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TOTAL EDGE IRREGULARITY STRENGTH OF SOME POLYTOPE STRUCTURES

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Abstract

Given a graph G with vertex set and edge set, a function defined from vertex set and edge set to $1, 2, \ldots, k$ is called an edge irregular total k-labeling if for every pair of distinct edges, the weight of the edges are all distinct. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength of G. The total edge irregularity strength of G is denoted by $tes(G)$. In our present study we have considered some graphs of the family of convex polytopes and have obtained its total edge irregularity strength.

1. Introduction

Graph theory has always been an interesting area of research. One of the key branches in it is graph labeling. Most graph labeling techniques trace their origin to one introduced by Rosa [8]. Rosa identified three types of labelings, which he called α-labeling, β-labeling and ρ-labeling [8]. The β-labeling were later renamed as graceful by Golomb and since then graceful labeling has been well studied [8].

Labeled graphs have broad range of applications such as coding theory, communication network, addressing, database management, secret sharing schemes, models for constraint programming over finite domains and network passwords [8].

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Baca, Jendrol, Miller and Ryan [9] introduced the total edge irregularity strength of a graph. Total edge irregularity strength has been well studied for honeycomb mesh networks [6], hexagonal networks [7], butterfly networks [2, 4], benes networks [2], series compositions of uniform theta graphs [3] and generalized uniform theta graph [5]. Umer et al. [12] applied the technique of 3-total edge product cordial labeling on some classes of convex polytopes. Syed Ahtshma Ul Haq Bokhary et al. [11] proved the total irregularity strength of convex polytope graphs S_n, T_n, U_n.

Definition 1.1 [8]. Given a graph $G = (V, E)$ a labeling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ is called an edge irregular total k-labeling if for every pair of distinct edges uv and xy, the weights $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x)$ $\partial(xy) + \partial(y)$. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength of G. The total edge irregularity strength of G is denoted by $tes(G)$.

We now begin with some known results on $tes(G)$ and basic definitions.

Theorem 1.1 [9]. *Let G be a graph with m edges. Then* $tes(G) \geq \lceil (m+2)/3 \rceil$.

Theorem 1.2 [9]. *Let G be a graph with maximum degree Δ. Then* $tes(G) \geq \lceil (\Delta + 2)/3 \rceil$.

In our study, we have considered some families of convex polytopes. Our results on edge irregular total k-labeling applied to these graphs of convex polytopes are presented in this paper. Further we have proved that the bound on tes is sharp as given in Theorem 1.1. In our paper we call the weight of the edges as edge sums.

Definition 1.2 [1]. For $m \geq 5$, convex polytope D_m consists of $2m$ 5-sided faces and a pair of m-sided faces. For our convenience we call the cycle induced by u_i the inner cycle, the vertices v_i are called interior vertices, the vertices w_i are called exterior vertices, cycle induced by z_i the outer cycle. The number of vertices of D_m is $4m$ and the number of edges of D_m is $6m$.

Notation 1. The vertex set and edge set of D_m are defined as follows:

$V(D_m) = \{u_i, v_i, w_i, z_i, 1 \leq i \leq m\}$ and $E(D_m) = \{u_i u_{i+1}, v_i v_{i+1}, z_i z_{i+1}, 1 \leq i \leq m\} \cup \{u_i, v_i, v_i, w_i, v_{i+1}, w_i, z_i, 1 \leq i \leq m\}$. See Figure 1.

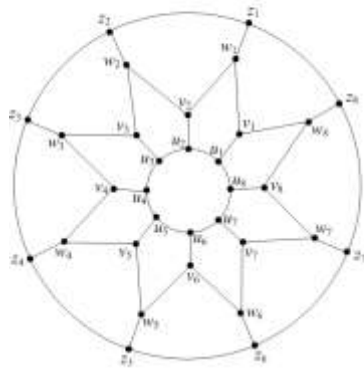

Figure 1. Convex Polytope D_8.

Definition 1.3 [10]. Convex polytope C_m consisting of $3m$ 3-sided faces, m 4-sided faces, m 5-sided faces and a pair of m-sided faces is obtained by the combination of the convex polytope Q_m and graph of an antiprism A_m. The convex polytope C_m can also be obtained from the convex polytope B_m by adding new edges $y_{i+1}z_i$ and having the same vertex set that is $V(C_m) = V(B_m)$ and $E(C_m) = E(B_m) \cup \{y_{i+1}z_i : 1 \leq i \leq m\}$. The vertex set and edge set of B_m are $V(B_m) = \{u_i, v_i, w_i, y_i, z_i, 1 \leq i \leq m\}$ and $E(B_m) = \{u_i u_{i+1}, v_i v_{i+1}, y_i y_{i+1}, z_i z_{i+1}, 1 \leq i \leq m\} \cup \{u_i v_i, v_i w_i, v_{i+1} w_i, w_i y_i, y_i z_i, 1 \leq i \leq m\}$. For our convenience, we call the cycle induced by $\{u_i, 1 \leq i \leq m\}$, the inner cycle, cycle induced by $\{v_i, 1 \leq i \leq m\}$, the interior cycle, the set of vertices $\{w_i, 1 \leq i \leq m\}$, the set of interior vertices, cycle induced by $\{y_i, 1 \leq i \leq m\}$, the exterior cycle, cycle induced by $\{z_i, 1 \leq i \leq m\}$, the outer cycle. The number of vertices of C_m is $5m$ and the number of edges of C_m is $10m$.

Notation 2. The vertex set and edge set of C_m are defined as follows: $V(C_m) = \{u_i, v_i, w_i, y_i, z_i, 1 \leq i \leq m\}$ and $E(C_m) = \{u_i u_{i+1}, v_i v_{i+1}, y_i y_{i+1}, z_i z_{i+1}, 1 \leq i \leq m\} \cup \{u_i v_i, v_i w_i, w_i y_i, y_i z_i, v_{i+1} w_i, y_{i+1} z_i, 1 \leq i \leq m\}$. See Figure 2.

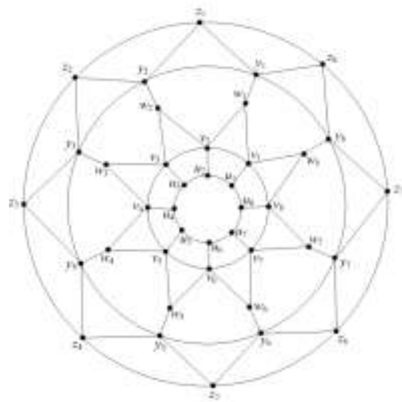

Figure 2. Convex Polytope C_8.

Definition 1.4 [13]**.** The graph G_m with 3, 5, 6 and m-sided faces. The order, size and faces of G_m, $5m$, $8m$ and $3m+2$. For our convenience, we call the cycle induced by $\{u_i, 1 \leq i \leq m\}$, the inner cycle, cycle induced by $\{v_i, 1 \leq i \leq m\}$, the central vertices, the set of vertices $\{w_i, 1 \leq i \leq m\}$ and $\{y_i, 1 \leq i \leq m\}$, the middle cycle vertices, cycle induced by $\{z_i, 1 \leq i \leq m\}$, the outer cycle. The number of vertices of G_m are $4m$ and the number of edges of G_m is $8m$.

Notation 3. The vertex set and edge set of G_m are defined as follows: $V(G_m) = \{u_i, v_i, w_i, y_i, z_i, 1 \leq i \leq m\}$ and $E(G_m) = \{u_i u_{i+1}, z_i z_{i+1}, 1 \leq i \leq m\} \cup \{u_i v_i, u_{i+1} v_i, v_i w_i, w_i y_i, w_{i+1} y_i, y_i z_i, 1 \leq i \leq m\}$. See Figure 3.

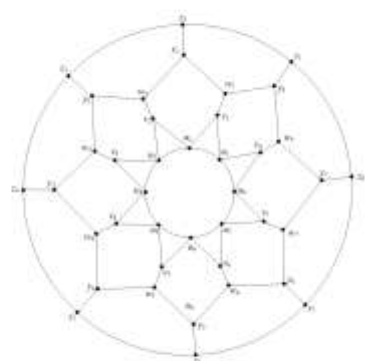

Figure 3. Convex Polytope G_8.

2. Main Results

Theorem 1. *For every* $m \geq 3$ *the total edge irregularity strength of convex polytope* D_m *is* $tes(D_m) = \lceil (6m+2)/3 \rceil = 2m+1$.

Proof of Theorem 1. The vertices and edges of D_m are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, the interior vertices then the exterior vertices followed by the vertices of the outer cycle. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input. The graph of convex polytope D_m, $m \geq 3$.

Algorithm.

Step 1. $f(u_i) = 1$

$f(v_i) = m+1$

$f(w_i) = m+1$

$f(z_i) = 2m+1, 1 \leq i \leq m.$

Step 2. $f(u_i u_{i+1}) = i, 1 \leq i \leq m-1$

$f(u_m u_1) = m.$

Thus the edge sums of the inner cycle are $3, 4, \ldots, m+2$.

Step 3.

$f(u_i v_i) = i, 1 \leq i \leq m.$

Thus the edge sums are $m+3$ to $2m+2$.

Step 4.

$f(u_i w_i) = 2i-1$

$f(v_{i+1} w_i) = 2i, 1 \leq i \leq m$

$f(v_1 w_m) = 2m.$

Thus the edge sums are from $2m+3$ to $4m+2$.

Step 5.

$f(w_i z_i) = m + i, 1 \leq i \leq m.$

Thus the edge sums are $4m + 3$ to $5m + 2$.

Step 6.

$f(z_i z_{i+1}) = m + i, 1 \leq i \leq m$

$f(z_m z_1) = 2m.$

Thus the edge sums of the outer cycle are $5m + 3$ to $6m + 2$.

Output. $tes(D_m) = \lceil (6m + 2)/3 \rceil = 2m + 1.$

Proof of Correctness. By the above stepwise procedure we see that the edge sums obtained are all unique. Hence D_m is total edge k-irregular. Labeling of D_5 is shown in Figure 4.

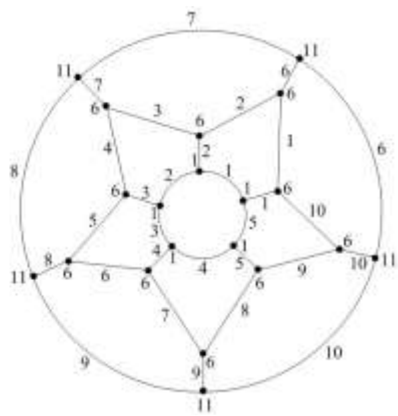

Figure 4. $tes(D_5) = 11.$

Theorem 2. *For every* $m \geq 3$ *the total edge irregularity strength of convex polytope* C_m *is* $tes(C_m) = \lceil (10m + 2)/3 \rceil = 3m + 3.$

Proof of Theorem 2. The vertices and edges of C_m are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, interior cycle, set of interior vertices then the exterior cycle followed by the vertices of the outer cycle. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input. The graph of convex polytope C_m for $m \geq 3$.

Algorithm.

Step 1.

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = m + 3$

$f(y_i) = 3m$

$f(z_i) = 3m + 3, 1 \leq i \leq m.$

Step 2.

$f(u_i u_{i+1}) = i, 1 \leq i \leq m - 1$

$f(u_m u_1) = m.$

Thus the edge sums of the inner cycle are $3, 4, \ldots, m + 2$.

Step 3.

$f(u_i v_i) = i, 1 \leq i \leq m$

Thus the edge sums are $m + 3$ to $2m + 2$.

Step 4.

$f(v_i v_{i+1}) = i, 1 \leq i \leq m - 1$

$f(v_m v_1) = m.$

Thus the edge sums of the interior cycle are $2m + 3$ to $3m + 2$.

Step 5.

$f(v_1 w_1) = m - 1$

$f(v_i w_i) = (m - 3) + 2i, 2 \leq i \leq m - 1$

$f(v_2 w_1) = m$

$f(v_{i+1}w_i) = (m-2) + 2i,\ 2 \le i \le m-2$

$f(v_1 w_m) = 3m - 2.$

Thus the edge sums of the alternating band of triangles are from $3m+3$ to $5m+2$.

Step 6.

$f(w_i y_i) = (m-1) + i,\ 1 \le i \le m.$

Thus the edge sums are $5m+3$ to $6m+2$.

Step 7.

$f(y_i y_{i+1}) = i + 2,\ 1 \le i \le m-1$

$f(y_1 y_m) = m + 2.$

Thus the edge sums of the exterior cycle are $6m+3$ to $7m+2$.

Step 8.

$f(y_1 z_1) = m$

$f(y_i z_i) = (m-3) + 2i,\ 2 \le i \le m-1$

$f(y_2 z_1) = m + 1$

$f(y_{i+1} z_i) = (m-1) + 2i,\ 2 \le i \le m-2$

$f(y_1 z_m) = 3m - 1.$

Thus the edge sums of the alternating band of triangles are from $7m+3$ to $9m+2$.

Step 9.

$f(z_i z_{i+1}) = 3m + i - 4,\ 1 \le i \le m-1$

$f(z_m z_1) = 4m - 4.$

Thus the edge sums of the outer cycle are $9m+3$ to $10m+2$.

Output. $tes(C_m) = \lceil (10m+2)/3 \rceil = 3m+3.$

Proof of Correctness. By the above stepwise procedure we see that the edge sums obtained are all unique. Hence C_m is total edge k-irregular. Labeling of C_5 is shown in Figure 5.

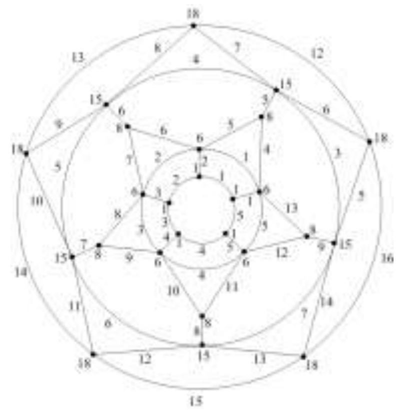

Figure 5. $tes(C_5) = 18$.

Theorem 3. *For every* $m \geq 3$ *the total edge irregularity strength of convex polytope* G_m *is* $tes(G_m) = \lceil (8m+2)/3 \rceil = 3m - 1$.

Proof of Theorem 3. The vertices and edges of G_m are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, the central vertices, then the middle cycle vertices followed by the vertices of the outer cycle. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input. The graph of convex polytope G_m for $m \geq 3$.

Algorithm.

Step 1.

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = m + 3$

$f(y_i) = 2m + 2$

$f(z_i) = 3m - 1, 1 \leq i \leq m.$

Step 2.

$f(u_i u_{i+1}) = i, 1 \leq i \leq m - 1$

$f(u_m u_1) = m.$

Thus the edge sums of the inner cycle are $3, 4, \ldots, m + 2$.

Step 3.

$f(u_i v_i) = 2i - 1, 1 \leq i \leq m$

$f(u_{i+1} v_i) = 2i, 1 \leq i \leq m$

$f(u_1 v_m) = 2m.$

Thus the edge sums of the alternating band of triangles are form $m + 3$ to $3m + 2$.

Step 4.

$f(v_i w_i) = (m - 2) + i, 1 \leq i \leq m.$

Thus the edge sums are $3m + 3$ to $4m + 2$.

Step 5.

$f(w_1 y_1) = 2i + 1$

$f(w_{i+1} y_i) = 2i + 2, 1 \leq i \leq m$

$f(w_1 y_m) = 2m + 2.$

Thus the edge sums are from $4m + 3$ to $6m + 2$.

Step 6.

$f(y_i z_i) = i + 6, 1 \leq i \leq m.$

Thus the edge sums are $6m + 3$ to $7m + 2$.

Step 7.

$f(z_i z_{i+1}) = (m + 4) + i, 1 \leq i \leq m - 1$

$f(z_m z_1) = 2m + 4.$

Thus the edge sums of the outer cycle are $7m + 3$ to $8m + 2$.

Output. $tes(G_m) = \lceil (8m + 2)/3 \rceil = 3m - 1.$

Proof of Correctness. By the above stepwise procedure we see that the edge sums obtained are all unique. Hence G_m is total edge k-irregular. Labeling of G_5 is shown in Figure 6.

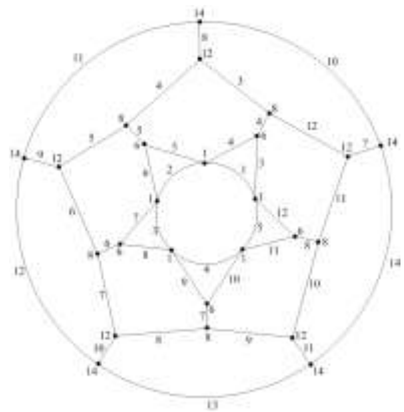

Figure 6. $tes(G_5) = 14.$

Conclusion

From the above mentioned findings discussed we have proved that some polytope structures D_m, C_m, G_m satisfies the conditions of edge irregular total k-labeling. Our future study is extended to other families of convex polytopes.

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