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EXISTENCE OF SOLUTIONS OF A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH FRACTIONAL DERIVATIVE AND FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract

This paper is concerned with the existence of solutions of the Fractional Boundary Value Problem

$$\begin{cases} D_0^p z(t) + f(t, z(t)) = 0, t \in [0, 1] \\ z(0) = z'(0) = 0, D_0^q z(1) + \lambda I_0^r z(1) = 0 \end{cases}$$

where D_0^p is Riemann-Liouville fractional derivative of order p, I_0^r denotes the Riemann-Liouville fractional integral of order $r, p \in (2, 3], q \in (0, 1], r > 0, f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $\lambda \neq -\frac{\Gamma(p+r)}{\Gamma(p-q)}$. Some existence results are obtained by means of Krasnoselskii's Fixed Point Theorem and Banach Fixed Point Theorem.

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1. Introduction

There are several kinds of fractional derivatives, such as Riemann-Liouville fractional derivative, Marchaud fractional derivative, Caputos derivative, Griinwald-Letnikov fractional derivative, etc. Fractional differential equations are being used in various fields of science and engineering such as control system, electrochemistry, electromagnetics, viscoelasticity, physics, biophysics, porous media, blood flow phenomena, electrical circuits, biology, fitting of experimental data etc. Due to these features, models of fractional order become more practical and realistic than the models of integer-order. There has been a significant development in the existence and uniqueness of boundary value problems for fractional differential equations, However, the theory of BVPs for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored see, [1]-[13] and the references therein. In [14], Li et al. considered the following three point BVPs of fractional order differential equations

$$\begin{cases} D_0^p z(t) + f(t, z(t)) = 0, \ t \in [0, 1] \\ z(0) = 0, \ D_0^q z(1) + D_0^q z(\xi) = 0. \end{cases}$$

Motivated by the work done in [14] [16], we are concerned with the nonlinear fractional boundary value problem

$$\begin{cases} D_0^p z(t) + f(t, z(t)) = 0, \ t \in [0, 1] \\ z(0) = z'(0) = 0, \ D_0^q z(1) + \lambda I_0^r z(1) = 0, \end{cases} \tag{1}$$

where D_0^p is Riemann-Liouville fractional derivative of order p, I_0^r denotes the Riemann-Liouville fractional integral of order $r, p \in (2, 3], q \in (0, 1]$, $r > 0, f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $\lambda \neq -\frac{\Gamma(p+r)}{\Gamma(p-q)}$.

2. Preliminaries

First of all, we introduce some notations, definitions and Lemmas.

Definition 1 [4]. The Fractional Integral of order $p > 0$ of a function $y : [0, \infty) \to \mathbb{R}$ is given by

$$I_0^p y(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s) ds$$

provided that Right integral exists

Definition 2 [4]. The Riemann-Liouville fractional Derivative of order $p > 0$ for a continuous function $y : [0, \infty) \to \mathbb{R}$ is defined as

$$D^p y(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-p-1} y(s) ds,$$

where n is the smallest integer greater than or equal to p.

Lemma 3 [9]. *Let* $p > 0$. *If we assume* $u \in C(0, 1) \cap L(0, 1)$, *then the fractional differential equation*

$$D_0^p u(t) = 0$$

has unique solution $u(t) = c_1 t^{p-1} + c_2 t^{p-2} + \ldots + c_N t^{p-N}$, $c_i \in \mathbb{R}, i = 1, 2, \ldots N$ *where N is the smallest integer greater than or equal to p*.

Lemma 4 [9]. *Assume that* $u \in C(0, 1) \cap L(0, 1)$ *with a fractional derivative of order* $p > 0$ *that belongs to* $C(0, 1) \cap L(0, 1)$. *Then*

$$I_0^p D_0^p u(t) = u(t) + c_1 t^{p-1} + c_2 t^{p-2} + \ldots + c_N t^{p-N}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$ *where N is the smallest integer greater than or equal to p*.

3. Auxiliary Result

In this section, we present supporting result needed in our main proofs.

Lemma 5. *Let* $y(t) \in L[0, 1]$ *and* $2 < p \leq 3$. *Then the problem*

$$\begin{cases} D_0^p z(t) + y(t) = 0, \; t \in [0, 1] \\ z(0) = z'(0) = 0, \; D_0^q z(1) + \lambda I_0^r z(1) = 0 \end{cases} \tag{2}$$

has a unique solution

$$z(t) = -\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} y(s)ds$$

$$+\frac{t^{p-1}}{\Gamma(p)\Delta\Gamma(p-q)}\int_0^1 (1-s)^{p-q-1} y(s)ds$$

$$+\frac{\lambda t^{p-1}}{\Gamma(p)\Delta\Gamma(p+r)}\int_0^1 (1-s)^{p+r-1} y(s)ds, \tag{3}$$

where $\Delta = \frac{1}{\Gamma(p-q)} + \frac{\lambda}{\Gamma(p+r)}$.

Proof. We know that by Lemma 4, (2) is equivalent to

$$z(t) = -I_0^p y(t) + c_1 t^{p-1} + c_2 t^{p-2} + c_3 t^{p-3} \tag{4}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, 3$.

From $z(0) = 0$ implies $c_3 = 0$. Therefore

$$z(t) = -I_0^p y(t) + c_1 t^{p-1} + c_2 t^{p-2}.$$

Also $z'(0) = 0 \Rightarrow c_2 = 0$.

Thus (4) becomes

$$z(t) = -I_0^p y(t) + c_1 t^{p-1}. \tag{5}$$

Now

$$D_0^q z(t) = -I_0^{p-q} y(t) + c_1 \frac{\Gamma(p)}{\Gamma(p-q)} t^{p-q-1}.$$

Also

$$I_0^r z(t) = -I_0^{p+r} y(t) + c_1 \frac{\Gamma(p)}{\Gamma(p+r)} t^{p+r-1}$$

using boundary condition $D_0^q z(1) + \lambda I_0^r z(1) = 0$

$$-I_0^{p-q}y(1)+c_1\frac{\Gamma(p)}{\Gamma(p-q)}+\lambda\left(-I_0^{p+r}y(1)+c_1\frac{\Gamma(p)}{\Gamma(p+r)}\right)=0$$

$$\Rightarrow c_1\left(\frac{\Gamma(p)}{\Gamma(p-q)}+\lambda\frac{\Gamma(p)}{\Gamma(p+r)}\right)=I_0^{p-q}y(1)+\lambda I_0^{p+r}y(1)$$

$$c_1=\frac{1}{\Gamma(p)\Delta}\left[\frac{1}{\Gamma(p-q)}\int_0^1(1-s)^{p+r-1}y(s)ds+\lambda\frac{1}{\Gamma(p+r)}\int_0^1(1-s)^{p-q-1}y(s)ds\right].$$

On putting the value of c_1 in (5), we obtain the solution (3). □

Let $Z=C[0,1]$, then $(Z,\|\cdot\|_Z)$ is a Banach space equipped with the norm

$$\|z\|_Z=\{\sup|z(\xi)|:\xi\in[0,1]\}$$

we define an operator $U:Z\to Z$ by

$$(Uu)(t)=-\int_0^t\frac{(t-s)^{p-1}}{\Gamma(p)}f(s,u(s))ds$$

$$+\frac{t^{p-1}}{\Gamma(p)\Delta\Gamma(p-q)}\int_0^1(1-s)^{p-q-1}f(s,u(s))ds$$

$$+\frac{\lambda t^{p-1}}{\Gamma(p)\Delta\Gamma(p+r)}\int_0^1(1-s)^{p+r-1}f(s,u(s))ds. \quad (6)$$

It follows from Lemma 5, the fixed point of U are the solution of (1).

For the forthcoming analysis, we need the following assumptions:

(A) $\|f(t,x)-f(t,y)\|\le L\|x-y\|$ for all $t\in[0,1]$

(B) $\|f(t,u)\|\le\|\theta\|$ for all $(t,u)\in[0,1]\times Z;\ \theta\in(L^1[0,1],\mathbb{R}^+)$

for convenience, let us set

$$\nabla=\left[\frac{1}{\Gamma(p+1)}+\left|\frac{1}{\Gamma(p)\Delta\Gamma(p-q+1)}\right|+\left|\frac{\lambda}{\Gamma(p)\Delta\Gamma(p+r+1)}\right|\right] \quad (7)$$

4. Existence Results

Theorem 6. *Let the function f satisfies assumption* (*A*) *with* $L < \frac{1}{\nabla}$; *where* ∇ *is given by* (7). *Then fractional boundary value problem given by* (1) *has a unique solution on* $[0, 1]$.

Proof. We shall use the Banach Contraction Principle to prove that U has a fixed point. For that, we shall prove U is a contraction. Let $z_1, z_2 \in C[0, 1]$. Then $\forall t \in [0, 1]$, we have

$$\| U(z_1(t)) - U(z_2(t)) \| \leq \frac{1}{\Gamma(p-1)} \int_0^t (t-s)^{p-1} \| f(s, z_1(t)) - f(s, z_2(t)) \| ds$$

$$+ \left| \frac{t^{p-1}}{\Gamma(p)\Delta\Gamma(p-q)} \right| \int_0^1 (1-s)^{p-q-1} \| f(s, z_1(t)) - f(s, z_2(t)) \| ds$$

$$+ \left| \frac{\lambda t^{p-1}}{\Gamma(p)\Delta\Gamma(p+r)} \right| \int_0^1 (1-s)^{p+r-1} \| f(s, z_1(t)) - f(s, z_2(t)) \| ds$$

$$\leq L \| z_1 - z_2 \| \left[\frac{1}{\Gamma(p+1)} + \left| \frac{t^{p-1}}{\Gamma(p)\Delta\Gamma(p-q+1)} \right| + \left| \frac{\lambda t^{p-1}}{\Gamma(p)\Delta\Gamma(p+r+1)} \right| \right]$$

$$\leq L \| z_1 - z_2 \| \left[\frac{1}{\Gamma(p+1)} + \left| \frac{1}{\Gamma(p)\Delta\Gamma(p-q+1)} \right| + \left| \frac{\lambda}{\Gamma(p)\Delta\Gamma(p+r+1)} \right| \right]$$

$$\leq L \| z_1 - z_2 \| \nabla.$$

Since $L\nabla < 1$, therefore, U is a contraction, which satisfies all the conditions of Banach Contraction Principle. Hence, U has a unique fixed point which is a solution of the problem (1). □

Next Result is based on Krasnoselskii's Fixed Point Theorem

Theorem 7 [26] (Krasnoselskii's fixed point theorem)**.** *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(1) $Ax + By \in M$ *whenever* $x, y \in M$;

(2) *A is compact and continuous*;

(3) *B is a contraction mapping.*

Then there exists $z \in M$ *such that* $z = Az + Bz.$

Theorem 8. *Let* $f : [0, 1] \times Z \to Z$ *be a continuous function maps bounded subsets of* $[0, 1] \times Z$ *into relatively compact subset of Z and A and B holds along with*

$$L\left[\left| \frac{1}{\Gamma(p)\Delta\Gamma(p-q+1)} \right| + \left| \frac{\lambda}{\Gamma(p)\Delta\Gamma(p+r+1)} \right|\right] < 1. \tag{8}$$

Then the boundary value problem (1) *has at least one solution defined on* $[0, 1]$.

Proof. Let $\sup_{t\in[0, 1]}| \theta(t) | = \| \theta \|$ fix $\bar{r} \geq \| \theta \| \nabla.$

Assume $B_{\bar{r}} = \{u \in Z; \| u \| \leq \bar{r}\}.$

Now define operators P and Q on $B_{\bar{r}}$ by

$$Pu(t) = -\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, u(s))ds$$

$$Qu(t) = \frac{t^{p-1}}{\Gamma(p)\Delta\Gamma(p-q)} \int_0^1 (1-s)^{p-q-1} f(s, u(s))ds$$

$$+ \frac{\lambda t^{p-1}}{\Gamma(p)\Delta\Gamma(p+r)} \int_0^1 (1-s)^{p+r-1} f(s, u(s))ds$$

for $z_1, z_2 \in B_{\bar{r}}$, we find that

$$\| Pz_1 + Qz_2 \| \leq \| \theta \| \left(\frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(p)\Delta\Gamma(p-q)} + \frac{\lambda}{\Gamma(p)\Delta\Gamma(p+r)} \right)$$

$$= \| \theta \| \nabla \leq \bar{r}.$$

Thus $Pz_1 + Qz_2 \in B_{\bar{r}}$. Also Q is a contraction mapping by (8). Continuity of f implies that P is continuous. Also, P is uniformly bounded on $B_{\bar{r}}$ as

$$\| Pu \| \leq \frac{\theta}{\Gamma(p+1)}.$$

In view of (A), define $\sup_{(t, u)\in[0, 1]\times B_{\bar{r}}} | f(t, u) | = \bar{f}$, we have

$$\| Pz(t_2) - Pz(t_1) \| = \| \frac{1}{\Gamma(p)} \int_0^{t_1} [(t_2 - s)^{p-1} - (t_1 - s)^{p-1}] f(s, u(s)) ds$$

$$+ \int_{t_1}^{t_2} (t_2 - s)^{p-1} f(s, u(s)) ds \|$$

$$\leq \frac{\bar{f}}{\Gamma(p+1)} | 2(t_2 - t_1)^p + t_1^p - t_2^p |$$

which is independent of u. Thus, P is equicontinuous. Using the fact, f maps bounded subsets into relatively compact subsets, we have that $P(V)$ is relatively compact in Z for every t, where V is a bounded subset of Z. So P is relatively compact on $B_{\bar{r}}$.

Hence by Arzela-Ascoli Theorem, P is compact on $B_{\bar{r}}$. Thus all assumptions of Theorem 7 are Satisfied. So bvp (1) has at least one solution on [0, 1]. □

5. Examples

Example 1. Consider the following fractional boundary value problem

$$\begin{cases} D_0^{\frac{5}{2}} u(t) = \frac{1}{t+1} \frac{| u(t) |}{1 + | u(t) |}, \\ u(0) = u'(0) = 0, \ (D_0^{\frac{1}{2}} u)(1) + \frac{2}{3} (I_0^{\frac{3}{2}} u)(1) = 0. \end{cases} \tag{9}$$

Here $p = \frac{5}{2}, q = \frac{1}{2}, r = \frac{3}{2}, \lambda = \frac{2}{3} \neq \frac{\Gamma(p+r)}{\Gamma(p-q)} = 6$ and $f(t, u(t)) = \frac{1}{t+1} \frac{| u(t) |}{1 + | u(t) |}$. As $| f(t, u) - f(t, v) | \leq | u - v |$, therefore (A) is satisfied along with $L = 1$ and

$$\Delta = \left[\frac{1}{\Gamma(p-q)} + \frac{\lambda}{\Gamma(p+r)} \right]$$

$$= 1.333.$$

Also

$$L\nabla = \left[\frac{1}{\Gamma\left(\frac{7}{2}\right)} + \left| \frac{1}{\frac{5}{2}\Delta\Gamma(3)} \right| + \left| \frac{2}{\frac{5}{2}\Delta\Gamma(5)} \right| \right]$$

$$= 0.0209 < 1.$$

Therefore all the conditions of Theorem (6) are satisfied. So fractional boundary value problem (9) has unique solution on [0, 1].

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