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INVESTIGATION OF TORSIONAL WAVES IN FIBRE REINFORCED MEDIUM EMBEDDED BETWEEN POROELASTIC MEDIUM AND POROELASTIC HALF SPACE UNDER GRAVITY

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Abstract

This paper deals with wave propagation in fibre reinforced medium embedded between isotropic porous medium and isotropic porous half space which is under gravitational force. The frequency equation is derived in the framework of Biot's theory. In absence of gravity and fibre reinforced medium, frequency equations are obtained as particular cases of the main frequency equation. Phase velocity is computed against wavenumber for various values of ratio of thickness of layers, and reinforced parameter. The numerical results are presented graphically.

1. Introduction

Fibre layer is usually useful in Civil Engineering as a building material, and its adjacent structures may involve other porous materials such as concrete mixture. Moreover, reinforced mediums are abundant on the Earth. Therefore, manmade and natural structures may contain fibre reinforced solid and porous solid. Kumar et al. [9] studied wave motion in an anisotropic fiber-reinforced thermo elastic solid. In the paper [9], the amplitude of displacements and temperature distribution were computed. The propagation of Love waves in a fiber-reinforced medium lying over an initially stressed orthotropic half-space was investigated [10]. Shear wave propagation in fiber-reinforced poroelastic medium sandwiched between two distinct dry sandy

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poroelastic half spaces was studied by Rajitha and Reddy [12], wherein, phase velocity was computed against wavenumber for various values of dry sandy parameters. It is well known that gravity plays a role in the motion of the structures, particularly, in Astrophysics. The gravity attraction or potential is governed by the distribution of mass within the Earth. Love waves in a fluid-saturated porous layer under a rigid boundary and lying over an elastic half space under gravity was studied [2]. Gupta et al. [15] studied torsional surface waves in an inhomogeneous layer over a gravitating anisotropic porous half-space, and, it was seen that phase velocity decreases as the wave number increases. In the same paper, it was concluded that as the gravity and porosity increase, phase velocity decreases, and as the anisotropy increases, phase velocity increases. Gravitational stresses in anisotropic ridges and valleys with small slopes were investigated by Liao et al. [8]. Abd-Alla et al. [1] studied propagation in fiber-reinforced anisotropic elastic media subjected to a gravity field. Love wave in the fiber-reinforced layer over a gravitating porous half-space was investigated by Ranjan and Samal [13], and, it was seen that phase velocity increases with increase of porosity of the half-space, and decreases with increase of gravity. Love wave propagation in a fiber-reinforced medium sandwiched between an isotropic layer and gravitating half space was studied [14]. In the present work, wave propagation in the fiber-reinforced medium embedded between isotropic porous medium, and porous half space under gravity is studied in the framework of Biot's theory of Poroelasticity [5].

This paper is organized as follows. In section 2, geometry and solution of the problem are presented. Boundary conditions and frequency equation are presented in section 3. In section 4, particular cases are discussed. Numerical results are discussed in section 5. Finally, conclusion is given in section 6.

2. Geometry and Solution of the Problem

Consider fibre reinforced medium (M_2, say) of thickness h_2 (say) embedded between isotropic porous medium (M_1) of thickness h_1 (say), and isotropic porous half space which is under gravity in the cartesian coordinate system. Assume that the direction of wave propagation is along in the direction of x-axis, and z-axis is taken to be the vertically downwards as shown in the figure 1. From the figure 1, it is clear that fibre reinforced medium occupies the space $-h_2 \leq z \leq 0$, upper isotropic porous medium occupies the space $-(h_1 + h_2) \leq z \leq 0$, and the lower isotropic porous half space occupies the space $z \geq 0$. The solutions for these three parts are presented separately in the following sub sections:

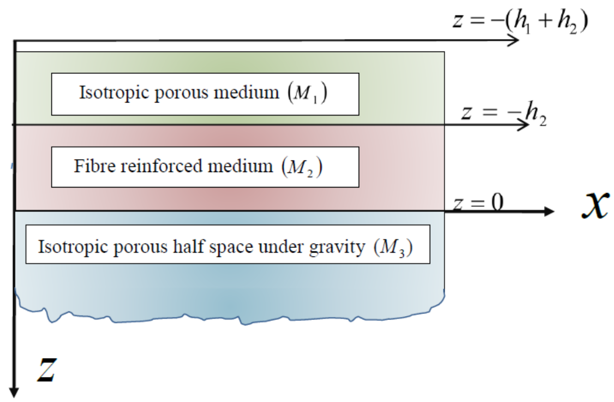

Figure 1. Geometry of the problem.

2.1 Isotropic Porous Medium

Let (u_1, v_1, w_1) and (U_1, V_1, W_1) be the solid and fluid displacement components, respectively, in the isotropic porous medium (M_1). For torsional waves the displacements are $u_1 = w_1 = 0, v_1 = v_1(x, z, t)$, and $U_1 = W_1 = 0, V_1 = V_1(x, z, t)$. The equations of motion [4] in this case reduce to

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho_{11} v_1 + \rho_{12} V_1), \frac{\partial^2}{\partial t^2}(\rho_{12} v_1 + \rho_{22} V_1) = 0. \quad (1)$$

In equation (1), ρ_{11}, ρ_{12}, and ρ_{22} are mass coefficients, the stress component and fluid pressure, are given by [5]

$$\sigma_{ij} = 2Ne_{ij} + (Ae + Q\varepsilon)\delta_{ij}, (i, j = 1, 2, 3) s = Qe + R\varepsilon. \quad (2)$$

In Equation (2), e_{ij} are strain displacements, N is the shear modulus, A, Q, R are poroelastic constants, e and ε are the dilatations of solids and fluids, respectively, and δ_{ij} is the kronecker delta function. The strain components e_{ij} are given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)(i,\ j = 1,\ 2,\ 3). \tag{3}$$

The solutions of (1) can be taken as

$$v_1 = v_1(z)e^{ik(x-ct)},\ V_1 = V_1(z)e^{ik(x-ct)}. \tag{4}$$

From the Equation (1), (2), (3), and (4), the following differential equation is obtained:

$$\frac{d^2 v_1}{dz^2} - A v_1 = 0. \tag{5}$$

The solution of Equation (5) is

$$v_1(z) = (C_1 e^{Az} + C_2 e^{-Az})\, e^{ik(x-ct)}. \tag{6}$$

In Equation (6), $A = k^2\left(1 - \frac{c^2 d_1}{N_1}\right),\ d_1 = \left(\rho_{11} - \frac{\rho_{12}^2}{\rho_{22}}\right),$ and C_1, C_2 are arbitrary constants. The shear stress component in yz plane is

$$(\sigma_{yz})_1 = N_1 A((C_1 e^{Az} - C_2 e^{-Az}))e^{ik(x-ct)}, \tag{7}$$

here N_1 is shear modulus is the medium M_1.

2.2 Fabre reinforced Medium

In a fibre-reinforced medium (M_2, say), with respect to preferred direction $\vec{a}$, the stress σ_{ij} components are given by [3]

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + a_i a_j e_{kk}) + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta(a_k a_m e_{km} a_i a_j). \tag{8}$$

$i, j, k, m = 1, 2, 3, a_1, a_2, a_3$ are the direction cosines of $\vec{a}$, therefore, $a_1^2 + a_2^2 + a_3^2 = 1$. The coefficients λ, α, β are reinforced parameters, and coefficients μ_T and μ_L are the transverse and longitudinal shear modulus, let (u_2, v_2, w_2) and (U_2, V_2, W_2) be the solid and fluid displacement components, respectively. For torsional waves, the displacements components are $u_2 = w_2 = 0, v_2 = v_2(x, z, t),$ and $U_2 = W_2 = 0, V_2 = V_2(x, z, t).$ The equations of motion in this case reduce to

$$S_1 \frac{\partial^2 v_2}{\partial x^2} + 2S_2 \frac{\partial^2 v_2}{\partial x \partial z} + S_3 \frac{\partial^2 v_2}{\partial z^2} = \frac{\partial^2}{\partial t^2}(\rho_{110} v_2 + \rho_{120} V_2),$$

$$\frac{\partial^2}{\partial t^2}(\rho_{120} v_2 + \rho_{220} V_2) = 0, \tag{9}$$

where

$$S_1 = \mu_T + (\mu_L - \mu_T)a_1^2,\ S_2 = a_1 a_3(\mu_L - \mu_T),\ S_3 = \mu_T + (\mu_L - \mu_T)a_3^2. \tag{10}$$

In Equation (9), ρ_{ij0} are mass coefficients. As in the earlier cases, the solution of Equation (9) is assumed as

$$v_2 = v_2(z)e^{ik(x-ct)},\ V_2 = V_2(z)e^{ik(x-ct)}. \tag{11}$$

From the equation (8), (9), (10), and (11), the following differential equation is obtained:

$$\frac{d^2 v_2}{dz^2} + iB_1 \frac{dv_2}{dz} - B_2^2 v_2 = 0. \tag{12}$$

In Equation (12),

$$B_1 = \frac{2kS_2}{S_3},\ B_2 = \frac{k^2}{S_3}(S_1 - c^2 d_2),\ d_2 = \rho_{110} - \frac{\rho_{120}^2}{\rho_{220}}.$$

The solution of Equation (12) is

$$v_2(z) = (e^{-\frac{D_1 z}{2}}(C_3 \cos(B_2 z) + C_4 \sin(B_2 z))e^{ik(x-ct)}. \tag{13}$$

In Equation (13), $D_1^2 = \frac{4B_2 - B_1^2}{2}$, C_3, C_4 are arbitrary constants. The shear stress component in yz plane

$(\sigma_{yz})_2 =$

$$N_2 e^{D_1 z}\left\{\left[-\sin\left(\frac{B_1 z}{2}\right)\frac{B_1}{2}+D_1\cos\left(\frac{B_1 z}{2}\right)\right]C_3-\left[\cos\left(\frac{B_1 z}{2}\right)\frac{B_1}{2}+D_1\sin\left(\frac{B_1 z}{2}\right)\right]C_4\right\}e^{ik(x-ct)}. \tag{14}$$

2.3 Isotropic Porous Half-space under Gravity

In the lower Isotropic porous half-space under gravity (M_3, say), let (u_3, v_3, w_3) and (U_3, V_3, W_3) be the solid and fluid displacement components, respectively. For torsional waves, the displacement components are $u_3 = w_3 = 0, v_3 = v_3(x, z, t)$, and $U_3 = W_3 = 0, V_3 = V_3(x, z, t)$. The equations of motion [4] in this case reduce to

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}-d_3 g\omega_{yz}+d_3 gz\left(\frac{\partial\omega_{xy}}{\partial x}-\frac{\partial\omega_{yz}}{\partial z}\right)$$

$$=\frac{\partial^2}{\partial t^2}(\rho_{1100}v_3+\rho_{1200}V_3),\ \frac{\partial^2}{\partial t^2}(\rho_{1200}v_3+\rho_{2200}V_3)=0. \tag{15}$$

In equation (15), ρ_{ij00} are mass coefficients, g is the acceleration due to gravity, ω_{xy}, ω_{yz} are rotational components given by

$$\omega_{xy}=\frac{1}{2}\left(\frac{\partial v_3}{\partial x}-\frac{\partial u_3}{\partial y}\right),\ \omega_{yz}=\frac{1}{2}\left(\frac{\partial w_3}{\partial y}-\frac{\partial v_3}{\partial z}\right). \tag{16}$$

As in the earlier cases, the solutions of Equation (15) are assumed as follows:

$$v_3=v_3(z)e^{ikx-ct},\ V_3=V_3(z)e^{ik(x-ct)}. \tag{17}$$

Equation (15), (2), (16), and (17) give

$$v_3''(z)+\left(\frac{a}{\beta_1^2+az}\right)v_3'(z)+k^2\left(\frac{c^2}{\beta_1^2+az}-1\right)v_3(z)=0. \tag{18}$$

In Equation (18), $a=\frac{-g}{2}$, and $\beta_1^2=-\frac{N_3}{d_3}$, the solution of Equation (18) is assumed as follows: $v_3(z)=\phi(z)(\beta_1^2+az)^{-\frac{1}{2}}$. (19)

From the Equation (18), and (19), the following differential equation is obtained:

$$\phi''(z) + \left[\frac{a^2}{4(\beta_1^2 + az)^2} + k^2\left(\frac{c^2}{\beta_1^2 + az} - 1\right)\right]\phi(z) = 0. \tag{20}$$

Introducing $z_1 = \frac{-2k}{a}(\beta_1^2 + az)$, and $s = -\frac{c^2 k}{2a}$, substituting $\phi(z) = \phi(z_1)$, the Equation (20) becomes

$$\phi''(z_1) + 4k^2\left[\frac{1}{4z_1^2} + \frac{s}{z_1} - \frac{1}{4}\right]\phi(z_1) = 0. \tag{21}$$

Equation (21) is the well-known form of Whittaker equation. The solution of Equation (21) is $\phi(z_1) = C_5 W_{-s,0}(-z_1) + C_6 W_{s,0}(z_1)$. The solution must vanish as $z \to \infty$ i.e., for $z_1 \to \infty$. In this case, the displacement component v_3 is given by

$$v_3 = C_5(\beta_1^2 + az)^{-\frac{1}{2}} \exp\left(\frac{2}{G} - kz\right)\left(2kz - \frac{4}{G}\right)^{-s}\left[1 - \frac{\left(s + \frac{1}{2}\right)}{\left(2kz - \frac{4}{G}\right)}\right]e^{ik(x-ct)}. \tag{22}$$

where $G = \frac{d_3 g}{N_3 k}$ is Biot's gravity parameter. The shear stress component in yz plane

$$(\sigma_{yz})_{M_3}$$

$$= N_3\left\{\beta_1^{\frac{-1}{2}} e^{\frac{2}{G}}\left(-\frac{G}{4}\right)^s\left[\frac{G}{4} + \frac{sG}{4} - 1\right]\left(1 + \frac{G(2s+1)^2}{16}\right) + \frac{G^2(2s+1)^2}{32}\right\}C_5 e^{ik(x-ct)}. \tag{23}$$

3. Boundary Conditions

The conditions at free boundary and two interfaces are prescribed as follows: At the stress free surface $z = -(h_1 + h_2)$, $(\sigma_{yz})_{M_1} = 0$, at $z = -h_2$, stress must be continuous i.e. $(\sigma_{yz})_{M_1} = (\sigma_{yz})_{M_2}$, and $(\sigma_{yz})_{M_1} = (\sigma_{yz})_{M_3}$ at

$z = 0,$ the displacement components are continuous at $z = -h_2$ i.e. $v_1 = v_2,$ and at rigid boundaries $z = 0$ i.e. $v_2 = v_3$. These boundary conditions lead to the following system of homogeneous equations in arbitrary constants C_l.

$$[D_{lm}][C_l] = 0,\ l,\ m = 1, 2, 3, 4, 5. \tag{24}$$

Equations. (24) result in a system of five homogeneous equations in five arbitrary constants C_1, C_2, C_3, C_4, C_5. In order to obtain a non-trivial solution, the determinant of coefficients must be zero. Accordingly, the following frequency equation is obtained:

$$|\ D_{lm}\ | = 0,\ l,\ m = 1, 2, 3, 4, 5. \tag{25}$$

4. Particular Cases

Case (i). If the gravitational force is neglected in lower porous half space. Then the frequency equation (25) reduces to

$$\begin{vmatrix} D_{11} & D_{12} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & D_{24} & 0 \\ 0 & 0 & D_{33} & D_{34} & D_{35} \\ D_{41} & D_{42} & D_{43} & D_{44} & 0 \\ 0 & 0 & D_{53} & 0 & D_{55} \end{vmatrix} = 0 \tag{26}$$

In Equation (26),

$$D_{11} = e^{-A(h_1+h_2)},\ D_{12} = -e^{-A(h_1+h_2)},\ D_{13} = D_{14} = D_{15} = 0,$$

$$D_{21} = -Ae^{-Ah_2},\ D_{22} = Ae^{-Ah_2},\ D_{23}$$

$$= e^{-D_1h_2}\left[-\sin\left(-\frac{B_1h_2}{2}\right)\frac{B_1}{2} + D_1\cos\left(-\frac{B_1h_2}{2}\right)\right],$$

$$D_{24} = -e^{-D_1h_2}\left[\cos\left(-\frac{B_1h_2}{2}\right)\frac{B_1}{2} + D_1\sin\left(-\frac{B_1h_2}{2}\right)\right],$$

$$D_{25} = 0,\ D_{31} = D_{32} = 0,\ D_{33} = D_1,\ D_{34} = -\frac{B_1}{2},\ D_{35} = 0,$$

$$D_{42} = e^{Ah_2},\ D_{43} = e^{-D_1 h_2}\cos\left(\frac{B_1 h_2}{2}\right),\ D_{44} = e^{-D_1 h_2}\sin\left(\frac{B_1 h_2}{2}\right),$$

$$D_{45} = 0,\ D_{51} = D_{52} = 0,\ D_{53} = 1, D_{54} = 0,\ D_{55} = 1.$$

This case was considered in the paper [2].

Case (ii). If the middle part fiber-reinforced medium is neglected, then the frequency equation in this case reduces to

$$\begin{vmatrix} D_{110} & D_{120} & 0 \\ D_{210} & D_{220} & D_{230} \\ D_{310} & D_{320} & D_{330} \end{vmatrix} = 0. \tag{27}$$

In Equation (27),

$$D_{110} = e^{-Ah_1},\ D_{120} = -e^{-Ah_1},\ a_{130} = 0,\ D_{210} = 1,\ D_{220} = -1,\ D_{230} = \chi_1,$$

$$D_{310} = 1,\ D_{320} = 1,\ D_{330} = \chi_2,$$

In the above,

$$\chi_1 = \left[\left(\frac{G}{4} + \frac{sG}{2} - 2\right)\left(1 + \frac{G(2s+1)^2}{16}\right) + \frac{G^2(2s+1)^2}{16}\right],$$

$$\chi_2 = \left[1 + \frac{G(2s+1)^2}{16}\right].$$

This case was considered in the paper [11].

5. Numerical Results

For numerical process, the following materials are used. For upper isotropic, sandstone saturated with kerosene [16], (Mat-I, say) is employed, and its material parameter values are $N_1 = 9.22 \times 10^{-10}$, $\rho_{11} = 18.8876 \times 10^{-8}$, $\rho_{12} = -19.6133 \times 10^{-11}$, $\rho_{22} = 21.0843 \times 10^{-9}$.

For middle fibre-reinforced half space, carbon fibre-epoxy resin [6], (Mat-II, say) is employed, and its material parameter values are $a_1^2 = 0.25$, $a_3^2 = 0.75$, $\mu_L = 2.46$, $\mu_T = 5.66$, $\rho_{110} = 10.8876 \times 10^{-8}$, ρ_{120}

$= -19.6133 \times 10^{-11}$, $\rho_{120} = -19.6133 \times 10^{-11}$, $\rho_{220} = 21.0843 \times 10^{-9}$. For the lower medium, sandstone saturated with water, [7] (Mat-III, say), is employed, and its material parameter values are $N_2 = 2.76 \times 10^{-10}$, $\rho_{1100} = 18.6620 \times 10^{-8}$, $\rho_{1200} = 0$, $\rho_{2200} = 22.1630 \times 10^{-9}$. For the above numerical process Biot's gravity (G) parameter is taken to be $0, 0.2, 0.4$. The figure 2 depicts the variation of phase velocity with wavenumber for the various values of thickness ratio (h_1/h_2). From the figure 2, it is clear that, as wavenumber increases the phase velocity decreases. Also it is clear that as thickness (h_1/h_2) ratio increases, phase velocity increases. The figure 3 depicts the variation of phase velocity with wavenumber for various values of fibre reinforced parameter a_1. In this figure, it is observed that as wavenumber increases phase velocity decreases. Also it is clear that as fibre reinforced parameter a_1 increases, phase velocity decreases. From the results, it is concluded that the phase velocity is constant for all the values of Biot's gravity (G) parameter. It is also seen that phase velocity is independent of Biot's gravity (G) parameter.

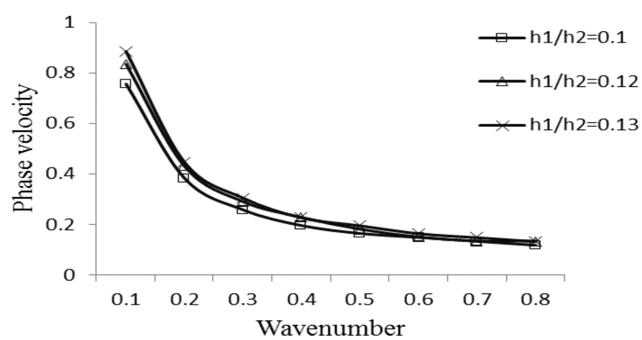

Figure 2. Variation of phase velocity with wavenumber for different (h_1/h_2) values of ratio of thickness.

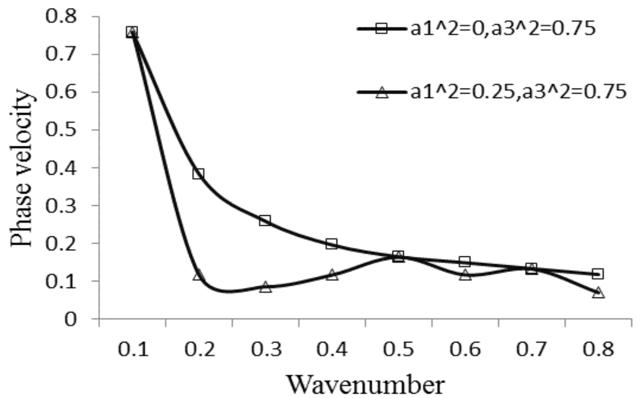

Figure 3. Variation of phase velocity with wavenumber for different a_1^2 values of fibre reinforced parameter.

6. Conclusion

Torsional waves in fibre reinforced medium embedded between isotropic porous medium, and porous half space under gravity. In the frame work of Biot's theory is investigated phase velocity is computed against wavenumber. From the numerical results, it is observed that as wavenumber increases, phase velocity decreases. It is also concluded that phase velocity is independent of Biot's gravity parameter. This kind of analysis is useful in Structural Engineering that are adjacent to the Earth.

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