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GENERALIZATION OF PRIME IDEALS ON FUZZY

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Abstract

In this paper, the notation of fuzzy prime ideals and maximum prime ideals are introduced by applying the general theory of algebraic fuzzy system.

Introduction

A fuzzy set is outlined mathematically by distribution to every doable individual within the universe of discourse a price representing its grade membership within the fuzzy set. This grade corresponds to the degree to that that individual is comparable or compatible with the construct diagrammatic by the fuzzy set. Thus individuals may belong to a greater or lesser degree as indicated by larger or smaller membership grade. These membership grades square measure fairly often diagrammatic by complex number values go within the interval between zero and one.

Fuzzy Prime ideals

Definition. Let μ be any fuzzy ideal of a ring R, $t \in [0, 1]$ and $t \leq \mu(0)$. The ideal μ_t is called a level ideal of μ.

Definition. A fuzzy ideal μ of a ring R is called prime, if the ideal μ_t, where $\mu(0) = t$, is a prime ideal of R.

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Theorem. *Each level ideal* μ_t, $t \in I_m \mu$ *is prime. If* $\mu(x) < \mu(y)$ *for some* $x, y \in R$. *Then*

$$\mu(xy) = \mu(y)$$

Proof. To prove that,

$$\mu(x) = \mu(y)$$

Given by the condition

$$\mu(x) < \mu(y) \ldots \text{(i)}$$

Consider,

$$\mu(x) = t, \; \mu(y) = t' \text{ and } \mu(xy) = S$$

Then,

$$\begin{aligned} S &= \mu(xy) \\ &\geq \max(\mu(x), \mu(y)) \text{ [by definition of fuzzy ideal]} \\ &= \mu(y) \\ &= t' \\ S &\geq t' \end{aligned}$$

Assume that,

$$S > t' \ldots \text{(ii)}$$

Consider,

$$\mu(x) < \mu(y) < \mu(xy)$$

That is,

$$t < t' < S$$

Then,

$$X \in \mu_s \text{ (or) } Y \in \mu_s$$

$XY \in \mu_s$ where μ_s is a prime ideal of R.

Hence,

$$t = \mu(x) \geq S$$

(or)

$$t = \mu(y) \geq S$$

Therefore, $t \geq S$. [by condition (i)]

Which is contradiction from (ii)

$$\mu(xy) = \mu(y)$$

Result

If μ is any fuzzy prime ideal of a ring R. Then $\mu(xy) = \max(\mu(x), \mu(y))$.

Result

If μ and θ are any fuzzy prime ideal of a ring R. Then $\mu \cap \theta$ is a fuzzy prime ideal of $R \leftrightarrow \mu \subseteq \theta$ (or) $\theta \subseteq \mu$.

Theorem. *$I \epsilon I_m \mu$ and let θ be any fuzzy prime ideal of R. Then $\mu \cap \theta$ is a fuzzy prime ideal of the ring.*

$\mu_t = \{x \epsilon R / \mu(x) = 1\}$

Proof. θ be any fuzzy prime ideal of R.

Case (i). If θ is constant

Let $\theta(r) = c$ for all $r \epsilon R$. [since for all $(x \epsilon \mu_t)$]

Then, we have

$$(\mu \cap \theta)x = \min(\mu(x), \theta(x))$$
$$= \min(1, c)$$
$$= c$$

Hence $\mu \cap \theta$ is a fuzzy prime ideal of μ_t.

Case (ii). If θ is non-constant

Then there exists $\alpha \epsilon [0, 1]$ such that

$$\theta(x) = \begin{cases} 1 & \text{if } x \epsilon \theta_t \\ \alpha & \text{if } x \epsilon R \sim \theta_t \end{cases}$$

Where $\theta_t = \{x \epsilon R / \theta(x) = 1\}$

Also θ_t is a prime ideal of R

$\theta_t \cap \mu_t$ is a prime ideal of μ_t.

$$(\mu \cap \theta) = \left\{\begin{cases} 1 & \text{if } x \epsilon \theta_t \cap \mu_t \\ \alpha & \text{if } x \epsilon \mu_t - (\theta_t \cap \mu_t) \end{cases}\right.$$

Hence $\mu \cap \theta$ is a fuzzy prime ideal of μ_t

Hence the proof.

Theorem. *If f is a homomorphism from a ring R onto a ring R' and* μ *is any f-invarient fuzzy prime ideal of R. Then* $f(\mu)$ *is fuzzy prime ideal of R'.*

(*or*)

$f(\mu)$ *is a fuzzy prime ideal of R'. Prove that* $\sigma' \subseteq f(\mu)$ (*or*) $\theta' \subseteq (\mu)$.

Proof. Let σ' and θ' be any two fuzzy ideal of R' such that $\sigma'\theta' \subseteq f(\mu)$

Then,

$$f^{-1}(\sigma'\theta') \subseteq f^{-1}(f(\mu)) = \mu \text{ [by result } f^{-1}(f(\mu)) = \mu]$$

$$f^{-1}(\sigma')f^{-1}(\theta') \subseteq \mu$$

Either,

$$\sigma' = f(f^{-1}(\sigma)) \subseteq f(\mu)$$

$$\sigma' \subseteq f(\mu) \text{ (or)}$$

$$\theta' = f(f^{-1}(\theta)) \subseteq f(\mu)$$

$$\theta' \subseteq f(\mu)$$

$$\sigma'\theta' \subseteq f(\mu)$$

Accordingly,

$f(\mu)$ is fuzzy prime ideal of R.

Hence the proof.

Theorem. *Let f be a homomorphism from a ring R onto a ring R' if* μ' *and* θ' *are any two fuzzy ideals of R' then the following holds*,

$$f^{-1}(\mu')f^{-1}(\theta') \subseteq f^{-1}(\mu'\theta').$$

Proof. Let $x \epsilon R$ and $\varepsilon > 0$ be given the set

$$\alpha = (f^{-1}(\mu')f^{-1}(\theta'))(x) \text{ and}$$

$$\beta = (f^{-1}(\mu'\theta'))(x)$$

Then,

$$\alpha - \varepsilon < \sup(\min(f^{-1}(\mu')), (f^{-1}(\theta'))(x_1))$$

$x = x_1x_2$

$$= \sup(\min \mu'(f(x_1)), \theta'(f(x_2)))$$

$x = x_1x_2$

$\alpha - \varepsilon < \min(\mu'(f(x_1)), \theta'(f(x_2))$ for some $x_1x_2 \epsilon R$ such that $x = x_1x_2$

$$\leq (\mu'\theta')(f(x_1x_2))$$

$$= f^{-1}(\mu'\theta')(x) = \beta$$

$\alpha - \varepsilon \leq \beta$

Since ε is an arbitrary.

Hence,

$$f^{-1}(\mu')f^{-1}(\theta') \subseteq f^{-1}(\mu'\theta')$$

Theorem. *If f is an homomorphism from a ring R onto a ring R' and* μ' *is* *any fuzzy prime ideal of R.*

Proof.

$$\mu\sigma \leq f^{-1}(\mu)$$

Then,

$$f(\mu\sigma) \subseteq f(f^{-1}(\mu')) = \mu'$$

By using theorem,

"Let f be a homomorphism from a ring R onto a ring and R' if μ' and θ' are any two fuzzy ideals of R' then the following holds,

$$f^{-1}(\mu')f^{-1}(\theta') \subseteq f^{-1}(\mu'\theta')\text{"}$$

$$f(\mu)f(\sigma) \subseteq \mu'$$

Either $f(\mu) \subseteq \mu'$ (or)

$$f(\sigma) \subseteq \mu'$$

Since μ' is fuzzy prime,

Either,

$$f^{-1}(f(\mu)) \subseteq f^{-1}(\mu')$$

(or)

$$f^{-1}(f(\sigma)) \subseteq f^{-1}(\mu')$$

$$\mu \subseteq f^{-1}(\mu')$$

(or)

$$\sigma \subseteq f^{-1}(\mu')$$

Hence,

$f^{-1}(\mu)$ is fuzzy prime ideal of R'.

Theorem. *If* μ *and* σ *be fuzzy semi prime ideals of R then* $\mu \times \sigma$ *is a fuzzy* *semi prime ideal of* $R \times R$.

Proof. We know that the cartesian product of any two fuzzy ideals is fuzzy ideal by (Malik and Mordeson, 1991).

So it is enough to show that for all $(a, b) \epsilon R \times R$.

Either,

$$\mu \times \sigma((a, b)^n) = \mu \times \sigma(a, b)$$

Since μ and σ is fuzzy semi prime ideals of R for all $a, b \epsilon R, \forall\ n \varepsilon^+$.

$$\mu(a)^n = \mu(a) \text{ and } \sigma(b)^n = \mu(b)$$

Then,

$$\begin{aligned} \mu \times \sigma((a, b)^n) &= \mu \times \sigma\,(a^n, b^n) \\ &= \min\,(\mu(a^n), \sigma(b^n)) \\ &= \mu(a^n) \\ &= \mu(a) \\ &= \mu \times \sigma(a, b) \end{aligned}$$

$$\begin{aligned} \mu \times \sigma((a, b)^n) &= \mu \times \sigma(a^n, b^n) \\ &= \min\,(\mu(a^n), \sigma(b^n)) \\ &= \sigma(b^n) \\ &= \sigma(b) \\ &= \mu \times \sigma(a, b) \end{aligned}$$

Therefore $\mu \times \sigma$ is a fuzzy semi prime ideal of $R \times R$.

Theorem. *If* μ *and* σ *be fuzzy semi primary ideals of* R *then* $\mu \times \sigma$ *is a fuzzy semi primary ideal of* $R \times R$.

Proof. Since, μ and σ be fuzzy semi primary ideals of R then for all $a, b, c, d \epsilon R$.

Either $\mu(ab) \leq \mu(a^n)$ or else $\mu(ab) \leq \mu(b^m)$ for some $m, n \varepsilon^+$.

Either $\sigma(cd) \leq \sigma(d^k)$ or else $\sigma(cd) \leq \sigma(d^l)$ for some k/ε^+.

Let $t = \max(m, n)$ and

$s = \max(k, l)$

Then,

$$
\begin{aligned}
\mu \times \sigma((a, c)(b, d)) &= \mu \times \sigma(ab, cd) \\
&= \min(\mu(ab), \sigma(cd)) \\
&= \mu(ab) \\
&\leq \mu(a^t) \\
&= \mu \times \sigma(a, c)^t
\end{aligned}
$$

or else,

$$
\begin{aligned}
\mu \times \sigma((a, c)(b, d)) &= \mu \times \sigma(ab, cd) \\
&= \min(\mu(ab), \sigma(cd)) \\
&= \sigma(cd) \\
&\leq \sigma(d^s) \\
&= \mu \times \sigma(b, d)^s
\end{aligned}
$$

Therefore $\mu \times \sigma$ is fuzzy semi primary ideal of $R \times R$.

Hence the proof.

Corollary. *A fuzzy ideal* $\mu \times \sigma$ *of* $R \times R$ *is fuzzy semi primary ideal if and only if the level ideals* $t \epsilon L_m \mu \times \sigma, (\mu \times \sigma)_t$ *are semi primary ideals of* $R \times R.s.$

Conclusion

The study of semi prime ideals in universal algebras is under investigation by the authors using the commentator of fuzzy ideals. moreover ,the radical of fuzzy ideals in universal algebra will be studied and would be applied to characterize the central properties of fuzzy semi prime ideals.

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