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ON A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL BOUNDARY CONDITIONS

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Abstract

This paper presents the following coupled system of fractional differential equations with fractional integral boundary conditions as well as integer and fractional derivative

$$\begin{cases} {}^cD^{\alpha_1}z_1(\xi) = h_1(\xi, z_2(\xi)), & \xi \in [0, 1] \\ {}^cD^{\alpha_2}z_2(\xi) = h_2(\xi, z_1(\xi)), & \xi \in [0, 1] \\ z_1(0) = z_1''(0) = 0, \\ ({}^cD^{q_1}z_1)(1) = \beta_1(J^{p_1}z_1)(1), \\ z_2(0) = z_2'(0) = z_2''(0) = 0, \\ ({}^cD^{q_2}z_2(1) = \beta_2(J^{p_2}z_2)(1), \end{cases}$$

where ${}^cD^{\alpha_i}$ are Caputo fractional derivative of order α_i, J^p denotes the Riemann-Liouville fractional integral of order $\alpha_1, \alpha_2 \in (3, 4]$, $q_1, q_2 \in (0, 3]$, $p_1, p_2 > 0$, $h_1, h_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\beta_i \neq \frac{\Gamma(p_i + 4)}{\Gamma(4 - q_i)}$, $i = 1, 2$. The main tools used are Banach fixed point theorem and Scheafer's fixed point theorem. To justify the results, we illustrate some examples.

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1. Introduction

Fractional differential equations are being used in various fields of science and engineering such as control system, electrochemistry, electromagnetics, viscoelasticity, physics, biophysics, porous media, blood flow phenomena, electrical circuits, biology, fitting of experimental data etc. Due to these features, models of fractional order become more practical and realistic than the models of integer order. There has been a significant development in the existence and uniqueness of boundary value problems for fractional differential equations; see, [1]-[11] and the references therein.

Many authors have studied the existence and uniqueness of solutions of coupled systems of fractional differential equations [12]-[19]. The coupled systems of fractional differential equations often exist in numerous models such as Chemostats and Microorganism Culturing, Brine Tank, Irregular Heartbeats, Chemical Kinetics, and Lidocaine and Pesticides, Predator Prey etc.

Cheng-Min Su et al. [20] study a boundary value problem of non-linear fractional differential equation with fractional integral boundary conditions as well as integer and fractional derivative. Motivated by the problem in [20], this paper is concerned with the existence and uniqueness of solutions for the following coupled system of nonlinear differential equations of fractional order

$$\begin{cases} {}^cD^{\alpha_1} z_1(\xi) = h_1(\xi, z_2(\xi)), \ \xi \in [0, 1] \\ {}^cD^{\alpha_2} z_2(\xi) = h_2(\xi, z_1(\xi)), \ \xi \in [0, 1] \\ z_1(0) = z_1'(0) = z_1''(0) = 0, \\ ({}^cD^{q_1} z_1)(1) = \beta_1 (J^{p_1} z_1)(1), \\ z_2(0) = z_2'(0) = z_2''(0) = 0, \\ ({}^cD^{q_2} z_2)(1) = \beta_2 (J^{p_2} z_2)(1), \end{cases} \tag{1}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative of order α, J^p is the Riemann-Liouville fractional integral of order p, $\alpha_1, \alpha_2 \in (3, 4]$, $q_1, q_2 \in (0, 3]$, $p_1, p_2 > 0$, $h_1, h_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\beta_i \neq \frac{\Gamma(p_i + 4)}{\Gamma(4 - q_i)}$, $i = 1, 2$.

2. Preliminaries

First of all, we introduce some notations, definitions and lemmas.

Definition 1. For a continuous function $h : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$ {}^{c}D^{\alpha} h(\xi) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{\xi} (\xi - s)^{n-\alpha-1} h^{(n)}(s) ds, \; n = [\alpha] + 1 $$

provided that $h^{(n)}(\xi)$ exists, where $[\alpha]$ denotes the integer part of the real number α and Γ is the Euler's Gamma function.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function $h : [0, \infty) \to \mathbb{R}$ is defined as

$$ J^{\alpha} h(\xi) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\xi} (\xi - s)^{\alpha-1} h(s) ds. $$

Lemma 3 ([21]). *Let* $m, n > 0$ *and* $h \in L_1[a, b]$ *then* $J^m J^n h = J^{m+n} h.$

Lemma 4 ([21]). *If h is continuous and* $n \geq 0$, *then*

$$ {}^{c}D^{n} J^{n} h = h. $$

It follows from Lemmas 3 and 4 that if h is continuous and $\beta > \alpha$, then ${}^{c}D^{\alpha} J^{\beta} h = J^{\beta-\alpha} h.$

Lemma 5 ([21]). *Let* $\beta > -1$ *and* $n > 0$. *Then*

$$ J^{n} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)} x^{n+\beta}. $$

Lemma 6 ([21]). *Let* $\beta \geq 0$ *and* $m = [n] + 1$, *then*

$$ {}^{c}D^{n} x^{\beta} = \begin{cases} 0, & \text{if } \beta \in \{0, 1, 2, \ldots, m-1\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} (x-a)^{\beta-n}, & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geq m \\ & \text{or } \beta \notin \mathbb{N} \text{ and } \beta > m-1. \end{cases} $$

Lemma 7 ([22]). *Let* $\alpha > 0$, *then*

$$ J^{\alpha}\, {}^{c}D^{\alpha} v(\xi) = v(\xi) + c_0 + c_1 \xi + c_2 \xi^2 + \ldots + c_{n-1} \xi^{n-1}, $$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, *where n is the smallest integer greater than or equal to* α.

3. Auxiliary Result

In this section, we present supporting result needed in our main proofs.

Lemma 8. *Let* $w \in C([0, 1], \mathbb{R})$ *and* $\beta \neq \frac{\Gamma(p+4)}{\Gamma(4-q)}$. *Then the problem*

$$\begin{cases} {}^cD^{\alpha}z(\xi) = w(\xi),\ \xi \in [0, 1] \\ z(0) = z'(0) = z''(0) = 0,\ ({}^cD^q z)(1) = \beta(J^p z)(1), \end{cases} \tag{2}$$

has a unique solution

$$z(\xi) = \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi - s)^{\alpha-1}w(s)ds$$

$$-\frac{\beta\Gamma(4-q)\Gamma(p+4)\xi^3}{6\Gamma(p+\alpha)[\beta\Gamma(4-q) - \Gamma(p+4)]}\int_0^1(1-s)^{p+\alpha-1}w(s)ds$$

$$+\frac{\Gamma(4-q)\Gamma(p+4)\xi^3}{6\Gamma(\alpha-q)[\beta\Gamma(4-q) - \Gamma(p+4)]}\int_0^1(1-s)^{\alpha-q-1}w(s)ds. \tag{3}$$

Proof. In view of Lemma 7, (2) is equivalent to

$$z(\xi) = J^{\alpha}w(\xi) - c_0 - c_1\xi - c_2\xi^2 - c_3\xi^3 \tag{4}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, 3$.

From $z(0) = 0$ it follows $c_0 = 0$. Also $z'(0) = 0 \Rightarrow c_1 = 0$ and $z''(0) = 0 \Rightarrow c_2 = 0$.

Thus (4) becomes

$$z(\xi) = J^{\alpha}w(\xi) - c_3\xi^3. \tag{5}$$

Now

$${}^cD^q z(\xi) = J^{\alpha-q}w(\xi) - c_3\frac{\Gamma(4)}{\Gamma(4-q)}\xi^{3-q}$$

$$J^p z(\xi) = J^{p+\alpha}w(\xi) - c_3\frac{\Gamma(4)}{\Gamma(4+p)}\xi^{p+3}.$$

From the boundary condition

$$({}^{c}D^{q}z)(1) = (J^{p}z)(1)$$

$$\Rightarrow J^{\alpha-q}w(1) - c_3\frac{\Gamma(4)}{\Gamma(4-q)} = \beta J^{p+\alpha}w(1) - \beta c_3\frac{\Gamma(4)}{\Gamma(4+p)}$$

$$\Rightarrow c_3\left[\frac{\Gamma(4)[\beta\Gamma(4-q) - \Gamma(p+4)]}{\Gamma(4-q)\Gamma(4+p)}\right] = \beta J^{p+\alpha}w(1) - J^{\alpha-q}w(1)$$

$$\Rightarrow c_3 = \frac{\Gamma(4-q)\Gamma(p+4)}{6[\beta\Gamma(4-q) - \Gamma(p+4)]}[\beta J^{p+\alpha}w(1) - J^{\alpha-q}w(1)].$$

On putting the value of c_3 in (5), we obtain the solution (3). □

It follows from Lemma 8 that the solution of coupled system (1) is given by the integral equations

$$z_1(\xi) = \frac{1}{\Gamma(\alpha_1)}\int_0^{\xi}(\xi - s)^{\alpha_1-1}h_1(s, z_2(s))ds$$

$$-\frac{\beta_1K_1\xi^3}{6\Gamma(p_1+\alpha_1)}\int_0^1(1-s)^{p_1+\alpha_1-1}h_1(s, z_2(s))ds$$

$$+\frac{K_1\xi^3}{6\Gamma(\alpha_1-q_1)}\int_0^1(1-s)^{\alpha_1-q_1-1}h_1(s, z_2(s))ds$$

$$z_2(\xi) = \frac{1}{\Gamma(\alpha_2)}\int_0^{\xi}(\xi - s)^{\alpha_2-1}h_2(s, z_1(s))ds$$

$$-\frac{\beta_2K_2\xi^3}{6\Gamma(p_2+\alpha_2)}\int_0^1(1-s)^{p_2+\alpha_2-1}h_2(s, z_1(s))ds$$

$$+\frac{K_2\xi^3}{6\Gamma(\alpha_2-q_2)}\int_0^1(1-s)^{\alpha_2-q_2-1}h_2(s, z_1(s))ds,$$

where

$$K_i = \frac{\Gamma(4-q_i)\Gamma(p_i+4)}{\beta_i\Gamma(4-q_i) - \Gamma(p_i+4)} \text{ for } i = 1, 2.$$

Let $Z = C[0, 1]$, then $(Z, \|\cdot\|_Z)$ is a Banach space equipped with the norm

$$\| z \|_Z = \{\sup | z(\xi) | : \xi \in [0, 1]\}.$$

Let $A = Z \times Z$. Then $(A, \|\cdot\|_A)$ is also a Banach space equipped with the norm

$$\| (z_1, z_2) \|_A = \| z_1 \|_Z + \| z_2 \|_Z.$$

Let us define an operator $P : A \to A$ as

$$P(z_1, z_2)(\xi) = (P_1 z_2(\xi), P_2 z_1(\xi)), \tag{6}$$

where

$$P_1 z_2(\xi) = \frac{1}{\Gamma(\alpha_1)} \int_0^\xi (\xi - s)^{\alpha_1 - 1} h_1(s, z_2(s)) ds$$

$$- \frac{\beta_1 K_1 \xi^3}{6\Gamma(p_1 + \alpha_1)} \int_0^1 (1 - s)^{p_1 + \alpha_1 - 1} h_1(s, z_2(s)) ds$$

$$+ \frac{K_1 \xi^3}{6\Gamma(\alpha_1 - q_1)} \int_0^1 (1 - s)^{\alpha_1 - q_1 - 1} h_1(s, z_2(s)) ds$$

$$P_2 z_1(\xi) = \frac{1}{\Gamma(\alpha_2)} \int_0^\xi (\xi - s)^{\alpha_2 - 1} h_2(s, z_1(s)) ds$$

$$- \frac{\beta_2 K_2 \xi^3}{6\Gamma(p_2 + \alpha_2)} \int_0^1 (1 - s)^{p_2 + \alpha_2 - 1} h_2(s, z_1(s)) ds$$

$$+ \frac{K_2 \xi^3}{6\Gamma(\alpha_2 - q_2)} \int_0^1 (1 - s)^{\alpha_2 - q_2 - 1} h_2(s, z_1(s)) ds.$$

Observe that the fixed point of P are the solution of coupled system (1).

To simplify and our convenience, we put

$$\Lambda_i = \frac{1}{\Gamma(\alpha_i + 1)} + \frac{\beta_i | K_i |}{6\Gamma(p_i + \alpha_i + 1)} + \frac{| K_i |}{6\Gamma(\alpha_i - q_i + 1)}, \text{ for } i = 1, 2. \tag{7}$$

4. Existence Results

We will use well known Banach fixed point theorem to prove our first result.

Theorem 9. *Suppose that* $\beta_i \neq \frac{\Gamma(p_i+4)}{\Gamma(4-q_i)}$, $i = 1, 2$ *and the following hypothesis holds*

(C1) *Assume that* $\exists$ *non-negative continuous functions* $r_i \in C[0, 1]$, $i = 1, 2$ *such that*

$$| h_i(\xi, w_1) - h_i(\xi, w_2) | \leq r_i(\xi) | w_1 - w_2 |, \forall w_1, w_2 \in \mathbb{R} \text{ and } \forall \xi \in [0, 1]$$

with $D_i = \sup_{\xi \in [0,1]} r_i(\xi)$, $i = 1, 2$ *and* $D = \max_i D_i$ *and if* $D(\Lambda_1 + \Lambda_2) < 1$, *where* Λ_i, $i = 1, 2$ *are defined by* (7), *then the coupled system* (1) *has a unique solution defined on* [0,1].

Proof. We shall prove P is a contraction.

Let $(z_1, z_2), (z_1', z_2') \in A$, then $\forall \xi \in [0, 1]$,

$$| P_1(z_2)(\xi) - P_1(z_2')(\xi) | \leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\xi} (\xi - s)^{\alpha_1 - 1} | h_1(s, z_2(s)) - h_1(s, z_2'(s)) | ds$$

$$+ \frac{\beta_1 | K_1 |}{6\Gamma(p_1 + \alpha_1)} \int_0^1 (1 - s)^{p_1 + \alpha_1 - 1} | h_1(s, z_2(s)) - h_1(s, z_2'(s)) | ds$$

$$+ \frac{| K_1 |}{6\Gamma(\alpha_1 - q_1)} \int_0^1 (1 - s)^{\alpha_1 - q_1 - 1} | h_1(s, z_2(s)) - h_1(s, z_2'(s)) | ds$$

$$\leq D \| z_2 - z_2' \|_Z \left[\frac{1}{\Gamma(\alpha_1)} \int_0^{\xi} (\xi - s)^{\alpha_1 - 1} ds + \frac{\beta_1 | K_1 |}{6\Gamma(p_1 + \alpha_1)} \int_0^1 (1 - s)^{p_1 + \alpha_1 - 1} ds \right.$$

$$\left. + \frac{| K_1 |}{6\Gamma(\alpha_1 - q_1)} \int_0^1 (1 - s)^{\alpha_1 - q_1 - 1} ds \right]$$

$$\leq D \| z_2 - z_2' \|_Z \left[\frac{1}{\Gamma(\alpha_1 + 1)} + \frac{\beta_1 | K_1 |}{6\Gamma(p_1 + \alpha_1 + 1)} + \frac{| K_1 |}{6\Gamma(\alpha_1 - q_1 + 1)} \right].$$

Thus

$$\| P_1(z_2) - P_1(z_2') \|_Z \leq D\Lambda_1 \| z_2 - z_2' \|_Z.$$

Similarly

$$\| P_2(z_1) - P_2(z_1') \|_Z \leq D\Lambda_2 \| z_1 - z_1' \|_Z.$$

$$\Rightarrow \| P(z_1, z_2) - P(z_1', z_2') \|_A \leq D(\Lambda_1 + \Lambda_2) \| (z_1, z_2) - (z_1' - z_2') \|_A.$$

As $D(\Lambda_1 + \Lambda_2) < 1$, therefore, P is a contraction. Hence, by Banach Fixed Point Theorem, P must have a unique fixed point i.e. the coupled system (1) has a unique solution. □

Theorem 10. *Suppose that* $\beta_i \neq \frac{\Gamma(p_i + 4)}{\Gamma(4 - q_i)}$, $i = 1, 2$ *and the following hypothesis holds*

(C2) $\exists$ *non-negative continuous functions* $b_1, b_2 \in C[0, 1]$ *such that* $| h_i(\xi, w) | \leq b_i(\xi)$, $\forall w \in \mathbb{R}$ and $\forall \xi \in [0, 1]$ with $B_i = \sup_{\xi \in [0, 1]} b_i(\xi)$, $i = 1, 2$.

Then the coupled system (1) *has at least one solution defined on* [0, 1].

Proof. We shall prove this result by Schaefer's fixed point theorem

Step I. P is continuous

Since h_1 and h_2 are continuous, therefore P is also continuous.

Step II. Bounded sets of A are mapped into bounded sets of A under the mapping P.

Let $B_\epsilon = \{(z_1, z_2) \in A; \| (z_1, z_2) \|_A \leq \epsilon\}$ where $\epsilon > 0$

Now, for $(z_1, z_2) \in B_\epsilon$ and $\forall \xi \in [0, 1]$,

$$| P_1(z_2)(\xi) | \leq \frac{1}{\Gamma(\alpha_1)} \int_0^\xi (\xi - s)^{\alpha_1 - 1} | h_1(s, z_2(s)) | ds$$

$$+ \frac{\beta_1 | K_1 |}{6\Gamma(p_1 + \alpha_1)} \int_0^1 (1 - s)^{p_1 + \alpha_1 - 1} | h_1(s, z_2(s)) | ds$$

$$+\frac{|K_1|}{6\Gamma(\alpha_1-q_1)}\int_0^1(1-s)^{\alpha_1-q_1-1}|h_1(s, z_2(s))|ds$$

$$\leq B_1\left[\frac{1}{\Gamma(\alpha_1)}\int_0^{\xi}(\xi-s)^{\alpha_1-1}ds+\frac{\beta_1|K_1|}{6\Gamma(p_1+\alpha_1)}\int_0^1(1-s)^{p_1+\alpha_1-1}ds\right.$$

$$\left.+\frac{|K_1|}{6\Gamma(\alpha_1-q_1)}\int_0^1(1-s)^{\alpha_1-q_1-1}\right]$$

$$\leq B_1\left[\frac{1}{\Gamma(\alpha_1+1)}+\frac{\beta_1|K_1|}{6\Gamma(p_1+\alpha_1+1)}+\frac{|K_1|}{6\Gamma(\alpha_1-q_1+1)}\right].$$

Thus,

$$\|P_1(z_2)\|_Z\leq B_1\Lambda_1.$$

Similarly

$$\|P_2(z_1)\|_Z\leq B_2\Lambda_2$$

$$\Rightarrow\|P(z_1, z_2)\|_Z\leq B_1\Lambda_1+B_2\Lambda_2$$

i.e.

$$\|P(z_1, z_2)\|_A<\infty.$$

Step III. $P: A\to A$ is completely continuous operator

Let $(z_1, z_2)\in B_\epsilon$ and $\xi_1, \xi_2\in[0, 1]$ with $\xi_1<\xi_2$, then

$$|P_1(z_2)(\xi_2)-P_1(z_2)(\xi_1)|\leq\frac{B_1}{\Gamma(\alpha_1)}\int_0^{\xi_1}[(\xi_2-s)^{\alpha_1-1}-(\xi_1-s)^{\alpha_1-1}]ds$$

$$+\frac{B_1}{\Gamma(\alpha_1)}\int_{\xi_1}^{\xi_2}(\xi_2-s)^{\alpha_1-1}ds$$

$$+\frac{B_1\beta_1|K_1||\xi_2^3-\xi_1^3|}{6\Gamma(p_1+\alpha_1)}\int_0^1(1-s)^{p_1+\alpha_1-1}ds+\frac{B_1|K_1||\xi_2^3-\xi_1^3|}{6\Gamma(\alpha_1-q_1)}\int_0^1(1-s)^{\alpha_1-q_1-1}ds$$

$$\leq\frac{B_1}{\Gamma(\alpha_1+1)}[(\xi_2-\xi_1)^{\alpha_1}+(\xi_2^{\alpha_1}-\xi_1^{\alpha_1})]+\frac{B_1(\xi_2-\xi_1)^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$+\frac{B_1\beta_1|K_1||\xi_2^3-\xi_1^3|}{6\Gamma(p_1+\alpha_1+1)}+\frac{B_1|K_1||\xi_2^3-\xi_1^3|}{6\Gamma(\alpha_1-q_1+1)}.$$

Now the right-hand side approaches to zero when ξ_1 approaches to ξ_2.

Thus

$$\| P_1 z_2(\xi_2) - P_1 z_2(\xi_1) \|_Z \to 0 \text{ as } \xi_1 \to \xi_2.$$

Similarly

$$\| P_2 z_1(\xi_2) - P_2 z_1(\xi_1) \|_Z \to 0, \text{ as } \xi_1 \to \xi_2.$$

Thus

$$\| P(z_1, z_2)(\xi_2) - P(z_1, z_2)(\xi_1) \|_A \to 0, \text{ as } \xi_1 \to \xi_2.$$

Combining Steps I to III and by the consequence of Arzela-Ascoli theorem, $P : A \to A$ is completely continuous operator.

Step IV. Let $\Theta = \{(z_1, z_2) \in A : (z_1, z_2) = \theta P(z_1, z_2)$ for some $\theta \in (0, 1)\}$.

We will show that the set Θ is bounded.

Let $(z_1, z_2) \in \Theta \Rightarrow (z_1, z_2)(\xi) = \theta P(z_1, z_2)(\xi)$ for some $\theta \in (0, 1)$.

Then we have

$$z_1(\xi) = \theta P_1 z_2(\xi),\ z_2(\xi) = \theta P_2 z_1(\xi),\ \forall \xi \in [0, 1].$$

Now

$$| z_1(\xi) | = | \theta P_1(z_2)(\xi) | \leq \theta B_1 \left[\frac{1}{\Gamma(\alpha_1)} \int_0^{\xi} (\xi - s)^{\alpha_1 - 1} ds + \frac{\beta_1 | K_1 |}{6\Gamma(p_1 + \alpha_1)} \int_0^1 (1 - s)^{p_1 + \alpha_1 - 1} ds \right.$$

$$\left. + \frac{| K_1 |}{6\Gamma(\alpha_1 - q_1)} \int_0^1 (1 - s)^{\alpha_1 - q_1 - 1} ds \right]$$

$$\leq B_1 \left[\frac{1}{\Gamma(\alpha_1 + 1)} + \frac{\beta_1 | K_1 |}{6\Gamma(p_1 + \alpha_1 + 1)} + \frac{| K_1 |}{6\Gamma(\alpha_1 - q_1 + 1)} \right].$$

Thus,

$$\| z_1 \|_Z \leq B_1 \Lambda_1.$$

Similarly

$$\| z_2 \|_Z \leq B_2 \Lambda_2.$$

Hence, we get

$$\| (z_1, z_2) \|_A \leq B_1 \Lambda_1 + B_2 \Lambda_2$$

i.e.

$$\| (z_1, z_2) \|_A < \infty$$

which implies that Θ is a bounded set. By Schaefer's fixed point theorem, P must have at least one fixed point which is a solution of the coupled system (1). □

5. Examples

Example 1. Consider the following coupled system

$$\begin{cases} {}^c D^{\frac{15}{4}} z_1(\xi) = \dfrac{1}{\xi^2 + 25} \dfrac{|z_2(\xi)|}{1 + |z_2(\xi)|}, \\ {}^c D^{\frac{7}{2}} z_2(\xi) = \dfrac{1}{\xi^2 + 36} \tan^{-1} z_1(\xi), \ \xi \in [0, 1], \\ z_1(0) = z_1'(0) = z_1''(0) = 0, \ ({}^c D^{\frac{1}{2}} z_1)(1) = \dfrac{16}{17} (J^{\frac{5}{2}} z_1)(1) \\ z_2(0) = z_2'(0) = z_2''(0) = 0, \ ({}^c D^{\frac{3}{2}} z_2)(1) = \dfrac{17}{18} (J^{\frac{5}{2}} z_2)(1). \end{cases} \tag{8}$$

Here $\alpha_1 = \frac{15}{4}$, $q_1 = \frac{1}{2}$, $p_1 = \frac{5}{2}$, $\beta_1 = \frac{16}{17} \neq \frac{\Gamma(p_1 + 4)}{\Gamma(4 - q_1)} = 86.62.$

Also $\alpha_2 = \frac{7}{2}$, $q_2 = \frac{3}{2}$, $p_2 = \frac{7}{2}$, $\beta_2 = \frac{17}{18} \neq \frac{\Gamma(p_2 + 4)}{\Gamma(4 - q_2)} = 1407.66.$

For $\xi \in [0, 1]$ and $w_1, w_2 \in \mathbb{R}$, we get:

$$|h_1(\xi, w_1) - h_1(\xi, w_2)| \leq \frac{1}{\xi^2 + 25} |w_1 - w_2|$$

$$|h_2(\xi, w_1) - h_2(\xi, w_2)| \leq \frac{1}{\xi^2 + 36} |w_1 - w_2|.$$

So, we can take

$$r_1(\xi) = \frac{1}{\xi^2 + 25}, \; r_2(\xi) = \frac{1}{\xi^2 + 36}.$$

Then

$$D_1 = \sup_{\xi\in[0, 1]} r_1(\xi) = \frac{1}{25}, \text{ and } D_2 = \sup_{\xi\in[0, 1]} r_2(\xi) = \frac{1}{36}.$$

And then, we have:

$$D = \max\{D_1, D_2\} = \frac{1}{25}.$$

Further

$$|K_1| = \frac{\Gamma(4 - q_1)\Gamma(p_1 + 4)}{|\beta_1\Gamma(4 - q_1) - \Gamma(p_1 + 4)|} = \frac{2650725\sqrt{\pi}}{1398360} = 3.35$$

and

$$|K_2| = \frac{\Gamma(4 - q_2)\Gamma(p_2 + 4)}{|\beta_2\Gamma(4 - q_2) - \Gamma(p_2 + 4)|} = \frac{7297290\sqrt{\pi}}{9723192} = 1.33.$$

Now

$$D\Lambda_1 = D\left[\frac{1}{\Gamma(\alpha_1 + 1)} + \frac{\beta_1|K_1|}{6\Gamma(p_1 + \alpha_1 + 1)} + \frac{|K_1|}{6\Gamma(\alpha_1 - q_1 + 1)}\right]$$

$$= \frac{1}{25}[0.0603 + 0.0004 + 0.0674]$$

$$= 0.0051$$

and

$$D\Lambda_2 = D\left[\frac{1}{\Gamma(\alpha_2 + 1)} + \frac{\beta_2|K_2|}{6\Gamma(p_2 + \alpha_2 + 1)} + \frac{|K_2|}{6\Gamma(\alpha_2 - q_2 + 1)}\right]$$

$$= \frac{1}{25}[0.08597 + 0.00004 + 0.11083]$$

$$= 0.0078.$$

And then

$$D(\Lambda_1 + \Lambda_2) = 0.0051 + 0.0078 = 0.0129 < 1.$$

Hence, all the assumptions of Theorem 9 are satisfied and consequently the coupled system (8) must have a unique solution defined on [0, 1].

Example 2. Now consider the following coupled system

$$\begin{cases} {}^cD^{\frac{7}{2}} z_1(\xi) = \dfrac{\cos z_2(\xi)}{8+\xi}, \\ {}^cD^{\frac{13}{4}} z_2(\xi) = \dfrac{\sin z_1(\xi)}{9+\xi^2}, \ \xi \in [0, 1] \\ z_1(0) = z_1'(0) = z_1''(0) = 0, \ ({}^cD^{\frac{5}{2}} z_1)(1) = \dfrac{15}{4}\left(J^{\frac{11}{2}} z_1\right)(1) \\ z_2(0) = z_2'(0) = z_2''(0) = 0, \ ({}^cD^{\frac{3}{2}} z_2)(1) = \dfrac{7}{8}\left(J^{\frac{9}{2}} z_2\right)(1). \end{cases} \tag{9}$$

Here $\alpha_1 = \dfrac{7}{2}, \ q_1 = \dfrac{5}{2}, \ p_1 = \dfrac{11}{2}, \ \beta_1 = \dfrac{15}{4} \neq \dfrac{\Gamma(p_1+4)}{\Gamma(4-q_1)} = 134607.13.$

Also $\alpha_2 = \dfrac{13}{4}, \ q_2 = \dfrac{3}{2}, \ p_2 = \dfrac{9}{2}, \ \beta_2 = \dfrac{7}{8} \neq \dfrac{\Gamma(p_2+4)}{\Gamma(4-q_2)} = 10557.42.$

For $\xi \in [0, 1]$ and $w \in \mathbb{R}$, we get:

$$|h_1(\xi, w)| = \left| \frac{\cos w}{8+\xi} \right| \leq \frac{1}{8+\xi}$$

$$|h_2(\xi, w)| = \left| \frac{\sin w}{9+\xi^2} \right| \leq \frac{1}{9+\xi^2}.$$

So, we can take

$$b_1(\xi) = \frac{1}{8+\xi}, \ b_2(\xi) = \frac{1}{9+\xi^2}.$$

And then, we have:

$$B_1 = \sup_{\xi \in [0,1]} b_1(\xi) = \frac{1}{8}, B_2 = \sup_{\xi \in [0,1]} b_2(\xi) = \frac{1}{9}.$$

Hence all the assumptions of theorem 10 are satisfied, therefore the coupled system (9) must have at least one solution defined on [0, 1].

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