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INEQUALITIES AMONG COMPLEMENTARY MEANS OF HERON MEAN AND CLASSICAL MEANS

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Abstract

In this paper, the complementary means of arithmetic, geometric, harmonic and contra harmonic with respect to Heron mean are defined and verified them as means. Further, inequalities among them and classical means are established.

I. Introduction

In Pythagorean school ten Greek means are defined based on proportion, the following are the familiar means in literature and are given as follows:

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For two real numbers u_1 and v_1 which are positive, $A(u_1, v_1) = \frac{u_1 + v_1}{2}$; $G(u_1, v_1) = \sqrt{u_1 v_1}$; $H(u_1, v_1) = \frac{2u_1 v_1}{u_1 + v_1}$ and $C(u_1, v_1) = \frac{u_1^2 + v_1^2}{u_1 + v_1}$. These are called Arithmetic, Geometric, Harmonic and Contra harmonic mean respectively.

The Hand book of Means and their Inequalities, by Bullen [1], gave the tremendous work on Mathematical means and the corresponding inequalities involving huge number of means. The authors in [2, 3, 4] discussed about the relations between the well-known means and series. The generalization of the means is discussed in [5, 6, 18, 19]. Relevant to this paper the authors in [7-17] established the good number of inequalities, double inequalities, introduced new means, studied homogeneous functions as an application inequalities are obtained.

In 1958, C. Gini introduced Complementary means and G. Toader in 1991 proposed a generalization of complementariness and inversion ([20], [21]). This work motivates us to introduce the following definitions and to establish some double inequalities in this paper.

Definition 1.1 [1]. For two real numbers $u_1, v_1 > 0$, the Heron mean is given by $H_e = \frac{u_1 + \sqrt{u_1 v_1} + v_1}{3}$.

Definition 1.2 [20, 21]. A mean N is called complementary to M with respect to P (or P-complementary to M) if it verifies $P(M, N) = P$, it is denoted by $M^{(P)} = N$.

(1) Complementary of Arithmetic mean with respect to Heron mean is denoted by $A^{(He)}$ and is given by $A^{(He)} = \frac{1}{2}[3A + 2G - \sqrt{4AG + 5A^2}]$

$$A^{(He)} = \frac{1}{4}[3u_1 + 3v_1 + 4\sqrt{u_1 v_1} - \sqrt{(u_1 + v_1)(5u_1 + 5v_1 + 8\sqrt{u_1 v_1})}].$$

(2) Complementary of Geometric mean with respect to Heron mean is denoted by $G^{(He)}$ and is given by $G^{(He)} = \frac{1}{2}[4A + G - \sqrt{8AG + G^2}]$

$$G^{(He)} = \frac{1}{2}[2u_1 + 2v_1 + \sqrt{u_1 v_1} - \sqrt{4(u_1 + v_1)\sqrt{u_1 v_1} + u_1 v_1}].$$

(3) Complementary of Harmonic mean with respect to Heron mean is denoted by $H^{(He)}$ and is given by $H^{(He)} = \frac{1}{2A}[4A^2 - G^2 + 2AG - G\sqrt{8A^2 - 3G^2 + 4AG}]$

$$H^{(He)} = u_1 + v_1 + \sqrt{u_1 v_1} - \frac{\sqrt{u_1 v_1 (2(u_1 + v_1)^2 - 3u_1 v_1 + 2\sqrt{u_1 v_1}(u_1 + v_1)}}{u_1 + v_1}.$$

(4) Complementary of Contra harmonic mean with respect to Heron mean is denoted by

$$C^{(He)} = \frac{1}{2A}(2A^2 + G^2 + 2AG) - \frac{1}{2A} - \sqrt{(2A^2 - G^2)(2A^2 + 3G^2 + 4AG)}$$

$$C^{(He)} = \frac{u_1 + v_1 + \sqrt{u_1 v_1} - u_1^2 - v_1^2}{2(u_1 + v_1)}$$

$$- \frac{\sqrt{(u_1^2 + v_1^2)^2 + 8u_1 v_1 (u_1^2 + v_1^2) + 4\sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1)}}{2(u_1 + v_1)}.$$

II. Main Results

In this section, it can be verified that above definitions are means. Inequalities involving these and classical means are established.

Theorem 2.1. *For* $u_1, v_1 > 0$, *complementary of geometric mean and arithmetic mean with respect to Heron mean satisfy the inequality* $A^{(He)} < G^{(He)}$.

Proof. Consider,

$$G^{(He)} - A^{(He)} = \frac{1}{2}[A - G + \sqrt{4AG + 5A^2} - \sqrt{8AG + G^2}]$$

$$= \frac{G}{2}\left[\frac{A}{G} - 1 + \sqrt{4\frac{A}{G} + 5\frac{A^2}{G}} - \sqrt{8\frac{A}{G} + 1}\right].$$

Let $\frac{A}{G} = k > 1$, $\frac{G}{2}[k - 1 + \sqrt{4k + 5k^2} - \sqrt{8k + 1}]$.

Put $k - 1 = z > 0 \Rightarrow k = z + 1$, $4k + 5k^2 = 5z^2 + 14z + 9$, $8k + 1 = 8z + 9$

$$G^{(He)} - A^{He} = \frac{G}{2} f(z)$$

where $f(z) = z + \sqrt{5z^2 + 14z + 9} - \sqrt{8z + 9}$.

To verify $f(z) > 0$. Clearly, $z^2 + 6z + 9 + 2z\sqrt{5z^2 + 14z + 9} > 0$ $\Rightarrow z^2 + 5z^2 + 14z + 9 + 2z\sqrt{5z^2 + 14z + 9} > 8z + 9$.

Therefore, $z + \sqrt{5z^2 + 14z + 9} > \sqrt{8z + 9}$. Hence $f(z) > 0, \Rightarrow G^{He} - A^{He} > 0$. Hence proved that $A^{(He)} < G^{(He)}$.

Theorem 2.2. *For $u_1, v_1 > 0$, complementary of harmonic mean and geometric mean with respect to Heron mean satisfy the inequality $H^{(He)} > G^{(He)}$.*

Proof. Consider,

$$H^{(He)} - G^{(He)} = \frac{1}{2A}[4A^2 - G^2 + 2AG - G\sqrt{8A^2 - 3G^2 + 4AG}]$$

$$- \frac{1}{2}[4A + G - \sqrt{8AG + G^2}]$$

$$= \frac{1}{2A}[AG - G^2 + A[\sqrt{8AG + G^2}] - G\sqrt{8A^2 - 3G^2 + 4AG}]$$

$$= \frac{G^2}{2A}\left[\frac{A}{G} - 1 + \frac{A}{G}\left[\sqrt{8\frac{A}{G} + 1}\right] - \sqrt{8\frac{A^2}{G^2} - 3 + 4\frac{A}{G}}\right].$$

Let $\frac{A}{G} = k > 1$

$$= \frac{G^2}{2A}[k - 1 + k[\sqrt{8k + 1}] - \sqrt{8k^2 - 3 + 4k}].$$

Put $k - 1 = z \Rightarrow k = z + 1,\ 8k^2 + 4k - 3 = 8z^2 + 20z + 9,\ 8k + 1 = 8z + 9 = \frac{G^2}{2A}[z + (z+1)[\sqrt{8z+9}] - \sqrt{8z^2 + 20z + 9}].$

To prove $z + (z+1)[\sqrt{8z+9}] - \sqrt{8z^2 + 20z + 9} > 0.$

Clearly, $8z^3 + 18z^2 + 6z + 2z(z+1)\sqrt{8z+9} > 0.$

Therefore, $z + (z+1)[\sqrt{8z+9}] > \sqrt{8z^2 + 20z + 9}$, thus $H^{(He)} - G^{(He)} > 0.$ Hence proved that $H^{(He)} > G^{(He)}$.

Theorem 2.3. *For* $u_1, v_1 > 0$, *complementary of arithmetic mean and contra harmonic mean with respect to Heron mean satisfy the inequality* $A^{(He)} > C^{(He)}$.

Proof. Consider

$$A^{(He)} - C^{(He)} = \frac{1}{2A}[A^2 - G^2 + \sqrt{(2A^2 - G^2)(2A^2 + 3G^2 + 4AG)}]$$

$$- \frac{1}{2A} A\sqrt{4AG + 5A^2}$$

dividing by G^2 and let $\frac{A}{G} = k > 1$, then

$$A^{(He)} - C^{(He)} = \frac{G^2}{2A}[k^2 - 1 + \sqrt{(2k^2 - 1)(2k^2 + 3 + 4k)} - k\sqrt{4k + 5k^2}]$$ put $k - 1 = z$ or $k = z + 1$, then

$$= \frac{G^2}{2A}[z^2 + 2z + \sqrt{(2z^2 + 4z + 1)(2z^2 + 8z + 9) - (z+1)\sqrt{5z^2 + 14z + 9}}]$$

$$= \frac{G^2}{2A} f(z),$$

To verify $f(z) > 0.$

Clearly, $4z^3 + 14z^2 + 12z + 2z(z+2)\sqrt{(2z^2 + 4z + 1)(2z^2 + 8z + 9)} > 0$

$$\Rightarrow 5z^4 + 28z^3 + 56z^2 + 44z + 9 + (2z^2 + 4z)\sqrt{(2z^2 + 4z + 1)(2z^2 + 8z + 9)}$$

$$> 5z^4 + 24z^3 + 42z^2 + 32z + 9$$

$$z^2 + 2z + \sqrt{(2z^2 + 4z + 1)(2z^2 + 8z + 9)} > (z + 1)\sqrt{5z^2 + 14z + 9}.$$

Therefore, $f(z) > 0$. Hence $A^{(He)} - C^{(He)} > 0$.

Hence proved that $C^{(He)} < A^{(He)}$.

Theorem 2.4. *For* $u_1, v_1 > 0$, *complementary of arithmetic mean with respect to Heron mean and arithmetic mean satisfy the inequality* $A^{(He)} < A$.

Proof. Consider

$$A - A^{(He)} = \frac{1}{4}[2u_1 + 2v_1 - 3u_1 - 3v_1 - 4\sqrt{v_1 v_1} + \sqrt{(u_1 + v_1)(5u_1 + 5v_1 + 8\sqrt{u_1 v_1})}]$$

$$= \frac{1}{4}[\sqrt{(u_1 + v_1)(5u_1 + 5v_1 + 8\sqrt{u_1 v_1})} - u_1 - v_1 - 4\sqrt{u_1 v_1}]$$

To verify whether, $\sqrt{(u_1 + v_1)(5u_1 + 5v_1 + 8\sqrt{u_1 v_1})} - u_1 - v_1 - 4\sqrt{u_1 v_1} > 0$. Clearly $(v_1 - u_1)^2 > 0, (u_1 + v_1)^2 + 16u_1 v_1 + 8(u_1 + v_1)\sqrt{u_1 v_1} > 5(u_1 + v_1)^2 + 8u_1 v_1\sqrt{u_1 v_1}$

$$\Rightarrow \sqrt{(u_1 + v_1)(5u_1 + 5v_1 + 8\sqrt{u_1 v_1})} > u_1 + v_1 + 4\sqrt{u_1 v_1}.$$

Hence proved that $A^{(He)} < A$.

Theorem 2.5. *For* $u_1, v_1 > 0$, *complementary of geometric mean with respect to Heron mean and arithmetic mean satisfy the inequality* $G^{(He)} > A$.

Proof. Consider $G^{(He)} - A = \frac{1}{2}[u_1 + v_1 + \sqrt{u_1 v_1} - \sqrt{4(u_1 + v_1)\sqrt{u_1 v_1} + u_1 v_1}]$.

To verify $u_1 + v_1 + \sqrt{u_1 v_1} - \sqrt{4(u_1 + v_1)\sqrt{u_1 v_1} + u_1 v_1} > 0$. Clearly $u_1 + v_1 + (\sqrt{u_1} - \sqrt{v_1})^2 > 0$

$$\Rightarrow (u_1 + b)^2 > 2\sqrt{u_1 v_1}(u_1 + v_1).$$

Therefore $u_1 + v_1 + \sqrt{u_1 v_1} > \sqrt{4(u_1 + v_1)\sqrt{u_1 v_1} + u_1 v_1}$.

Hence proved that $G^{(He)} > A$.

Theorem 2.6. *For* $u_1, v_1 > 0$, *complementary of contraharmonic mean with respect to Heron mean and geometric mean satisfy the inequality* $C^{(He)} < G$.

Proof. Consider

$$C^{(He)} - G = \frac{u_1^2 + v_1^2 + 4u_1 v_1}{2(u_1 + v_1)}$$

$$- \frac{\sqrt{(u_1^2 + v_1^2)^2 + 8u_1 v_1 (u_1^2 + v_1^2) + 4\sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1)}}{2(u_1 + v_1)}.$$

Clearly

$$u_1^2 + v_1^2 + 4u_1 v_1 < \sqrt{(u_1^2 + v_1^2)^2 + 8u_1 v_1 (u_1^2 + v_1^2) + 4\sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1)}.$$

As $(u_1^2 + v_1^2 + 4u_1 v_1)^2 < (u_1^2 + v_1^2)^2 + 8u_1 v_1 (u_1^2 + v_1^2) + 4\sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1)$.

Simplifying gives $4u_1^2 v_1^2 < \sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1)$ which is $4G^4 < 4GA^2C$, which is always true as $G < A$ and $G < C$.

Therefore $C^{(He)} - G < 0$.

Hence proved that $C^{(He)} < G$.

Theorem 2.7. *For* $u_1, v_1 > 0$, *complementary of contraharmonic mean with respect to Heron mean and harmonic mean satisfy the inequality* $C^{(He)} < H$.

Proof. Consider

$$H - C^{(He)} = \frac{2u_1 v_1}{u_1 + v_1}$$

$$- \frac{u_1 + v_1 + \sqrt{u_1 v_1} - u_1^2 - v_1^2 + \sqrt{(u_1^2 + v_1^2)^2 + 8u_1 v_1 (u_1^2 + v_1^2) + 4\sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1)}}{2(u_1 + v_1)}$$

$$= \frac{\sqrt{(u_1^2 + v_1^2)^2 + 8u_1 v_1 (u_1^2 + v_1^2) + 4\sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1)}}{2(u_1 + v_1)}$$

$$- \frac{u_1^2 + v_1^2 + 2\sqrt{u_1 v_1}(u_1 + v_1)}{2(u_1 + v_1)}.$$

Clearly $4u_1 v_1 (u_1^2 + v_1^2) > 0$

$$\Rightarrow u_1^2 + v_1^2 + 2\sqrt{u_1 v_1}(u_1 + v_1) < 8u_1 v_1 (u_1^2 + v_1^2) + 4\sqrt{u_1 v_1}(u_1^2 + v_1^2)(u_1 + v_1).$$

Hence proved that $C^{(He)} < H$.

Theorem 2.8. *For* $u_1, v_1 > 0$, *complementary of harmonic mean with respect to Heron mean and contraharmonic mean satisfy the inequality* $H^{(He)} < C$.

Proof. Consider,

$$C - H^{(He)} = \frac{u_1^2 + v_1^2}{u_1 + v_1} - u_1 - v_1 - \sqrt{u_1 v_1} + \frac{\sqrt{u_1 v_1 2(u_1 + v_1)^2 - 3u_1 v_1 + 2\sqrt{u_1 v_1}(u_1 + v_1)}}{u_1 + v_1}$$

$$= \frac{\sqrt{u_1 v_1 2(u_1 + v_1)^2 - 3u_1 v_1 + 2u_1 v_1 (u_1 + v_1)} - u_1 v_1 - (u_1 + v_1)\sqrt{u_1 v_1}}{u_1 + v_1}.$$

Need to verify,

$$\sqrt{u_1 v_1 2(u_1 + v_1)^2 - 3u_1 v_1 + 2u_1 v_1 (u_1 + v_1)} - u_1 v_1 - (u_1 + v_1)\sqrt{u_1 v_1} > 0.$$

Clearly $(v_1 - u_1)^2 > 0$

$$\Rightarrow 2u_1 v_1 (u_1 + v_1)^2 - 3u_1^2 v_1^2 > u_1^2 v_1^2 + (u_1 + v_1)^2 u_1 v_1$$

$$\Rightarrow \sqrt{u_1 v_1 2(u_1 + v_1)^2 - 3u_1 v_1 + 2c(u_1 + v_1)} > u_1 v_1 + (u_1 + v_1)\sqrt{u_1 v_1}.$$

Hence proved that. $H^{(He)} < C$.

Conclusion

The inequality $H < G < A < C$ is well known in literature. By studying the complementary means of arithmetic, geometric, harmonic and contra harmonic with respect to Heron mean the following inequality chain is established.

$$C^{(He)} < H < G < A^{(He)} < A < G^{(He)} < H^{(He)} < C.$$

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