi fuzzy subfield of a field F. Then*

(i) $H(A_i^+) = 1$ *if and only if* ${}^{\oplus}A_i^+(x) = A_i^+(x)$ *for all x in F*

(ii) $H(A_i^-) = -1$ *if and only if* ${}^{\oplus}A_i^-(x) = A_i^-(x)$ *for all x in F.*

(iii) ${}^{\oplus}A_i^+(x) = 1$ if and only if $H(A_i^+) = A_i^+(x)$ for all x in F

(iv) ${}^{\oplus}A_i^-(x) = -1$ *if and only if* $H(A_i^-) = A_i^-(x)$ *for all x in F.*

(v) ${}^{\oplus}({}^{\oplus}A) = {}^{\oplus}A.$

Proof. It is trivial.

Theorem 3.5. *If A is a bipolar valued multi fuzzy subfield of a field F, then* ${}^{0}A$ *is a bipolar valued multi fuzzy subfield of a field F.*

Proof. For any x in F, we have ${}^{0}A_i^{+}(x-y) = A_i^{+}(x-y)H(A_i^{+}) \geq \min\{A_i^{+}(x), A_i^{+}(y)\}H(A_i^{+}) = \min\{A_i^{+}(x)H(A_i^{+}), A_i^{+}(y)H(A_i^{+})\} = \min\{{}^{0}A_i^{+}(x), {}^{0}A_i^{+}(y)\}$ which implies that ${}^{0}A_i^{+}(x-y) \geq \min\{{}^{0}A_i^{+}(x), {}^{0}A_i^{+}(y)\}$ for all x, y in F and for all i. And ${}^{0}A_i^{-}(x-y) = -A_i^{-}(x-y)H(A_i^{-}) \leq (-)\max\{A_i^{-}(x), A_i^{-}(y)\}H(A_i^{-}) = \max\{-A_i^{-}(x)H(A_i^{-}), -A_i^{-}(y)H(A_i^{-})\} = \max\{{}^{0}A_i^{-}(x), {}^{0}A_i^{-}(y)\}$ which implies that ${}^{0}A_i^{-}(x-y) \leq \max\{A_i^{-}(x), A_i^{-}(y)\}$ for all x, y in F and for all i. Also ${}^{0}A_i^{+}(xy^{-1}) = A_i^{+}(xy^{-1})H(A_i^{+}) \geq \min\{A_i^{+}(x), A_i^{+}(y)\}H(A_i^{+}) = \min\{A_i^{+}(x)H(A_i^{+}), A_i^{+}(y)H(A_i^{+})\} = \min\{{}^{0}A_i^{+}(x), {}^{0}A_i^{+}(y)\}$. Therefore ${}^{0}A_i^{+}(xy^{-1}) \geq \min\{{}^{0}A_i^{+}(x), {}^{0}A_i^{+}(y)\}$ for all x and $y \neq 0$ in F and for all i. And ${}^{0}A_i^{-}(xy^{-1}) = -A_i^{-}(xy^{-1})H(A_i^{-}) \leq (-)\max\{A_i^{-}(x), A_i^{-}(y)\}H(A_i^{-}) = \max\{-A_i^{-}(x), H(A_i^{-}), -A_i^{-}(y)H(A_i^{-})\} = \max\{{}^{0}A_i^{-}(x), {}^{0}A_i^{-}(y)\}$. Therefore ${}^{0}A_i^{+}(xy^{-1}) \leq \max\{{}^{0}A_i^{-}(x), {}^{0}A_i^{-}(y)\}$ for all x and $y \neq 0$ in F and for all i. Hence ${}^{0}A$ is a bipolar valued multi fuzzy subfield of a field F.

Theorem 3.6. *If A is a bipolar valued multi fuzzy subfield of a field F, then ${}^{\Delta}A$ is a bipolar valued multi fuzzy subfield of F.*

Proof. For any x in F, we have ${}^{\Delta}A_i^{+}(x-y) = A_i^{+}(x-y)/H(A_i^{+}) \geq \min\{A_i^{+}(x), A_i^{+}(y)\}/H(A_i^{+}) = \min\{A_i^{+}(x)/H(A_i^{+}), A_i^{+}(y)/H(A_i^{+})\} = \min\{{}^{\Delta}A_i^{+}(x), {}^{\Delta}A_i^{+}(y)\}$ which implies that ${}^{\Delta}A_i^{+}(x-y) \geq \min\{{}^{\Delta}A_i^{+}(x), {}^{\Delta}A_i^{+}(y)\}$ for all x, y in F and for all i. And ${}^{\Delta}A_i^{-}(x-y) = -A_i^{-}(x-y)/H(A_i^{-}) \leq \max\{A_i^{-}(x), A_i^{-}(y)\}/H(A_i^{-}) = \max\{-A_i^{-}(x)/H(A_i^{-}), -A_i^{-}(y)/H(A_i^{-})\} = \max\{{}^{\Delta}A_i^{-}(x), {}^{\Delta}A_i^{-}(y)\}$ which implies that ${}^{\Delta}A_i^{-}(x-y) \leq \max\{{}^{\Delta}A_i^{-}(x), {}^{\Delta}A_i^{-}(y)\}$ for all x, y in F and for all i. Also ${}^{\Delta}A_i^{+}(xy^{-1}) = A_i^{+}(xy^{-1})/H(A_i^{+}) \geq \min\{A_i^{+}(x), A_i^{+}(y)\}/H(A_i^{+}) = \min\{A_i^{+}(x)/H(A_i^{+}), A_i^{+}(y)/H(A_i^{+})\} = \min\{{}^{\Delta}A_i^{+}(x), {}^{\Delta}A_i^{+}(y)\}$. Therefore ${}^{\Delta}A_i^{+}(xy^{-1}) \geq \min\{{}^{\Delta}A_i^{+}(x), {}^{\Delta}A_i^{+}(y)\}$ for all x, $y \neq 0$ in F and for all i. And ${}^{\Delta}A_i^{-}(xy^{-1}) = -A_i^{-}(xy^{-1})/H(A_i^{-}) \leq (-)\max\{A_i^{-}(x), A_i^{-}(y)\}/H(A_i^{-}) = \max\{-A_i^{-}(x)$

$/H(A_i^-), -A_i^-(y)/H(A_i^-)\} = \max\{{}^{\Delta}A_i^-(x), {}^{\Delta}A_i^-(y)\}$. Therefore ${}^{\Delta}A_i^+(xy^{-1}) \leq \max\{{}^{\Delta}A_i^-(x), {}^{\Delta}A_i^-(y)\}$ for all x and $y \neq 0$ in F and for all i. Hence ${}^{\Delta}A$ is a bipolar valued multi fuzzy subfield of a field F.

Theorem 3.7. *Let A be a bipolar valued multi fuzzy subfield of a field F,*

(i) *If* $H(A_i^+) < 1$, *then* ${}^{0}A_i^+ < A_i^+$.

(ii) $H(A_i^-) > -1$, *then* ${}^{0}A_i^- > A_i^-$.

(iii) $H(A_i^+) < 1$, *and* $H(A_i^-) > -1$, *then* ${}^{0}A < A$.

Proof. It is trivial.

4. (λ, μ)- Level Subsets of Bipolar Valued Multi Fuzzy Subfields

Definition 4.1. Let A be a bipolar valued multi fuzzy subset of X. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, in $[0, 1]$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n), \mu_i$ in $[-1, 0]$, then the (λ, μ)- level subset of A is the set $A_{(\lambda, \mu)} = \{x \in X : A_i^+(x) \geq \lambda_i$ and $A_i^-(x) \leq \mu_i$ for all $i\}$.

Example 4.2. Consider the set $X = \{0, 1, 2, 3, 4\}$. Let $A = \{(0, 0.5, 0.6, 0.3, -0.1, -0.5, -0.7), (1, 0.4, 0.5, 0.8, -0.3, -0.5, -0.6), (2, 0.6, 0.4, 0.8, -0.05, -0.4, -0.5), (3, 0.45, 0.6, 0.9, -0.2, -0.4, -0.7), (4, 0.2, 0.4, 0.5, -0.5, -0.6, -0.7)\}$ be a bipolar valued multi fuzzy subset of X and $\lambda_1 = 0.4, \lambda_2 = 0.3, \lambda_3 = 0.4, \mu_1 = -0.1, \mu_2 = -0.2, \mu_3 = -0.1$. Then $((0.4, 0.3, 0.4), (-0.1, -0.2, -0.1))$- level subset of A is $A_{((0.4, 0.3, 0.4, -0.1, -0.2, -0.1))} = \{1, 3\}$.

Definition 4.3. Let A be a bipolar valued multi fuzzy subset of X. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_i$ in $[0, 1]$, the A^+-level λ-cut of A is the set $P(A^+, \lambda) = \{x \in X : A_i^+(x) \geq \lambda_i$ for all $i\}$.

Example 4.4. Consider the set $X = \{0, 1, 2, 3, 4\}$. Let $A = \{(0, 0.5, 0.6, 0.3, -0.1, -0.5, -0.7), (1, 0.4, 0.5, 0.8, -0.3, -0.5, -0.6), (2, 0.6, 0.4, 0.8, -0.05, -0.4, -0.5), (3, 0.45, 0.6, 0.9, -0.2, -0.4, -0.7), (4, 0.2, 0.4, 0.5, -0.5, -0.6, -0.7)\}$ be a bipolar valued multi fuzzy subset of X and $\lambda_1 = 0.4, \lambda_2 = 0.3, \lambda_3 = 0.4$. Then A^+-level $(0.4, 0.3, 0.4)$-cut of A is $P(A_i^+, (0.4, 0.3, 0.4)) = \{1, 2, 3\}$.

Definition 4.5. Let A be a bipolar valued multi fuzzy subset of X. For $\mu_1 = (\mu_1, \mu_2, \ldots, \mu_n)$, μ_i in $[-1, 0]$, the A^--level μ-cut of A is the set $N(A^-, \mu) = \{x \in X : A_i^-(x) \leq \mu_i$ for all $i\}$.

Example 4.6. Consider the set $X = \{0, 1, 2, 3, 4\}$. Let $A = \{(0, 0.5, 0.6, 0.3, -0.1, -0.5, -0.7), (1, 0.4, 0.5, 0.8, -0.3, -0.5, -0.6), (2, 0.6, 0.4, 0.8, -0.05, -0.4, -0.5), (3, 0.45, 0.6, 0.9, -0.2, -0.4, -0.7), (4, 0.2, 0.4, 0.5, -0.5, -0.6, -0.7)\}$ be a bipolar valued multi fuzzy subset of X and $\mu_1 = -0.1$, $\mu_2 = -0.2$, $\mu_3 = -0.1$. Then A^--level $(-0.1, -0.2, -0.1)$-cut of A is $N(A_i^-, (-0.1, -0.2, -0.1)) = \{0, 1, 3, 4\}$.

Theorem 4.7. *Let A be a bipolar valued multi fuzzy subfield of a field F. Then for λ_i in $[0, 1]$ and μ_i in $[-1, 0]$ such that $\lambda_i \leq A_i^+(e)$ and $\mu_i \geq A_i^-(e)$ for all i, $A_{(\lambda, \mu)}$ is a (λ, μ)-level subfield of F.*

Proof. For all x and y in $A_{(\lambda,\mu)}$, we have, $A_i^+(x) \geq \lambda_i$ and $A_i^-(x) \leq \mu_i$ and $A_i^+(y) \geq \lambda_i$ and $A_i^-(y) \leq \mu_i$ for all i. Now $A_i^+(x-y) \geq \min\{A_i^+(x), A_i^+(y)\} \geq \min\{\lambda_i, \lambda_i\} = \lambda_i$, which implies that $A_i^+(x-y) \geq \lambda_i$ for all i. And $A_i^+(xy^{-1}) \geq \min\{A_i^+(x), A_i^+(y)\} \geq \min\{\lambda_i, \lambda_i\} = \lambda_i$, which implies that $A_i^+(xy^{-1}) \geq \lambda_i$ for all i and for $x, y \neq 0$. Also $A_i^-(x-y) \leq \max\{A_i^-(x), A_i^-(y)\} \leq \max\{\mu_i, \mu_i\} = \mu_i$, which implies that $A_i^-(x-y) \leq \mu_i$ for all i. And for x and $y \neq 0$, we have $A_i^-(xy^{-1}) \leq \max\{A_i^-(x), A_i^-(y)\} \leq \max\{\mu_i, \mu_i\} = \mu_i$, which implies that $A_i^-(xy^{-1}) \leq \mu_i$ for all i. Therefore $x - y$, xy^{-1} in $A_{(\lambda, \mu)}$. Hence $A_{(\lambda, \mu)}$ is a (λ, μ)-level subfield of F.

Theorem 4.8. *Let A be a bipolar valued multi fuzzy subfield of a field F. Then for λ_i, γ_i in $[0, 1]$, $[-1, 0]$, μ_i, δ_i in $[0, 1]$, $\lambda_i \leq A_i^+(e)$,*

$\gamma_i \leq A_i^+(e)$, $\mu_i \geq A_i^-(e)$, $\delta_i \geq A_i^-(e)$, $\gamma_i < \lambda_i$ *and* $\mu_i < \delta_i$, *for all i, the two* (λ, μ)- *level subfields* $A_{(\lambda, \mu)}$ *and* $A_{(\gamma, \delta)}$ *of A are equal if and only if there is no x in F such that* $\lambda_i > A_i^+(x) > \gamma_i$ *and* $\mu_i < A_i^-(x) < \delta_i$ *for all i.*

Proof. Assume that $A_{(\lambda, \mu)} = A_{(\gamma, \delta)}$. Suppose there exists x in F such that $\lambda_i > A_i^+(x) > \gamma_i$ and $\mu_i < A_i^-(x) < \delta_i$ for all i. Then $A_{(\lambda, \mu)} \subseteq A_{(\gamma, \delta)}$ implies x belongs to $A_{(\gamma, \delta)}$, but not in $A_{(\lambda, \mu)}$. This is contradiction to $A_{(\lambda, \mu)} = A_{(\gamma, \delta)}$. Therefore there is no x in F such that $\lambda_i > A_i^+(x) > \gamma_i$ and $\mu_i < A_i^-(x) < \delta_i$ for all i. Conversely, if there is no x in F such that $\lambda_i > A_i^+(x) > \gamma_i$ and $\mu_i < A_i^-(x) < \delta_i$. Then $A_{(\lambda, \mu)} = A_{(\gamma, \delta)}$ (By the definition of (λ, μ)- level subset).

Theorem 4.9. *Let A be a bipolar valued multi fuzzy subfields of a subfield F. If any two* (λ, μ)- *level subfields of A belongs to F, then their intersection is also* (λ, μ)- *level subfield of A in F.*

Proof. Let λ_i, γ_i in $[0, 1]$, μ_i, δ_i in $[-1, 0]$, $\lambda_i \leq A_i^+(e)$, $\gamma_i \leq A_i^+(e)$, $\mu_i \geq A_i^-(e)$, $\delta_i \geq A_i^-(e)$, for all i.

Case (i). If $\lambda_i > A_i^+(x) > \gamma_i$ and $\mu_i < A_i^-(x) < \delta_i$ for all i, then $A_{(\lambda, \mu)} \subseteq A_{(\gamma, \delta)}$.

Therefore $A_{(\lambda,\mu)} \cap A_{(\gamma,\delta)} = A_{(\lambda,\mu)}$, but $A_{(\lambda,\mu)}$ is a (λ, μ)- level subfield of A.

Case (ii). If $\lambda_i < A_i^+(x) < \gamma_i$ and $\mu_i > A_i^-(x) > \delta_i$ for all i, then $A_{(\gamma, \delta)} \subseteq A_{(\lambda, \mu)}$.

Therefore $A_{(\lambda,\mu)} \cap A_{(\gamma,\delta)} = A_{(\gamma,\delta)}$, but $A_{(\gamma,\delta)}$ is a (λ, μ)- level subfield of A.

Case (iii). If $\lambda_i < A_i^+(x) < \gamma_i$ and $\mu_i < A_i^-(x) < \delta_i$ for all i, then $A_{(\gamma, \mu)} \subseteq A_{(\lambda, \delta)}$.

Therefore $A_{(\gamma,\mu)} \cap A_{(\lambda,\delta)} = A_{(\gamma,\mu)}$, but $A_{(\gamma,\mu)}$ is a (λ, μ)- level subfield of A.

Case (iv). If $\lambda_i > A_i^+(x) < \gamma_i$ and $\mu_i > A_i^-(x) > \delta_i$ for all i, then $A_{(\lambda, \delta)} \subseteq A_{(\gamma, \mu)}$.

Therefore $A_{(\lambda,\delta)} \cap A_{(\gamma,\mu)} = A_{(\lambda,\delta)}$, but $A_{(\lambda,\delta)}$ is a (λ, μ)- level subfield of A.

Case (v). If $\lambda_i = \gamma_i$ and $\mu_i = \delta_i$, then $A_{(\lambda, \mu)} = A_{(\gamma, \delta)}$.

The other cases are true, so, in all the cases, intersection of any two (λ, μ)- level subfields is a (λ, μ)- level subfield of A.

Theorem 4.10. *Let A be a bipolar valued multi fuzzy subfield of a field F. The intersection of a collection of* (λ, μ)- *level subfields of A is also a* (λ, μ)- *level subfield of A.*

Proof. It is trivial.

Theorem 4.11. *Let A be a bipolar valued multi fuzzy subfield of a field F. If any two* (λ, μ)- *level subfields of A belongs to F, then their union is also* (λ, μ)- *level subfield of A in F.*

Proof. Let λ_i, γ_i in $[0, 1]$, μ_i, δ_i in $[-1, 0]$, $\lambda_i \leq A_i^+(e)$, $\gamma_i \leq A_i^+(e)$, $\mu_i \geq A_i^-(e)$, $\delta_i \geq A_i^-(e)$ for all i.

Case (i). If $\lambda_i > A_i^+(x) > \gamma_i$ and $\mu_i < A_i^-(x) < \delta_i$ for all i, then $A_{(\lambda, \mu)} \subseteq A_{(\gamma, \delta)}$.

Therefore $A_{(\lambda,\mu)} \cup A_{(\gamma,\delta)} = A_{(\gamma,\delta)}$, but $A_{(\lambda,\delta)}$ is a (λ, μ)- level subfield of A.

Case (ii). If $\lambda_i < A_i^+(x) < \gamma_i$ and $\mu_i > A_i^-(x) > \delta_i$ for all i, then $A_{(\gamma, \delta)} \subseteq A_{(\lambda, \mu)}$.

Therefore $A_{(\lambda,\mu)} \cup A_{(\gamma,\delta)} = A_{(\lambda,\mu)}$, but $A_{(\lambda,\mu)}$, is a (λ, μ)- level subfield of A.

Case (iii). If $\lambda_i < A_i^+(x) < \gamma_i$ and $\mu_i < A_i^-(x) < \delta_i$ for all i, then $A_{(\gamma, \mu)} \subseteq A_{(\lambda, \delta)}$.

Therefore $A_{(\gamma,\mu)} \cup A_{(\lambda,\delta)} = A_{(\lambda,\delta)}$, but $A_{(\lambda,\delta)}$ is a (λ, μ)- level subfield of A.

Case (iv). If $\lambda_i > A_i^+(x) > \gamma_i$ and $\mu_i > A_i^-(x) > \delta_i$ for all i, then $A_{(\lambda, \delta)} \subseteq A_{(\gamma, \mu)}$.

Therefore $A_{(\lambda,\delta)} \cup A_{(\gamma,\mu)} = A_{(\gamma,\mu)}$, but $A_{(\lambda,\mu)}$ is a (λ, μ)- level subfield of A.

Case (v). If $\lambda_i = \gamma_i$ and $\mu_i = \delta_i$, then $A_{(\lambda,\mu)} = A_{(\gamma,\delta)}$.

In other cases are true, so, in all the cases, union of any two (λ, μ)- level subfield is a (λ, μ)- level subfield of A.

Theorem 4.12. *Let A be a bipolar valued multi fuzzy subfield of a field F. The union of a collection of* (λ, μ)- *level subfields of A is also a* (λ, μ)- *level subfield of A.*

Proof. It is trivial.

Theorem 4.13. *The homomorphic image of a* (λ, μ)- *level subfield of a bipolar valued multi fuzzy subfield of a field F is a* (λ, μ)- *level subfield of a bipolar valued multi fuzzy subfield of a field* F'.

Proof. Let $U = g(A)$. Here $A = \langle A_1^+, A_2^+, \ldots, A_n^+, \ldots, A_1^-, A_2^-, \ldots, A_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F, and $U = \langle U_1^+, U_2^+, \ldots, U_n^+, \ldots, U_1^-, U_2^-, \ldots, U_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F'. Let x and y in F. Then $g(x)$ and $g(y)$ in F'. Let $A_{(\lambda,\mu)}$ be a $(\lambda, \mu) = ((\lambda_1, \lambda_2, \ldots, \lambda_n), (\mu_1, \mu_2, \ldots, \mu_n))$- level subfield of A. That is, $A_i^+(x) \geq \lambda_i$ and $A_i^-(x) \leq \mu_i$; $A_i^+(y) \geq \lambda_i$ and $A_i^-(y) \leq \mu_i$; $A_i^+(x - y) \geq \lambda_i$, $A_i^-(x - y) \leq \mu_i$, $A_i^+(xy^{-1}) \geq \lambda_i$, $A_i^-(xy^{-1}) \leq \mu_i$ for all i. We have to prove that $g(A_{(\lambda,\mu)})$ is a (λ, μ)- level subfield of U. Now $U_i^+(g(x)) \geq A_i^+(x) \geq \lambda_i$ which implies that $U_i^+(g(x)) \geq \lambda_i$; and $U_i^+(g(y)) \geq A_i^+(y) \geq \lambda_i$ which implies that $U_i^+(g(y)) \geq \lambda_i$ for all i. Then $U_i^+(g(x) - g(y)) = U_i^+(g(x - y)) \geq A_i^+(x - y) \geq \lambda_i$, which implies that $U_i^+(g(x) - g(y)) \geq \lambda_i$ for all i. And $U_i^-(g(x)) \leq A_i^-(x) \leq \mu_i$ which implies that $U_i^-(g(x)) \leq \mu_i$; and $U_i^-(g(y)) \leq A_i^-(y) \leq \mu_i$ which implies that $U_i^-(g(y)) \leq \mu_i$ for all i. Then $U_i^-(g(x) - g(y)) = U_i^-(g(x - y))$

$\leq A_i^-(x-y) \leq \mu_i$, which implies that $U_i^-(g(x)-g(y)) \leq \mu_i$ for all i. And for all $U_i^+(g(x)) \geq \lambda_i$ and $U_i^+(g(y)) \geq \lambda_i$ for all i, $U_i^+(g(x)\,g(y)^{-1}) = U_i^+(g(xy^{-1}))$ $\geq A_i^+(xy^{-1}) \geq \lambda_i$, which implies that $U_i^+(g(x)g(y)^{-1}) \geq \lambda_i$ for all i. And for all $U_i^-(g(x)) \leq \mu_i$ and $U_i^-(g(y)) \leq \mu_i$ for all i, $U_i^-(g(x)g(y)^{-1}) = U_i^-(g(xy^{-1}))$ $\leq A_i^-(xy^{-1}) \leq \mu_i$, which implies that $U_i^-(g(x)g(y)^{-1}) \leq \mu_i$ for all i. Hence $g(A_{(\lambda, \mu)})$ is a (λ, μ)-level subfield of a bipolar valued multi fuzzy subfield U of F'.

Theorem 4.14. *The homomorphic pre-image of a* (λ, μ)*-level subfield of a bipolar valued multi fuzzy subfield of a field* F' *is a* (λ, μ)*-level subfield of a bipolar valued multi fuzzy subfield of a field F.*

Proof. Let $U = g(A)$. Here $U = \langle U_1^+, U_2^+, \ldots, U_n^+, U_1^-, U_2^- \ldots U_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F', and $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^- \ldots A_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F. Let $g(x)$ and $g(y)$ in F'. Then x and y in F. Let $g(A_{(\lambda, \mu)})$ be a (λ, μ)-level subfield of U. That is $U_i^+(g(x)) \geq \lambda_i$ and $U_i^-(g(x)) \leq \mu_i$; $U_i^+(g(y)) \geq \lambda_i$ and $U_i^-(g(y)) \leq \mu_i$; $U_i^+(g(x)-g(y)) \geq \lambda_i$, $U_i^-(g(x)-g(y)) \leq \mu_i, U_i^+(g(x)g(y)^{-1}) \geq \lambda_i, U_i^-(g(x)g(y)^{-1}) \leq \mu_i$ for all i. We have to prove that $A_{(\lambda, \mu)}$ is a (λ, μ)-level subfield of A. Now $A_i^+(x) = U_i^+(g(x)) \geq \lambda_i$ implies that $A_i^+(x) \geq \lambda_i$; $A_i^+(y) = U_i^+(g(y)) \geq \lambda_i$ implies that $A_i^+(y) \geq \lambda_i$ for all i. Then $A_i^+(x-y) = U_i^+(g(x-y)) = U_i^+(g(x)-g(y)) \geq \lambda_i$, which implies that $A_i^+(x-y) \geq \lambda_i$ for all i. And $A_i^-(x) = U_i^-(g(x)) \leq \mu_i$ implies that $A_i^-(x) \leq \mu_i$; $A_i^-(y) = U_i^-(g(y)) \leq \mu_i$ implies that $A_i^-(y) \leq \mu_i$ for all i. $A_i^-(x-y) = U_i^-(g(x-y)) = U_i^-(g(x)-g(y)) \leq \mu_i$, which implies that $A_i^-(x-y) \leq \mu_i$ for all i. And for $A_i^+(x) \geq \lambda_i$ and $A_i^+(y) \geq \lambda_i$ for all i, $A_i^+(xy^{-1}) = U_i^+(g(xy^{-1})) = U_i^+(g(x)g(y)^{-1}) \geq \lambda_i$ which implies that $A_i^+(xy^{-1}) \geq \lambda_i$ for all i. Also for all $A_i^-(x) \leq \mu_i$ and $A_i^-(y) \leq \mu_i$ for all

i, $A_i^-(xy^{-1}) = U_i^-(g(xy^{-1})) = U_i^-(g(x)g(y)^{-1}) \leq \mu_i$, which implies that $A_i^-(xy^{-1}) \leq \mu_i$ for all i. Hence $A_{(\lambda,\mu)}$ is a (λ, μ)- level subfield of bipolar valued multi fuzzy subfield A of F.

Theorem 4.15. *The anti-homomorphic image of a* (λ, μ)- *level subfield of a bipolar valued multi fuzzy subfield of a field F is a* (λ, μ)- *level subfield of a bipolar valued multi fuzzy subfield of a field* F'.

Proof. Let $U = g(A)$. Here $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^- \ldots A_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F, and $U = \langle U_1^+, U_2^+, \ldots, U_n^+, \ldots, U_1^-, U_2^-, \ldots, U_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F'. Let x and y in F. Then $g(x)$ and $g(y)$ in F'. Let $A_{(\lambda,\mu)}$ be a (λ, μ)- level subfield of A. That is $A_i^+(x) \geq \lambda_i$ and $A_i^-(x) \leq \mu_i$; $A_i^+(y) \geq \lambda_i$ and $A_i^-(y) \leq \mu_i$ for all i. And $A_i^+(y - x) \geq \lambda_i$ and $A_i^+(yx^{-1}) \geq \lambda_i$ and $A_i^-(y - x) \leq \mu_i$, $A_i^-(yx^{-1}) \leq \mu_i$ for all i. We have to prove that $g(A_{(\lambda,\mu)})$ is a (λ, μ)- level subfield of U. Now $U_i^+(g(x)) \geq A_i^+(x) \geq \lambda_i$ which implies that $U_i^+(g(x)) \geq \lambda_i$; and $U_i^+(g(y)) \geq A_i^+(y) \geq \lambda_i$ which implies that $U_i^+(g(y)) \geq \lambda_i$ for all i. Also $U_i^+(g(x) - g(y)) = U_i^+(g(y - x)) \geq A_i^+(y - x) \geq \lambda_i$ which implies that $U_i^+(g(x) - g(y)) \geq \lambda_i$ for all i. And $U_i^-(g(x)) \leq A_i^-(x) \leq \mu_i$ which implies that $U_i^-(g(x)) \leq \mu_i$; and $U_i^-(g(y)) \leq A_i^-(y) \leq \mu_i$ which implies that $U_i^-(g(y)) \leq \mu_i$ for all i. Also $U_i^-(g(x) - g(y)) = U_i^-(g(y - x)) \leq A_i^-(y - x) \leq \mu_i$ which implies that $U_i^-(g(x) - g(y)) \leq \mu_i$ for all i. For $U_i^+(g(x)) \geq \lambda_i$ and $U_i^+(g(y)) \geq \lambda_i$ for all i, And i, $U_i^+(g(x)g(y)^{-1}) = U_i^+(g(yx^{-1})) \geq A_i^+(yx^{-1})$ $\geq \lambda_i$ which implies that $U_i^+(g(x)g(y)^{-1}) \geq \lambda_i$ for all i. Also For $U_i^-(g(x)) \leq \mu_i$ and $U_i^-(g(y)) \leq \mu_i$ for all i. And $U_i^-(g(x)g(y)^{-1}) = U_i^-(g(yx^{-1}))$ $\leq A_i^-(yx^{-1}) \leq \mu_i$ which implies that $U_i^-(g(x)g(y)^{-1}) \leq \mu_i$ for all i. Hence $g(A_{(\lambda,\mu)})$ is a (λ, μ)- level subfield of bipolar valued multi fuzzy subfield U of F'.

Theorem 4.16. *The anti-homomorphic pre-image of a* (λ, μ)- *level subfield of a bipolar valued multi fuzzy subfield of a field* F' *is a* (λ, μ)- *level subfield of a bipolar valued multi fuzzy subfield of a field F.*

Proof. Let $U = g(A)$. Here $U = \langle U_1^+, U_2^+, \ldots, U_n^+, U_1^-, U_2^- \ldots U_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F', and $A = \langle A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^- \ldots A_n^- \rangle$ is a bipolar valued multi fuzzy subfield of F. Let $g(x)$ and $g(y)$ in F'. Then x and y in F. Let $g(A_{(\lambda, \mu)})$ be a (λ, μ)- level subfield of U. That is $U_i^+(g(x)) \geq \lambda_i$ and $U_i^-(g(x)) \leq \mu_i$; $U_i^+(g(y)) \geq \lambda_i$ and $U_i^-(g(y)) \leq \mu_i; U_i^+(g(y) - g(x)) \geq \lambda_i, U_i^-(g(y) - g(x)) \leq \mu_i$, $U_i^+(g(y)\, g(x)^{-1}) \geq \lambda_i, U_i^-(g(y)\, g(x)^{-1}) \leq \mu_i$ for all i. We have to prove that $A_{(\lambda, \mu)}$ is a (λ, μ)- level subfield of A. Now $A_i^+(x) = U_i^+(g(x)) \geq \lambda_i$ which implies that $A_i^+(x) \geq \lambda_i$ and $A_i^+(y) = U_i^+(g(y)) \geq \lambda_i$ which implies that $A_i^+(y) \geq \lambda_i$ for all i. Then $A_i^+(x - y) = U_i^+(g(x - y)) = U_i^+(g(y) - g(x)) \geq \lambda_i$ which implies that $A_i^+(x - y) \geq \lambda_i$ for all i. And $A_i^-(x) = U_i^-(g(x)) \leq \mu_i$ which implies that $A_i^-(x) \leq \mu_i$ and $A_i^-(y) = U_i^-(g(y)) \leq \mu_i$ which implies that $A_i^-(y) \leq \mu_i$ for all i. Also $A_i^-(x - y) = U_i^-(g(x - y)) = U_i^-(g(y) - g(x)) \leq \mu_i$ which implies that $A_i^-(x - y) \leq \mu_i$ for all i. For $A_i^+(x) \geq \lambda_i$ and $A_i^+(y) \geq \lambda_i$ for all i, $A_i^+(xy^{-1}) = U_i^+(g(xy^{-1})) = U_i^+(g(x)g(y)^{-1}) \geq \lambda_i$ which implies that $A_i^+(xy^{-1}) \geq \lambda_i$ for all i. And $A_i^-(x) \leq \mu_i$ and $A_i^-(y) \leq \mu_i$ for all i, $A_i^-(xy^{-1}) = U_i^-(g(xy^{-1})) = U_i^-(g(y)g(x)^{-1}) \leq \mu_i$, which implies that $A_i^-(xy) \leq \mu_i$ for all i. Hence $A_{(\lambda, \mu)}$ is a (λ, μ)- level subfield of bipolar valued multi fuzzy subfield A of F.

Theorem 4.17. *Let A be a bipolar valued multi fuzzy subfield of a field F. Then for* λ_i *in* $[0, 1]$ *for all* i, A_i^+*- level* λ*-cut* $P(A_i^+, \lambda)$ *is a* A_i^+*- level* λ*-cut subfield of F.*

Proof. For all x and y in $P(A_i^+, \lambda)$, we have $A_i^+(x) \geq \lambda_i$ and $A_i^+(y) \geq \lambda_i$ for all i. Now $A_i^+(x-y) \geq \min\{A_i^+(x), A_i^+(y)\} \geq \min\{\lambda_i, \lambda_i\} = \lambda_i$, which implies that $A_i^+(x-y) \geq \lambda_i$ for all i. And $A_i^+(xy^{-1}) \geq \min\{A_i^+(x), A_i^+(y)\} \geq \min\{\lambda_i, \lambda_i\} = \lambda_i$, which implies that $A_i^+(xy^{-1}) \geq \lambda_i$ for all i. Therefore $x - y$, xy^{-1} in $P(A_i^+, \lambda)$. Hence $P(A_i^+, \lambda)$ is a A_i^+-level λ-cut subfield of F.

Theorem 4.18. *Let A be a bipolar valued multi fuzzy subfield of a field F. Then for* μ_i *in* $[-1, 0]$ *for all* i, A_i^-*-level* μ*-cut* $N(A_i^-, \mu)$ *is a* A_i^-*-level* μ*-cut subfield of F.*

Proof. For all x and y in $N(A_i^-, \mu)$, we have $A_i^-(x) \leq \mu_i$ and $A_i^-(y) \leq \mu_i$ for all i. Now $A_i^-(x-y) \leq \max\{A_i^-(x), A_i^-(y)\} \leq \max\{\mu_i, \mu_i\} = \mu_i$, which implies that $A_i^-(x-y) \leq \mu_i$ for all i. And $A_i^-(xy^{-1}) \leq \max\{A_i^-(x), A_i^-(y)\}$ $\leq \max\{\mu_i, \mu_i\} = \mu_i$, which implies that $A_i^-(xy^{-1}) \leq \mu_i$ for all i. Therefore $x - y$, xy^{-1} in $N(A_i^-, \mu)$. Hence $N(A_i^-, \mu)$ is a A_i^--level μ-cut subfield of F.

References

[1] M. S. Anitha, Murugananatha Prasad and K. Arjunan, Homomorphism and Antihomomorphism of Bipolar-valued fuzzy subgroups of a group, International journal of Mathematical Archive 4(12) (2013), 274-276.

[2] Arsham Borumand Saeid, Bipolar-valued fuzzy BCK/BCI-algebras, World Applied Sciences Journal 7(11) (2009), 1404-1411.

[3] K. Chandrasekar Rao and V. Swaminathan, Anti-homomorphism in Fuzzy Ideals, World Academy of Science, engineering and Technology, 44 (2010).

[4] F. P. Choudhury, A. B. Charaborty and S. S. Khare , A note on fuzzy subgroups and fuzzy homomorphisms, Journal of Mathematical Analysis and Applications131 (1988), 537-553.

[5] Kyoung Ja Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc. (2) 32(3) (2009), 361-373.

[6] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, (2000), 307-312.

[7] K. M. Lee, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar valued fuzzy sets, J. fuzzy Logic Intelligent Systems, 14(2) (2004), 125-129.

[8] Sabu Sebastian and T. V. Ramakrishnan, Multi-fuzzy sets, International Mathematical Forum 6(50) (2010), 2471-2476.

[9] Samit Kumar Majumder, Bipolar Valued fuzzy Sets in Γ-Semigroups, Mathematica Aeterna 2(3) (2012), 203-213.

[10] V. K. Shanthi and G. Shyamala, Notes on Bipolar-valued multi fuzzy subgroups of a group, International journal of Mathematical Archive-6(6) (2015), 234-238.

[11] P. Sivaramakrishna Das, Fuzzy groups and level subgroups, Journal of Mathematical Analysis and Applications 84 (1981), 264-269.

[12] M. Vasu and D. Sivakumar, Lower Level Subsets of Anti *L*-Fuzzy Subfield of a Field, International Journal of Engineering Research & Technology (IJERT), vol 2, issue 9, September (2013).

[13] C. Yamini, K. Arjunan and B. Ananth, Bipolar valued multi fuzzy subfield of a field, International Journal of Management, Technology And Engineering, ISSN NO : 2249-7455,volume 8,issue XI, November (2018).

[14] B. Yasodara and K. E. Sathappan, Bipolar-valued multi fuzzy subsemirings of a semiring, International Journal of Mathematical Archive, 6(9) (2015), 75-80.

[15] B. Yasodara and K. E. Sathappa, Homomorphism and anti-homomorphism of bipolar-valued multi fuzzy subsemirings of a semiring, Bulletin of Mathematics and Statistics Research 3(3) (2015), 229-233.

[16] L. A. Zadeh, Fuzzy sets, Inform. And Control 8 (1965), 338-353.

[17] W. R. Zhang, Bipolar Fuzzy sets and Relations, a computational Frame work for cognitive modeling and multiple decision Analysis, proceedings of Fuzzy IEEE conferences, (1994), 305-309.

[18] W. R. Zhang, Bipolar Fuzzy sets, Proceedings of Fuzzy IEEE Conferences (1998), 835-840.