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SYSTEMATIC REVIEW ON GENERALIZED CONVERGENCE IN PROBABILISTIC NORMED SPACE

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Abstract

In this review paper, the concept of Statistical Convergence is systematically studied. The idea of Statistical Convergence was started nearly 68 years ago. The notion Statistical Convergence was introduced to deal with the issues of series summation. To check the behavior of the sequence (x_n) which does not converge usually, is the basic idea behind finding the notion Statistical Convergence. So the notion Statistical Convergence is much more generalized as compared to the Usual Convergence. In this systematic review the notion is studied in the PN-space also.

Introduction

The notion of Statistical Convergence came into consideration in 1935 in a monograph of Zygmund [40]. In 1951, formally it was presented by Henry Fast [8] and Steinhaus [37] in the same year. In fact, Fast [8] got this concept from Steinhaus [37]. It was introduced by Schoenberg [32]. This notion was introduced to deal with the issues of series summation. To check the behavior of a sequence (x_n) which does not converge usually is the basic idea behind the finding of the notion statistical convergence. So in comparison to usual convergence, the idea of statistical convergence is more general.

In a book namely Trigonometric Series [40], Statistical Convergence of Fourier Series was proved for the first time by Antony Zygmund [40], which was the first edition in 1935. In that book, the term Almost Convergence was used rather than Statistical Convergence.

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After the papers of Fridy [9] and Salat [27], this concept became an active area of research. Various extensions, generalizations, variants and applications of the notionhas been given by several authors in this field.

Further, important generalization of the notion such as Lacunary Statistical Convergence was given by Fridy and Orhan [10]. Using (V, λ)-summability Mursaleen [21] presented the generalization of Statistical Convergence as λ-Statistical Convergence. The concept of $\mathcal{J}$-Convergence was presented by Kostyorko, Salat and Wilcznski [17]. Colak [4] has given the impression of Statistical Convergence of order α.

Further, the notion of Statistical Convergence for Double Sequences came into consideration after the work of Bromwich [3]. Double sequences also were of keen interest of many researchers like Hardy [11], Tripathy [38], Mursaleen and Edely [6], Mursaleen and Mohiuddine [22, 23], Kostyrko [17, 18] et al., Sava's and Patterson [28] in the area of Statistical Convergence.

The notion was also studied in different spaces. The generalized metric space named as Statistical Metric Space was presented by Menger [20]. Now days, it is known as Probabilistic Metric Space and become an active area of research. It has so many applications in functional analysis. These types of sequences have motivated Karakus [13, 14] to define a new concept Statistical Convergence For Double Sequence in Probabilistic Normed Space.

In the present study we would systematically review all the data about the notion Statistical Convergence. All the extensions, generalizations, variants and applications of the notion Statistical Convergence and in the set up of Probabilistic Normed Space would be presented in this review.

Now lets recall the concept of Natural/Asymptotic Density and Statistical Convergence:

1.1. Natural Density

The probability to measure "how large is a set?" is termed as the Natural Density (Asymptotic Density). Suppose A is the subset of $\mathbb{N}$. Then Natural Density of set A is described as:

$$\delta(\mathrm{A}) = \frac{1}{n}|\{a \in A : a \leq n\}|, \text{ when } n \to \infty,$$

where $| * |$ denotes cardinality of thin closed set.

Remark. If $\delta(A) = 0$, then A is known as thin subset of $\mathbb{N}$ and if $\delta(A) \neq 0$ then it is called a non-thin subset of $\mathbb{N}$.

1.2. Statistical Convergence

The notion Statistical Convergence is mainly established using the concept of natural density for the subsets of natural numbers.

Definition 1.2.1. A sequence $x = (x_g)$ is known as Statistical Convergent to a number L if for each $\varepsilon > 0$,

$$\delta(\{g \leq n : | x_g - L | \geq \varepsilon\}) = 0.$$

Hence, we can write it as $S - \lim_{g \to \infty} x_g = L$ or $x_g \to L(S)$ as $g \to \infty$.

The L is known as Statistical Limit for the sequence (x_g) and S represents the collection of those sequences which are statistically convergent.

1.3. Probabilistic Normed Space *PN*-space

Firstly, Karl Menger [20] presented probabilistic metric space as the generalization of a metric space in the category of statistical metric space. The technique of the *PN*-Space helps in those circumstances where we are not able to find exact distance between two different points. He proposed the probabilistic concept for the distance between the points p, q by the distribution function $F_{pq}(x)$ in place of the number $d(p, q)$. In 1962, Serstnev [36] also proposed *PN*-space as an important family of *PN*-Space

Some basic definitions related to probabilistic normed spaces are given as below:

If $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$, then the distribution function and triangular norm are defined as follow:

Definition 1.3.1. A distribution function $f : \mathbb{R} \to \mathbb{R}_0^+$ is a non-decreasing and left continuous with $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$. Also, D^+ denotes all the collection of all distribution functions on $\mathbb{R}$.

For example of a unit step function $G(x)$ treated as distribution function:

$$G(x) = \begin{cases} 0 & if x \leq 0 \\ 1 & if x > 0. \end{cases}$$

Definition 1.3.2. A t-norm is the continuous mapping $\lozenge : [0, 1] \times [0, 1] \to [0, 1]$. For each x, y, z and $d \in [0, 1]$, we have

(a) $x \lozenge 0 = x$;

(b) $x \lozenge y = y \lozenge x$;

(c) $z \lozenge d \geq x \lozenge y$ if $z \geq x, d \geq y$;

(d) $(x \lozenge y) \lozenge z = x \lozenge (y \lozenge z)$.

With the help of above definitions the probabilistic normed space is defined as follows.

Definition 1.3.3. The triplet $(X, F, *)$ is said to be probabilistic normed space if X is a vector space and F is a map from X into D^+ (for $x \in X$, the distribution function $F(x)$ is represented by F_x, $F_x(t)$ is the value of F_x at $t \in \mathbb{R}$) and $*$ is the t-norm holds the next mentioned properties;

(i) $F_x(0) = 0$.

(ii) $F_x(t) = 1$ for all $t > 0 \Leftrightarrow x = 0$.

(iii) $F_{ax}(t) = F_x\left(\frac{t}{|a|}\right)$ for every $a \in \mathbb{R} - \{0\}$.

(iv) $F_{x+y}(S + t) \geq F_x(S) * F_y(t)$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+$.

Karakus [13] defined Statistical Convergence as well as Statistical Cauchy in *PN*-space and gave some important characterization for statistically convergent sequences, which are given below:

Definition 1.3.4. Let $(X, F, *)$ be a A PN-space sequence $x = (x_n)$ is called convergent to $L \in X$ w.r.t. probabilistic norm F if for any $\varepsilon > 0$ and $\lambda \in (0, 1)$, there is an integer $k_0 > 0$ with $F_{x_n - L}(\varepsilon) > 1 - \lambda$ for all $n \geq k_0$.

Symbolically, $F - \lim_{n\to\infty} x_n = L$ or $x_n \xrightarrow{F} L$ as $n \to \infty$.

2. Literature Survey

In 2008, Sencimen and Pehlivan investigated statistical continuity in *PN*-space in their paper Statistical continuity in *PN*-space. The authors also examined the statistical continuity properties of probabilistic norm [33].

In 2009, Rahmat and Harikrishan studied the notions *I*-convergence of sequences, *I*-convergence of sequences of functions and *I*-Cauchy sequences in *PN*-space and proved some important results in their paper On *I*-convergence in the Topology Induced by Probabilistic Norms [26].

In 2009, Lafuerza-Guillen and Rafi the notion of Statistical Convergence and Cauchy's sequence in *PN*-space which is endowed in strong topology in their paper Statistical Convergence in Strong Topology of *PN*-space [19].

In 2009, Sencimen and Pehlivan studied the notion of Statistically *D*-bounded sequences in the setup of *PN*-space which is endowed in strong topology in their paper Statistically *D*-bounded sequences in *PN*-space [35].

In 2010, Mursaleen and Mohiuddine studied and extended the notion of *I*-convergence, $I\{*\}$- convergence, *I*-limit points and *I*-cluster pointsfor double sequences in *PN*-space [23].

In 2011, Esi and Ozdemir generalized the results given by Karakus and Alotaibi in their paper Generalized Delta (*m*)-Statistical Convergence in. In their study the authors defined the concepts of S-Delta m (lambda) – statistical Convergence and *S*-Delta *m* (lambda)-statistically Cauchy in *PN*-space in [2].

In 2011, Mursaleen and Lohani studied statistical bounded sequences in the setup of *PN*-space in their paper Statistical limit superior and limit inferior in *PN*-space [24].

In 2012, Esi studied and explained the important properties of the notion asymptotically double lacunary statistical convergent sequences in *PN*-space in their paper On Asymptotically Double Lacunary Statistical Equivalent Sequences in *PN*-space [7].

In 2012, Mursaleen and Mohiuddine in the setup of *PN*-space studied the notion of *I*-convergence, $I\{*\}$- convergence, *I*-limit points and *I*-cluster points in their paper on ideal convergence in *PN*-space. The authors also defined relationship between *I*-convergence and $I\{*\}$- convergence. The authors proved that $I\{*\}$- convergence implies the *I*-convergence in *PN*-space but converse is not true. [25].

In 2012, Sava and Esi studied the notion of statistical convergence of triple sequences in *PN*-space [29].

In 2012, Savas and Mohiuddine studied am important generalization of statistical convergence in his paper "lambda-Statistically Convergent Double Sequences in Probabilistic Normed Spaces" [30].

In 2015, Savas and Gurdal introduced a new type of summability notion, *I*-statistical convergence and *I*-lacunary statistical convergence for double sequences in *PN*-space in their paper *I*-Statistical Convergence In *PN*-space [34].

In 2016, Hazarika has given lacunary *I*-convergence of double Sequences in *PN*-space in their paper Lacunary ideal convergence of multiple sequences in *PN*-space [31].

In 2016, Sencien and Pehlivan studied the notion statistical cluster point of a sequence. In their paper the authors investigated some properties of the set of all strong statistical cluster points of a sequence in *PN*-space in their paper Strong Gamma-Statistical Convergence in *PN*-space [35].

In 2016, Tripathy and Goswami studied multiple sequences and its statistical convergence in their paper Statistically Convergent Multiple Sequences in *PN*-space [39].

In 2017, Aldhaifallah, Nisar, Srivastava et al. defined a new notion Statistical Lambda-Convergence which is an important generalization in their paper Statistical Lambda-Convergence in Probabilistic Normed Spaces [1].

In 2017, Kilicman, Borgohain studied the relationship of lacunary (lambda)-statistical convergence along with lacunary (lambda)-summable sequences in their paper some new lacunary statistical convergence with ideals [16]. The authors also studied the notion (lambda)-lacunary statistical convergence in *PN*-space and discussed some topological properties.

In 2018, Khan, Altaf, Khan motivated by the work of Kizmaz proposed his paper. A New Type of Ideal Convergence of Difference Sequence in Probabilistic Normed Space [15].

Conclusion

In this review paper, the notion of Statistical Convergence and Statistical Convergence in *PN*-space is systematically reviewed. All the important papers are considered and analysed till date. Although this notion is being an active area of research but still there are some gaps in the research in this direction which can be easier to find with the help of this review.

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