Advances and Applications in Mathematical Sciences
Volume 20, Issue 8, June 2021, Pages 1609-1622

ϕ-CONTRACTION AND ITS APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATION

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Abstract

In this article we defined ϕ contraction, which is more globally than previously defined θ contraction and generalized Khan contraction [4]. More general phenomena have also been shown by giving the suitable examples. The rearmost part of this article consists of the application of this contraction to fractional differential equation.

1. Introduction and Preliminaries

Firstly, the idea of θ contraction in 2014 introduced by Jleli et al. [2] and defined generalization of Banach Contraction. After that many researchers (see [1], [3], [5]) developed work on fixed point. In 2017 Piri et al. [4] defined generalized Khan contraction and they settled the existence and uniqueness of fixed point. In this work with the concept of θ contraction and Khan contraction we defined new type of ϕ contraction and furnished fixed point theorem, supporting examples for the newly defined concept and application to fractional differential equations is the important part of this article.

2010 Mathematics Subject Classification: 55M20, 46B85.
Keywords: fixed point, metric space, contraction.
Received June 6, 2020

For the results we recall some basic definitions:

Definition 1.1 [2]. Let Φ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfy the conditions

θ_1. θ is non decreasing,

θ_2. for every sequence $\{\alpha_n\} \subset (0, \infty)$

$\lim_{n \to \infty} \alpha_n = 1 \Leftrightarrow \lim_{n \to \infty} \alpha_n = 0^+$

θ_3. there exists $s \in (0, 1)$ and $L \in (0, \infty]$ such that

$$\lim_{\alpha \to 0} \frac{\theta(\alpha) - 1}{\alpha^s} = L$$

and they prove the results.

Theorem 1.1 [2]. *Let (V, ρ) be a complete metric space and $A : V \to V$ be a mapping let there exists $\theta \in \Phi$ such that*

$$u, v \in V, \rho(Au, Av) \neq 0 \Rightarrow \theta(\rho(Au, Av)) \leq [\theta(\rho(u, v))]^k. \tag{1.1}$$

Then A has a unique fixed point.

In 2017 Piri et al. [4] defined generalized Khan contraction.

Definition 1.2. Let (V, ρ) be a metric space. A mapping $A : V \to V$ is called generalized Khan Contraction if satisfies

$$\rho(Au, Av) \leq \begin{cases} k \dfrac{\rho(u, Au)\rho(u, Av) + \rho(v, Av)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}} & \text{if } \max\{\rho(u, Av), \rho(Au, v)\} \neq 0 \\ 0 & \text{if } \max\{\rho(u, Av), \rho(Au, v)\} = 0 \end{cases}$$

where $k \in [0, 1)$ and $u, v \in V$.

Theorem 1.2 [4]. *Let (V, ρ) be a complete metric space and $A : V \to V$ satisfy*

$$\rho(Au, Av) \leq \begin{cases} k \dfrac{\rho(u, Au)\rho(u, Av) + \rho(v, Av)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}} & \textit{if } \max\{\rho(u, Av), \rho(Au, v)\} \neq 0 \\ 0 & \textit{if } \max\{\rho(u, Av), \rho(Au, v)\} = 0 \end{cases}$$

where $k \in [0, 1)$ and $u, v \in V$. Then A has a unique fixed point.

2. Main Results

As the main part of our paper, we introduced new type of ϕ contraction defined as:

Let (V, ρ) be a complete metric space and $A : V \to V$ be a mapping and there exists $\phi \in \Phi$ and $k \in (0, 1)$ such that

$$\phi(\sigma(Au, Av)) \leq (\phi(M(u, v)))^k, \tag{2.1}$$

where

$$M(u, v) = \begin{cases} \dfrac{\rho(u, Au)\rho(u, Av) + \rho(v, Av)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}} & \text{if } \max\{\rho(u, Av), \rho(Au, v)\} \neq 0 \\ 0, & \text{if } \max\{\rho(u, Av), \rho(Au, v)\} = 0 \end{cases}$$

If we take $\phi(u) = e^u$ then,

$$e^{(\rho(Au, Av))} \leq (e^{M(u, v)})^k$$

$$e^{(\rho(Au, Av))} \leq (e^{\frac{\rho(u, Au)\rho(u, Av) + \rho(v, A_v)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}}})^k$$

$$e^{(\rho(Au, Av))} \leq (e^{k\frac{\rho(u, Au)\rho(u, Av) + \rho(v, Av)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}}})$$

$$\rho(Au, Av) \leq k\frac{\rho(u, Au)\rho(u, Av) + \rho(v, A)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}}$$

which is Khan contraction [4], i.e., Khan contraction is a special case of our newly defined contraction.

Theorem 2.1. *Let* (V, ρ) *be a complete metric space and* $A : V \to V$ *be a mapping satisfy* 2.1. *Then A has a unique fixed point.*

Proof. Let $u \in V$ and u_n be any sequence of V such that

$$u_{n+1} = Au_n \;\; \forall n = 0, 1, 2, 3, \ldots.$$

Now

$$\phi(\rho(u_{n+1}, u_n)) = \phi(\rho(Au_n, Au_{n-1})) \leq (\phi(M(u_n, u_{n-1}))^k \tag{2.2}$$

where

$$M(u_n, u_{n-1}) = \frac{\rho(u_n, u_{n+1})\rho(u_n, u_n) + \rho(u_{n-1}, u_n)\rho(u_{n-1}, u_{n+1})}{\max\{\rho(u_n, u_n), \rho(u_{n+1}, u_{n-1})\}}$$

$$= \frac{\rho(u_{n-1}, u_n)\rho(u_{n-1}, u_{n+1})}{\max\{(u_{n+1}, u_{n-1})\}}$$

$$= \rho(u_n, u_{n-1})$$

this together we get

$$\phi(\rho(u_n, u_{n+1})) \leq (\phi(\rho(u_{n-1}, u_n)))^k$$

therefore by continue this process we get

$$1 < \phi(\rho(u_n, u_{n+1})) \leq (\phi(\rho(u_{n-2}, u_{n-1})))^{k^2}$$

$$\leq (\phi(\rho(u_{n-2}, u_{n-1})))^{k^3}$$

$$\leq \dots$$

$$\dots$$

$$\leq (\phi(\rho(u_0, u_1)))^{k^n}$$

Taken $n \to \infty$ we get

$$(\phi(\rho(u_n, u_{n+1})) \to 1$$

therefore by θ_2 we obtain

$$\lim_{n\to\infty}(u_n, u_{n+1}) = 0. \tag{2.3}$$

Now we shall show that $\{u_n\}$ is a cauchy sequence in V.

So from θ_3, $\exists\, 0 < p < 1$ and $0 < l < \infty$ such that

$$\lim_{n\to\infty}\frac{\rho(u_n, u_{n+1}) - 1}{(\rho(u_n, u_{n+1}))^s)} = l. \tag{2.4}$$

If $l < \infty$ and $v = \frac{1}{2}$ therefore there exists $m \in N$ such that

$$\left| \frac{\rho(u_n, u_{n+1})-1}{(\rho(u_n, u_{n+1}))^s} - l \right| \leq v, \forall n \geq m.$$

So

$$\frac{\rho(u_n, u_{n+1})-1}{(\rho(u_n, u_{n+1}))^s} \geq l - v = v \;\; \forall n \geq m.$$

Hence

$$n(\rho(u_n, u_{n+1}))^s) \geq n\beta[(\rho(u_n, u_{n+1}))-1] \, \forall n \geq m,$$

where $\beta = \frac{1}{v}$.

If $l \to \infty$. Let $v > 0$ be a given real number then by there exists $m \in N$ such that

$$\frac{\rho(u_n, u_{n+1})-1}{(\rho(u_n, u_{n+1}))^s} \geq v \;\; \forall n \geq m$$

so we get

$$n(\rho(u_n, u_{n+1}))^s) \leq n\beta[(\rho(u_n, u_{n+1}))-1] \, \forall n \geq m$$

Where $\beta = \frac{1}{v}$.

Hence, for all the cases there exists $\beta > 0$ and $m \in N$ such that

$$n(\rho(u_n, u_{n+1}))^s \leq n\beta[(\rho(u_n, u_{n+1}))-1] \;\; \forall n \geq m$$

taking limit $n \to \infty$ both side we get

$$\lim_{n\to\infty} n(\rho(u_n, u_{n+1}))^s) = 0$$

therefore by the definition of limit there exists $m_1 \in N$ such that

$$n(\rho(u_n, u_{n+1}))^s) \leq 1 \;\; \forall n > m_1$$

then

$$\rho(u_n, u_{n+1}) \leq \frac{1}{(n)^{\frac{1}{s}}} \, \forall n > m_1.$$

Now, for $p > q > m$, we have

$$\rho(u_p, u_q) \leq \sum_{j=p}^{q-1} \rho(u_j, u_{j+1})$$

$$\leq \sum_{j=p}^{q-1} \frac{1}{j^{1/s}}$$

because $0 < s < 1$, therefore by P-series test $\sum_{j=p}^{q-1} \frac{1}{j^{1/s}}$ is convergent.

Hence $\rho(u_p, u_q) \to 0$ as $p, q \to \infty$ therefore $\{u_n\}$ is Cauchy sequence. Since V is complete metric space then $\exists\, u \in V$ such that $u_n \to u$. Now we show that u is a fixed point of S

$$\rho(Au, u) = \lim_{n\to\infty} \rho(Au_n, u_n)$$

$$= \lim_{n\to\infty} \rho(u_{n+1}, u_n)$$

$$\rho(Au, u) = 0$$

therefore u is fixed point of A.

For the Uniqueness. Let if possible there two fixed point u_1 and u_2 such that $u_1 \neq u_2$, because u_1 and u_2 is fixed point so $Au_1 = u_1$ and $Au_2 = u_2$

Now

$$\phi(\rho(Au_1, Au_2)) \leq (\phi(M(u_1, u_2))^k$$

$$\phi(\rho(u_1, u_2)) \leq (\phi(\frac{\rho(u_1, Au_1)\rho(u_1, Au_2) + \rho(u_2, Au_2)\rho(u_2, Au_1)}{\max\{\rho(u_1, Au_2), \rho(Au_1, u_2)\}})^k$$

$$\leq \phi\left(\frac{\rho(u_1, u_1)\rho(u_1, u_2) + \rho(u_2, u_2)\rho(u_2, u_1)}{\max\{\rho(u_1, u_2), \rho(u_1, u_2)\}}\right)^k$$

$$\phi(\rho(u_1, u_2)) \leq (\phi(\rho(u_1, u_2)))^k$$

which is contradiction.

So

$$u_1 = u_2.$$

Hence A has a unique fixed point.

Corollary 2.1. *Let (V, ρ) be a complete metric space and $A : V \to V$ be a mapping, and there exists $\phi \in \Phi$ and $k \in (0, 1)$ such that*

$$u, v \in V, \rho(Au, Av) \neq 0 \; \phi(\rho(Au, Av)) \leq (\phi(N(u, v)))^k \tag{2.5}$$

where $N(u, v) = \max\{M(u, v), \rho(u, v)\}$ and

$$M(u, v) = \begin{cases} \dfrac{\rho(u, Au)\rho(u, Av) + \rho(v, Av)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}} & \text{if } \max\{\rho(u, Av), \rho(Au, v)\} \neq 0 \\ 0 & \text{if } \max\{\rho(u, Av), \rho(Au, v)\} = 0. \end{cases}$$

Then A has a unique fixed point.

Proof. We have

$$N(u, v) = \max\{M(u, v), \rho(u, v)\}$$

so two cases arrive:

Case 1. If

$$\max\{M(u, v), \rho(u, v)\} = M(u, v).$$

Then the proof is similar to theorem 2.1.

Case 2. If

$$\max\{M(u, v), \rho(u, v)\} = \rho(u, v).$$

Then we get

$$\phi(\rho(Au, Av)) \leq [\phi(\rho(u, v))]^k.$$

Then proof is similarly the Theorem 1.1

Example 2.1. Let $V = \{0, 1, 2, 3\}$ and $\rho(u, v) = |u - v| \; \forall u, v \in V$. Then Clearly is (V, ρ) complete metric space Let $A : V \to V$ be defined by

$$A(2) = A(1) = 1, A(0) = 2, A(3) = 0.$$

Now we taken $\phi(u) = e^u$ when $u = 0, v = 1$ then

$$A(0) = 2,\ A(1) = 1$$

$$\phi(\rho(Au,\ Av)) \le (\phi(M(u,\ v)))^k$$

$$\phi(\rho(Au,\ Av)) \le \left(\phi \frac{\rho(u,\ Au)\rho(u,\ Av) + \rho(v,\ Av)\rho(v,\ Au)}{\max\{\rho(u,\ Av),\ \rho(Au,\ v)\}}\right)^k$$

$$\phi(1) \le \left(\phi\left(\frac{2 \times 1 + 0 \times 1}{\max\{1,\ 1\}}\right)\right)^k$$

$$e^1 \le e^{2k} \Rightarrow k \ge 1/2.$$

When $u = 0, v = 2$ then $A(0) = 2,\ A(2) = 1$ we get

$$\phi(1) \le \left(\phi\left(\frac{2 \times 1 + 1 \times 1}{\max\{1,\ 0\}}\right)\right)^k$$

$$e^1 \le e^{2k} \Rightarrow k \ge 1/2$$

When $u = 0, v = 3$ then $A(0) = 2,\ A(3) = 0$ so we get

$$\phi(2) \le \left(\phi\left(\frac{2 \times 0 + 3 \times 1}{\max\{0,\ 1\}}\right)\right)^k$$

$$e^1 \le e^{3k} \Rightarrow k \ge 2/3$$

When $u = 1, v = 3$ then $A(1) = 1,\ A(3) = 0$ so we get

$$\phi(1) \le \left(\phi\left(\frac{0 + 3 \times 2}{\max\{1,\ 2\}}\right)\right)^k$$

$$e^1 \le e^{3k} \Rightarrow k \ge 1/3$$

When $u = 2, v = 3$ then $A(2) = 1,\ A(3) = 0$ so we get

$$\phi(1) \le \left(\phi\left(\frac{1 \times 2 + 3 \times 2}{\max\{2,\ 2\}}\right)\right)^k$$

$$e^1 \le e^{4k} \Rightarrow k \ge 1/4.$$

In all the above cases, the condition of our theorem satisfies and it is clear that 1 is a fixed point of A.

Note: Example 2.1 satisfy our contraction but not satisfy Jleli [2] so Jleli is a special case of our contraction. Because in Theorem 1.1 we take when $u = 2$, $v = 3$ then $A(2) = 1$, $A(3) = 0$ and $\phi(u) = e^{u}$ so we get

$$\phi(\rho(A(2), A(3))) \leq (\phi(\rho(2, 3)))^{k}$$

$$e^{1} \leq e^{k}.$$

So

$$k \geq 1$$

which is contradiction.

Hence our contraction is more generalized to Jleli [2]

Example 2.2. Let $V = \{\gamma_n : n \in N\}$ where $\gamma_n = \frac{n(n+1)}{2}$ and $\sigma : V \times V \to [0, \infty)$ defined by $\sigma(u, v) = \max\{u, v\}$ if $u \neq v$ and $\rho(u, v) = 0$ if $u = v$ Then (V, ρ) is complete metric space and consider the mapping $S : V \to V$ by $A(\gamma_1) = \gamma_1$ and $A(\gamma_n) = \gamma_{n-1}$ where $n > 1$ we taken $\phi(u) = e^{u}$.

Then we have to show that A satisfies the condition 2.1.

Now

$$\phi(\rho(Au, Sv)) \leq (\phi(M(u, v)))^{k}$$

$$\phi(\rho(Au, Av)) \leq \left(\phi\left(\frac{\rho(u, Au)\rho(u, Av) + \rho(v, Av)\rho(v, Au)}{\max\{\rho(u, Av), \rho(Au, v)\}}\right)\right)^{k}.$$

Take $u = \gamma_n$ and $v = \gamma_m$ we get

$$e^{\rho(A\gamma_n, A\gamma_m)} \leq \left(e^{\frac{\rho(\gamma_n, A\gamma_n)\rho(\gamma_n, A\gamma_m) + \rho(\gamma_m, A\gamma_m)\rho(\gamma_m, A\gamma_1)}{\max\{\rho(\gamma_n, A\gamma_m), \rho(A\gamma_n, A\gamma_m)\}}}\right)^{k}$$

$$\rho(A\gamma_n, A\gamma_m) \leq k\left(\frac{\rho(\gamma_n, A\gamma_n)\rho(\gamma_n, A\gamma_m) + \rho(\gamma_m, A\gamma_m)\rho(\gamma_m, A\gamma_n)}{\max\{\rho(\gamma_n, A\gamma_m), \rho(A\gamma_n, \gamma_m)\}}\right)$$

There are two cases arises:

Case I. when $n = 1$ and $m > 2$ in this condition $u = \gamma_1$ and $v = \gamma_m$ we get

$$\rho(A\gamma_1, A\gamma_m) \leq k\left(\frac{\rho(\gamma_1, A\gamma_1)\rho(\gamma_1, A\gamma_m)\rho(\gamma_m, A\gamma_m)\rho(\gamma_m, A\gamma_1)}{\max\{\rho(\gamma_1, A\gamma_m), \rho(A\gamma_1, \gamma_m)\}}\right)$$

$$\rho(A\gamma_1, A\gamma_m) \leq k\left(\frac{\rho(\gamma_m, \gamma_{m-1})\rho(\gamma_m, \gamma_1)}{\max\{\rho(\gamma_1, \gamma_{m-1}), \rho(\gamma_1, \gamma_m)\}}\right)$$

$$\rho(A\gamma_1, A\gamma_m) \leq k\,\rho(\gamma_m, \gamma_{m-1})$$

$$k \geq \frac{\gamma_{m-1}}{\gamma_m}.$$

So

$$k \geq \frac{m-1}{m+1}.$$

Hence $k \in (0, 1)$.

Case II. When $m > n > 1$

$$\rho(A\gamma_n, A\gamma_m) \leq k\left(\frac{\rho(\gamma_n, A\gamma_n)\rho(\gamma_n, A\gamma_m) + \rho(\gamma_m, A\gamma_m)\rho(\gamma_m, A\gamma_n)}{\max\{\rho(\gamma_n, A\gamma_m), \rho(A\gamma_{n-1}, \gamma_m)\}}\right)$$

$$\rho(A\gamma_n, A\gamma_m) \leq k\left(\frac{\rho(\gamma_n, \gamma_{n-1})\rho(\gamma_n, \gamma_{m-1}) + \rho(\gamma_m, \gamma_{m-1})\rho(\gamma_m, \gamma_{n-1})}{\max\{\rho(\gamma_n, \gamma_{m-1}), \rho(\gamma_{n-1}, \gamma_m)\}}\right)$$

$$\rho(\gamma_{n-1}, \gamma_{m-1}) \leq k\left(\frac{\gamma_n\rho(\gamma_n, \gamma_{m-1}) + \gamma_m\gamma_m}{\max\{\rho(\gamma_n, \gamma_{m-1}), \gamma_m\}}\right).$$

Since $m > n$ so $m - 1 \geq n$ there two situations arrive $\gamma_{m-1} = \gamma_n$ or $\gamma_{m-1} > \gamma_n$.

So $\rho(\gamma_n, \gamma_{m-1}) = 0$ or $\rho(\gamma_n, \gamma_{m-1}) = \gamma_{m-1}$.

Now if $\rho(\gamma_n, \gamma_{m-1}) = 0$ then

$$\gamma_{m-1} \leq k\gamma_m$$

$$k \geq \frac{\gamma_{m-1}}{\gamma_m}.$$

So

$$k \geq \frac{m-1}{m+1}.$$

Hence $k \in (0, 1)$.

Now if $\rho(\gamma_n, \gamma_{m-1}) = \gamma_{m-1}$ then

$$\gamma_{m-1} \leq k \frac{\gamma_m \gamma_{m-1} + \gamma_m^2}{\gamma_m}$$

$$k \geq \frac{\gamma_m \gamma_{m-1}}{\gamma_m \gamma_{m-1} + \gamma_m^2}.$$

Hence $k \in (0, 1)$.

In all the above cases, the condition of our theorem satisfies and it is clear that 1 is a fixed point of A.

3. Application to Fractional Calculus

Let $(C[0,1], \rho)$ be a metric space defined by

$$\rho(u_1, v_1) = \| u_1 - v_1 \|_\infty = \max_{t \in [0, 1]} | u_1(t) - v_1(t) |.$$

Where $C([0, 1])$ be a collection of continuous functions.

First we consider the following fractional differential equation

$$^{n}D^{\gamma}(x(t)) + g(t)x(t) = 0 \, (0 \leq t \leq 1, \gamma < 1) \tag{3.1}$$

with $x(0) = x(1) = 0$ and $g : [0, 1] \to R$ is continuous function and problem 3.1 is equivalent to the integral

$$x(t) = \int_0^1 G(t, v) g(v) x(v) dv$$

where Green's function associated with the problem 3.1 is given by

$$G(t, v) = \begin{cases} \dfrac{(t(1-v))^{c-1} - (t-v)^{c-1}}{\Gamma(c)} & \text{if } 0 \le t \le v \le 1 \\ \dfrac{(t(1-v))^{c-1}}{\Gamma(c)} & \text{if } 0 \le v \le t \le 1. \end{cases}$$

Assume that the subsequent conditions hold:

1. $| g(u_1) - g(u_2) | \le K(u_1, u_2)$ for each $t \in [0, 1]$

where

$$K(u_1, u_2) = \frac{| u_1 - Au_1 || u_1 - Au_2 | + | u_2 - Au_2 || u_2 - Au_1 |}{\max \{| u_1 - Au_2 |, | u_2 - Au_1 |\}}$$

2. There exists $\tau > 1$ such that $| x(t) | \le e^{-\tau} \forall\, t \in [0, 1]$.

Assume that mapping $A : C[0, 1] \to C[0, 1]$ is defined by

$$A(x(t)) = \int_0^1 G(t, v) g(v) x(v) dv. \tag{3.2}$$

Now we have to prove that existence of a solution of the fractional differential equation 3.1

Theorem 3.1. *Under the Assumption of condition* (1)-(2), 3.1 *has a solution.*

Proof. The solution of 3.1 is equivalent to $x \in C[0,1]$ which is a solution of the integral equation

$$x(t) = \int_0^1 G(t, v) g(v) x(v) dv \ \forall t \in [0, 1]$$

we consider

$$| Au_1(t) - Au_2(t) | = | \int_0^1 G(t, s) g_1(s) u_1(s) ds - \int_0^1 G(t, v) g_2(s) u_2(s) ds |$$

$$| Au_1(t) - Au_2(t) | \le \int_0^1 G(t, s) (g_1(s) u_1(s) - g_2(s) u_2(s) ds |$$

$$\leq \int_0^1 | G(x, s) || g_1(s)u_1(s) - g_2(s)u_2(s) | ds$$

$$\leq \int_0^1 | G(x, s) |(| g_1(s) || u_1(s) | - | g_2(s)| u_2(s) |) ds$$

$$\leq \int_0^1 | G(x, s) | e^{-\tau}(| g_1(s) - g_2(s) |) ds$$

$$\leq \int_0^1 | G(x, s) | e^{-\tau} K(u_1, u_2) ds$$

$$\leq \int_0^1 | G(x, s) | e^{-\tau} \frac{| u_1 - Au_1 || u_1 - Au_2 | + | u_2 - Au_2 || u_2 - Au_1 |}{\max \{| u_1 - Au_2 |, | u_1 - Au_1 |\}} ds$$

$$\leq e^{-\tau} \frac{| u_1 - Au_1 || u_1 - Au_2 | + | u_2 - Au_2 || u_2 - Au_1 |}{\max \{| u_1 - Au_2 |, | u_2 - Au_1 |\}} \int_0^1 (G(x, s)) ds$$

$$\leq e^{-\tau} \frac{| u_1 - Au_1 || u_1 - Au_2 | + | u_2 - Au_2 || u_2 - Au_1 |}{\max \{| u_1 - Au_2 |, | u_1 - Au_1 |\}}$$

$$\sup_{t \in [0, 1]} \left(\int_0^1 (G(x, s)) ds \right)$$

$$\leq e^{-\tau} M(u_1, u_2) \sup_{t \in [0, 1]} \left(\int_0^1 (G(x, s)) ds \right)$$

$$\leq e^{-\tau} M(u_1, u_2)$$

where

$$M(u, v) = \frac{\rho(u_1, Au_1)\rho(u_1, Au_2) + \rho(u_2, Au_2)\rho(u_2, Au_1)}{\max \{\rho(u_1, Au_2), \rho(u_1, u_2)\}}$$

Therefore $\forall\ u_1, u_2 \in V$ and $\forall\ t \in [0, 1]$ we have

$$\rho(Au_1, Au_2) \leq (e^{-\tau} M(u_1, u_2))$$

Taking both side exponential we get

$$e^{\rho(Au_1, Au_2)} \leq e^{(e^{-\tau} M(u_1, u_2))}$$

$$e^{\rho(Au_1,\, Au_2)} \leq \left(e^{(M(u_1,\, u_2))}\right)^{e^{-\tau}}.$$

Now we consider the function $\phi(u) = e^{u}$

We get

$$\phi(\rho(Au_1,\, Au_2)) \leq (\phi(M(u_1,\, u_2))^{k}$$

where $k = e^{-\tau}$.

Therefore the mapping A is a new type ϕ contraction.

It follows from the Theorem 2.1, A defined by equation 3.2, has a fixed point in $(C[0, 1])$ which in turns is the solution of problem 3.1.

References

[1] J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho and Y. O. Yang, Fixed point results for generalized-contractions, J. Nonlinear Sci. Appl. 10 (2017), 2350-2358.

[2] M. Jleli and B. Samet, A new generalizations of the Banach contraction principle, J. Inequal. Appl. 38 (2014).

[3] X. Liu, S. Chang, Y. Xiao and L. C. Zhao, Existence of fixed points for θ-type contraction and θ-type Suzuki contraction in complete metric spaces, Fixed Point Theory Appl. 8 (2016).

[4] H. Piri, H. Marasi and P. Kumam, A fixed point theorem for F-Khan-contractions on complete metric spaces and application to integral equations, J. Nonlinear Sci. Appl. 10 (2017), 4564-4573.

[5] V. Parvaneh, F. Golkarmanesh, N. Hussain and P. Salimi, New fixed point theorems for- $\alpha - H\theta$- contractions in ordered metric spaces, J. Fixed Point Theory Appl. 18 (2016), 905-925.