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THE FORCING STAR CHROMATIC NUMBER OF A GRAPH

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Abstract

Let S be a χ_s-set of G. A subset $T \subseteq S$ is said to be a forcing subset for S if S is the unique χ_s-set containing T. The forcing star chromatic number $f_{\chi_s}(S)$ of S in G is the minimum cardinality of a forcing subset for S. The forcing star chromatic number $f_{\chi_s}(G)$ of G is the smallest forcing number of all χ_s-sets of G. Some general properties satisfied by this concept are studied. The forcing star chromatic number of some standard graphs are determined. Connected graphs of order $n \geq 2$ with star chromatic number 0 or 1 or $\chi_s(\mathrm{G})$ are characterized.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be adjacent if uv is an edge of G. If $uv \in E(G)$,

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we say that u is a neighbor of v and denote by $N(v)$, the set of neighbors of v. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. A vertex v is said to be a universal vertex if $\deg(v) = n - 1$. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A vertex x is said to lie on a $u - v$ geodesic P if x is a vertex of P including the vertices u and v. The eccentricity $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. $e(v) = \max\{d(v, u) : u \in V(G)\}$. The minimum eccentricity among the vertices of G is the radius, $radG$ or $r(G)$ and the maximum eccentricity is its diameter, $diamG$. We denote $rad(G)$ by r and $diamG$ by d. The diameter of a graph is the maximum distance between a pair of vertices of G.

A double star is a tree with diameter 3. It is denoted by $K_{2,\, r,\, s}$. The vertex set of $K_{2,\, r,\, s}$ where uv is the internal edge of $K_{2,\, r,\, s}$. Therefore $K_{2,\, r,\, s} = K_{1, r} \cup K_{1,\, s} \cup \{uv\}$, where the centre vertex of $K_{1,\, s}$ is u and the centre vertex of $K_{1,\, s}$ is v. Let $G = (V, E)$ be a connected graph. We define the distance as the minimum length of path connecting vertices u and v in G, denoted by $d(u, v)$. A k-coloring of G is a function $c : V(G) \to \{1, 2, \ldots k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. Thus, the coloring c induces a partition Q of $V(G)$ into k color classes (independent sets) $C_1, C_2, \ldots, C_k$, where C_i is the set of all vertices colored by the color i for $1 \leq i \leq k$. A p-vertex coloring of is an assignment of p colors, $1, 2, \ldots, p$ to the vertices of G, the coloring is proper if no two distinct adjacent vertices have the same color. If $\chi(G) = p$, G is said to be p-chromatic, where $p \leq k$. A set $C \subseteq V(G)$ is called chromatic set if C contains all vertices of distinct colors in G. The chromatic number of G is the minimum cardinality among all the chromatic sets of G. That is $\chi(G) = \min\{|C_i| / C_i$ is a chromatic set of $G\}$. The concept of the chromatic number was studied in [1, 2, 7-9]. A star colouring of a graph G is proper colouring such that no path of length 4 is bicolourable. The minimum colours needed for a star coloring of G is called star chromatic number and is denoted by $\chi_s(G)$. Let G be a star colourable. A set $S \subseteq V(G)$ is called a star chromatic set if S contains all vertices of distinct colours in G. Any star chromatic set of order $\chi_s(G)$ is called a χ_s-set of G. The concept of the star chromatic number was studied in [5, 6]. The chromatic number has application in Time Table Scheduling, Map coloring, channel assignment problem in radio technology, town planning, GSM mobile phone networks etc. [4, 7].

2. The Forcing Star Chromatic Number of a Graph

Theorem 2.1. *Let S be a χ_s-set of G. A subset $T \subseteq S$ is said to be a forcing subset for S if S is the unique χ_s-set containing T. The forcing star chromatic number $f_{\chi_s}(S)$ of S in G is the minimum cardinality of a forcing subset for S. The forcing star chromatic number $f_{\chi_s}(S)$ of G is the smallest forcing number of all χ_s-sets of G.*

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2, v_3\}$ and $S_2 = \{v_2, v_3, v_4\}$ are the only two χ_s-sets of G so that $\chi_s(\mathrm{G}) = 3$. It is clear that $f_{\chi_s}(S_1) = 1$, $f_{\chi_s}(S_2) = 1$ so that $f_{\chi_s}(G) = 1$.

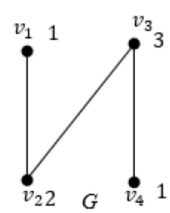

Figure 2.1

Observation 2.3. For every connected graph G, $0 \leq f_{\chi_s}(G) \leq \chi_s(G)$.

Remark 2.4. The bounds in the Observation 2.3 are sharp. For the complete graph $G = K_n (n \geq 2)$, $S = V(G)$ is the unique χ_s-set of G so that $f_{\chi_s}(G) = 0$. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_2, v_6\}$, $S_3 = \{v_1, v_3, v_5\}$, $S_4 = \{v_1, v_5, v_6\}$, $S_5 = \{v_4, v_5, v_6\}$, $S_6 = \{v_2, v_3, v_4\}$, $S_7 = \{v_3, v_4, v_5\}$, and $S_8 = \{v_3, v_4, v_6\}$ such that $f_{\chi_s}(S_i) = 3$ and $\chi_s(G) = 3$ for $i = 1$ to 8 so that $f_{\chi_s}(G) = \chi_s(G) = 3$. Also the bounds are strict. For the graph G given in Figure 2.1, $\chi_s(G) = 3$, $f_{\chi_s}(G) = 1$. Thus $0 < f_{\chi_s}(G) < \chi_s(G)$.

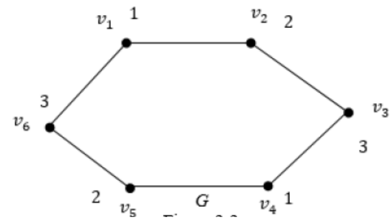

Figure 2.2

Theorem 2.5. *Let G be a connected graph. Then*

(a) $f_{\chi_s}(G) = 0$ *if and only if G has a unique χ_s-set.*

(b) $f_{\chi_s}(G) = 1$ *if and only if G has at least two χ_s-sets, one of which is a unique χ_s-set containing one of its elements, and*

(c) $f_{\chi_s}(G) = \chi_s(G)$ *if and only if no χ_s-set of G is the unique χ_s-set containing any of its proper subsets.*

Proof. (a) Let $f_{\chi_s}(G) = 0$. Then, by definition, $f_{\chi_s}(S) = 0$ for some χ_s-set S of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique χ_s-set of G. The converse is clear.

(b) Let $f_{\chi_s}(G) = 1$. Then by Theorem 2.5(a), G has at least two χ_s-sets. Also, since $f_{\chi_s}(G) = 1$, there is a singleton subset T of a χ_s-set S of G such that T is not a subset of any other χ_s-set of G. Thus S is the unique χ_s-set containing one of its elements. The converse is clear.

(c) Let $f_{\chi_s}(G) = \chi_s(G)$. Then $f_{\chi_s}(G) = \chi_s(G)$ for every χ_s-set S in G. Also, by Theorem 2.3, $\chi_s(\mathrm{G}) \geq 2$ and hence $f_{\chi_s}(G) \geq 2$. Then by Theorem 2.5(a), G has at least two χ_s-sets and so the empty set ϕ is not a forcing subset for any χ_s-set of G. Since $f_{\chi_s}(G) = \chi_s(G)$, no proper subset of S is a forcing subset of S. Thus no χ_s-set of G is the unique χ_s-set containing any of its proper subsets. Conversely, the data implies that G contains more than one χ_s-set and no subset of any χ_s-set S other than S is a forcing subset for S. Hence it follows that $f_{\chi_s}(G) = \chi_s(G)$. ■

Definition 2.6. A vertex v of a graph G is said to be a star chromatic vertex of G if v belongs to every χ_s-set of G.

Example 2.7. For the graph G given in Figure 2.3, $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_3, v_4\}$, $S_3 = \{v_1, v_3, v_6\}$, $S_4 = \{v_2, v_3, v_5\}$, $S_5 = \{v_3, v_4, v_5\}$ and $S_6 = \{v_3, v_5, v_6\}$ are the only χ_s-sets of G such that v_3 is a star chromatic vertex of G.

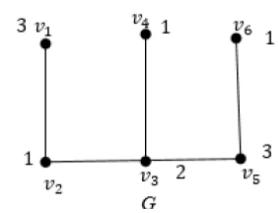

Figure 2.3

Theorem 2.8. *Let G be a connected graph and W be the set of all star chromatic vertices of G. Then $f_{\chi_s}(G) \leq \chi_s(G) - |W|$.*

Proof. Let S be any χ_s-set of G. Then $\chi_s(\text{G}) = |S|$, $W \subseteq S$ and S is the unique χ_s-set containing $S - W$. Thus $f_{\chi_s}(G) \leq |S - W| = |S| - |W| = \chi_s(G) - |W|$. ■

In the following we determine the forcing star chromatic number of some standard graphs.

Theorem 2.9. *For the complete graph $G = K_n (n \geq 2)$. Then $f_{\chi_s}(G) = 0$.*

Proof. Let $S = V(G)$ is the unique χ_s-sets of G, the result follows from Theorem 2.5(a). ■

Theorem 2.10. *For the star graph $G = K_{1,n} (n \geq 3)$, $f_{\chi_s}(G) = 1$.*

Proof. Let $V = \{x, v_1, v_2, \ldots, v_{n-1}\}$ be the vertex set of G where x is the central vertex of G. Then $S_i = \{x, v_i\}, (1 \leq i \leq n-1)$ is the χ_s-set of G so that $\chi_s(\mathrm{G}) = 2$. Since x is a star chromatic vertex of G, by Observation 2.8(c), $f_{\chi_s}(G) \leq 2-1 = 1$. Since $n \geq 3$, χ_s-set is not unique. Hence by Observation 2.5 (b), $f_{\chi_s}(G) = 1$. ■

Theorem 2.11. *For the double star graph* $G = K_{2, r, s}$, $f_{\chi_s}(G) = 3$.

Proof. Let $V = \{x, v_1, v_2, \ldots, v_r\} \cup \{y, u_1, u_2, \ldots, u_s\}$ be the vertex set of G such that $xv_i, xy, yu_j \in E(G)$ for all $(1 \leq i \leq r)$ and $(1 \leq j \leq s)$ where $r + s = n-2$. Then $S_1 = \{x, y, v_i\}$ and $S_2 = \{x, y, u_j\}$ $(1 \leq i \leq r)$ and $(1 \leq j \leq s)$ are the only χ_s-sets of G such that $f_{\chi_s}(S_1) = f_{\chi_s}(S_2) = 3$ so that $f_{\chi_s}(G) = 3$. ■

Theorem 2.12. *For the complete bipartite graph* $G = K_{r, s}(1 \leq r \leq s)$,

$$f_{\chi_s}(G) = \begin{cases} 0 & \textit{if } r = s = 1 \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. If $r = s = 1$, then the result follows from Theorem 2.9. For $r = 1, s \geq 2$ then the result follows from Theorem 2.10. So let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the bipartite sets of G. Then $S_i = X \cup \{y_i\} (2 \leq i \leq s)$ is a χ_s-set of G such that $f_{\chi_s}(S_i) = 1$ for all $(2 \leq i \leq s)$ so that $f_{\chi_s}(G) = 1$. ■

Theorem 2.13. *For the path* $G = P_n(n \geq 4)$, $f_{\chi_s}(G) = \begin{cases} 1 & \textit{if } n = 4 \\ 2 & \textit{if } n = 5 \\ 3 & \textit{otherwise.} \end{cases}$

Proof. Let P_n be $v_1, v_2, \ldots, v_n$. We consider the following cases.

Case (i) $n = 3r, r \geq 2$.

Assign $C(v_i) = 1, i = 1, 4, \ldots, 3r+1, C(v_j) = 2, j = 2, 5, \ldots, 3r-1, C(v_k) = 3$, $k = 3, 6, \ldots, 3r$. Then $S_{ijk} = \{v_i, v_j, v_k\}$ is a χ_s-set of G such that $\chi_s(S_{ijk}) = 3$ for $i, j, k(i = 1, 4, \ldots, 3r-2, j = 2, 3, \ldots, 3r-1, k = 3, 6,$

$\ldots, 3r)$ so that $\chi_s(G) = 3$. By Observation 2.3, $0 \leq f_{\chi_s}(G) \leq 3$. Since χ_s-set of G is not unique $f_{\chi_s}(G) \geq 1$. It is easily verified that no singleton subsets or two element subsets of S_{ijk} for all $i, j, k(i = 1, 4, \ldots, 3r-2,$ $j = 1, 4, 3r-1, k = 3, 6, \ldots, 3r)$ is not a forcing subset of S_{ijk} so that $f_{\chi_s}(S_{ijk}) = 3$. Since this is true for all χ_s-set S_{ijk} for all $i, j, k(i = 1, 4, \ldots, 3r-2, j = 2, 3, \ldots, 3r-1, k = 3, 6, \ldots, 3r), f_{\chi_s}(G) = 1$.

Case (ii) $n = 3r+1, r \geq 1$.

Assign $C(v_i) = 1, i = 1, 4, \ldots, 3r+1, C(v_j) = 2, j = 2, 5, \ldots, 3r-1, C(v_k) = 3,$ $k = 3, 6, \ldots, 3r$. For $r = 1, S_1 = \{v_1, v_2, v_3\}$ and $S_2 = \{v_2, v_3, v_4\}$ are the only two χ_s-sets of G such that $\chi_s(\mathrm{S}_1) = \chi_s(\mathrm{S}_2) = 1$ and so $\chi_s(\mathrm{G}) = 1, f_{\chi_s}(G) = 1$. Let $r \geq 2$. Then $S_{ijk} = \{v_i, v_j, v_k\}$ and $S_{ijk} = \{v_i, v_{3r+1}, v_k\}$ are the only χ_s-sets of G such that $\chi_s(S_{ijk})$ $= \chi_s(S_{ijk}) = 3$ for $i, j, k(i = 1, 4, \ldots, 3r+1, j = 2, 3, \ldots, 3r-1, k = 3, 6, \ldots, 3r)$ so that $\chi_s(G) = 3$. By Observation 2.3, $0 \leq f_{\chi_s}(G) \leq 3$. Since χ_s-set of G is not unique $f_{\chi_s}(G) \geq 1$. It is easily verified that no singleton subsets or two element subsets of S_{ijk} for all $i, j, k(i = 1, 4, \ldots, 3r+1\ j = 2, 5, \ldots, 3r-1,$ $k = 3, 6, \ldots, 3r)$ is not a forcing subset of S_{ijk} so that $f_{\chi_s}(S_{ijk}) = 3$. Similarly no singleton subsets or two element subsets of S_{ik} for all $i, k(i = 1, 4, \ldots, 3r+1, k = 3, 6, \ldots, 3r)$ is not a forcing subset of S_{ik} so that $f_{\chi_s}(S_{ik}) = 3$. Since this is true for all χ_s-sets S_{ijk} and S_{ik} for all $i, j, k(i = 1, 4, \ldots, 3r+1, j = 2, 3, \ldots, 3r-1, k = 3, 6, \ldots, 3r), f_{\chi_s}(G) = 3$.

Case (iii) $n = 3r+2, r \geq 1$.

Assign $C(v_i) = 1, i = 1, 4, \ldots, 3r+1, C(v_j) = 2, j = 2, 5, \ldots, 3r+2, C(v_k) = 3,$ $k = 3, 6, \ldots, 3r$. For $r = 1, S_1 = \{v_1, v_2, v_3\}, S_2 = \{v_1, v_3, v_5\}$ and $S_3 = \{v_2, v_3, v_4\},$ $S_4 = \{v_3, v_4, v_5\}$ are the χ_s-sets of G such that $f_{\chi_s}(G) = 2$. Let $r \geq 2$. Then $S_{ijk} = \{v_i, v_j, v_k\}, S_{ik} = \{v_i, v_{3r+1}, v_k\}, S_{ij} = \{v_i, v_j, v_{3r+2}\}, S_i = \{v_i, v_{3r+1}, v_{3r+2}\}$ are the only χ_s-sets of G such that $\chi_s(S_{ijk}) = \chi_s(S_{ijk}) = \chi_s(S_{ik}) = \chi_s(S_{ij}) = 3$

for all $i, j, k(i = 1, 4, \ldots, 3r + 1, j = 2, 5, \ldots, 3r + 2, k = 3, 6, \ldots, 3r)$ so that $\chi_s(G) = 3$. By Observation 2.3, $0 \leq f_{\chi_s}(G) \leq 3$. Since χ_s-set of G is not unique $f_{\chi_s}(G) \leq 1$. It is easily verified that no singleton subsets or two element subsets of S_{ijk} for all $i, j, k(i = 1, 4, \ldots, 3r + 1, j = 2, 5, \ldots, 3r + 2, k = 3, 6, \ldots, 3r)$ is not a forcing subset of S_{ijk} so that $f_{\chi_s}(S_{ijk}) = 3$. Similarly no singleton subsets or two element subsets of S_{ik} for all $i, k(i = 1, 4, \ldots, 3r + 1, k = 3, 6, \ldots, 3r)$ is not a forcing subset of S_{ik} so that $f_{\chi_s}(S_{ijk}) = 3$. Similarly no singleton or two element subsets of S_{ij} for all $i, j(i = 1, 4, \ldots, 3r + 1, j = 2, 5, \ldots, 3r + 2)$ is not a forcing subset of S_{ij} so that $f_{\chi_s}(S_{ij}) = 3$. Similarly no singleton subsets or two element subsets of S_i for all $i(i = 1, 4, \ldots, 3r + 1)$ is not a forcing subset of S_i so that $f_{\chi_s}(S_i) = 3$. Since this is true for all χ_s-sets $S_{ijk}, S_{ik}, S_{ij}, S_i$ for all $(i = 1, 4, \ldots, 3r + 1, j = 2, 3, \ldots, 3r + 2, k = 3, 6, \ldots, 3r), f_{\chi_s}(G) = 3$. ■

Theorem 2.14. *For the cycle* $G = C_n (n \geq 4)$, $f_{\chi_s}(G) = \begin{cases} 0 & if\ n = 4 \\ 2 & if\ n = 5 \\ 3 & if\ n \geq 6. \end{cases}$

Proof. The proof is similar to that of Theorem 2.13. ■

Theorem 2.15. *Let G be a connected graph of order $n \geq 2$ with $\Delta(G) = n - 1$. Let v be a universal vertex of G. Then v is a star chromatic vertex of G.*

Proof. On the contrary, suppose that v is not a star chromatic vertex of G. Then there exists a χ_s-set S of G such that $v \notin S$. It follows that there exists at least one vertex, say $x \in S$ such that $vx \notin E(G)$. Hence it follows that v is not a universal vertex of G, which is a contradiction. Therefore v is a star chromatic vertex of G. ■

Theorem 2.16. *Let G be a connected graph of order $n \geq 2$ with $\Delta(G) = n - 1$. Then $f_{\chi_s}(G) = 0$.*

Proof. Let v be a vertex of G such that $\deg(v) = n - 1$. Since any induced paths P_4 is not bicolor able, assign each vertex of G with distinct colours.

Hence it follows that $S = V(G)$ is the unique χ_s-set of G. Therefore $f_{\chi_s}(G) = 0$. ■

Corollary 2.17. *Let* $G = K_1 + P_{n-1} (n \geq 4)$, $f_{\chi_s}(G) = 0$.

Corollary 2.18. *Let* $G = K_1 + C_{n-1} (n \geq 4)$, $f_{\chi_s}(G) = 0$.

Theorem 2.19. *For every positive integers* $a \geq 3$, *there exists a connected graph G such that* $\chi_s(G) = f_{\chi_s}(G) = a$.

Proof. Let $P_{ia} : u_{i1} : u_{i2}, \ldots, u_{ia} (1 \leq i \leq a)$ be a copy of path on a vertices. Let G be the graph obtained from $P_{ia} (1 \leq i \leq a)$ by joining the edges u_{ij} with u_{ik} where $|j - k| \geq 2$ for all $(1 \leq i \leq a)$ and join u_{ij} with u_{kr} for all $(1 \leq i, j, k, r \leq a, j \neq r)$. The graph G is shown in Figure 2.4.

First we prove that $\chi_s(G) = a$. Since u_{i1} is adjacent to u_{ij} for $1 \leq i \leq a$ and $1 \leq j \leq a$, assign $c(u_{i1}) = 1$. Since u_{i2} is adjacent to u_{ij} for $1 \leq i \leq a$ and $1 \leq j \leq a$, assign $c(u_{i2}) = 2$. Similarly u_{ia} is adjacent to u_{ij} for $1 \leq i \leq a$ and $1 \leq j \leq a$, assign $c(u_{ia}) = a$. Since no path with four vertex is bicolourable, $\chi_s(G) = a$.

Next we prove that $f_{\chi_s}(G) = a$. It is easily seen that any χ_s-set of G is of the form $S_i = \{u_{i1}, u_{i2}, \ldots, u_{ia}\}$ for $1 \leq i \leq a$. On the contrary suppose that $f_{\chi_s}(G) < a$. Then there exists a χ_s-set say S_1 with a proper subset T of S_1 such that $|T| < a$. Then there exists $x \in S_1$ such that $x \notin T$. Without loss of generality, let $x = u_{11}$. Let $S_i' = S_1 \cup \{u_{11}\} \cup \{u_{21}\}$. Then S_i' is a χ_s-set of G with $T \subset S_1$ which is a contradiction. Therefore $f_{\chi_s}(G) = a$. ■

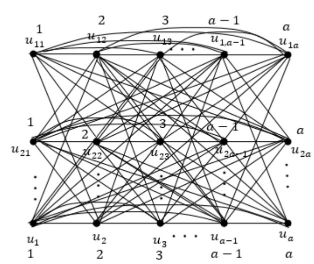

Figure 2.4

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