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GEOMETRIC MEAN CORDIAL LABELING OF BUTTERFLY GRAPH

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Abstract

We consider a graph $G = (V, E)$. Let $f : V(G) \to \{0, 1, 2\}$ be a surjective function. For each edge uv designate the label $\lceil \sqrt{f(u)f(v)} \rceil$, f is called a geometric mean cordial (GMC) labeling if $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with $x, x \in \{0, 1, 2\}$. A graph with a geometric mean cordial labeling is called geometric mean cordial (GMC) graph [3]. In this research paper we investigate the GMC labeling of Butterfly graph and also we establish the GMC labeling of some graph operations on Butterfly graph such as Fusion and Duplication.

1. Introduction

In this research paper we consider only finite, simple, connected and undirected graphs. A graph labeling is an assignment of integers to the vertices or edges, or both subject to some conditions. The concept of geometric mean cordial labeling was developed by K. Chitra Lakshmi and K. Nagarajan [2]. Geometric mean cordial labeling has various applications, one of its prominent application being the communications network addressing. The vertices of the graph can be considered as the nodes of the communication

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network, which helps in transmission of messages over the links of communication. In this research article we investigate the GMC labeling of Butterfly graph and also we establish the GMC labeling of some graph operations on Butterfly graph such as Fusion and Duplication.

2. Preliminaries

Definition 2.1 [4]. A shell graph is formed from a cycle C_n, by choosing a particular vertex as an apex vertex and connecting all the non-adjacent vertices of the cycle to the chosen apex vertex.

Definition 2.2 [4]. A butterfly graph is obtained by appending two pendent edges at the apex of a double shell graph which consists of two disjoint shells with a common apex where each shell is of any order.

Definition 2.3 [6]. For every vertex $v \in V(G)$, the open neighbourhood set $N(v)$ is the set of all vertices adjacent to v in G.

Definition 2.4 [3]. Let u and v be two distinct vertices of a graph G. A new graph G' is constructed by identifying (fusing) two vertices u and v by a single vertex x is such that every edge which was incident with either u or v in G is now incident with x in G'.

Definition 2.5 [3]. The duplication of vertex u is obtained by introducing a new vertex w in such a way that the vertex w becomes adjacent to those vertices which are adjacent to the vertex u.

3. Results

Theorem 3.1. *The Butterfly graph* $BF(m, n), (m \geq n)$ *is a GMC graph if* $m \equiv 0, 1 (\text{mod}\, 3)$ *for* $n = m - 1$ *and if* $m \equiv 2 (\text{mod}\, 3)$ *for* $n = m - 2$.

Proof. Let G be a Butterfly graph $BF(m, n), (m \geq n)$ with vertex set $V(G) = \{u, v, w, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(G) = \{uw, vw, wu_i, wv_j, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq m-1\} \cup \{v_j v_{j+1} : 1 \leq j \leq n-1\}$.

Let $|V(G)| = k$ and $|E(G)| = l$. Then $k = m + n + 3$ and $l = 2m + 2n$.

Then the Butterfly graph $BF(m, n)$ is as in figure 3.1 (a)

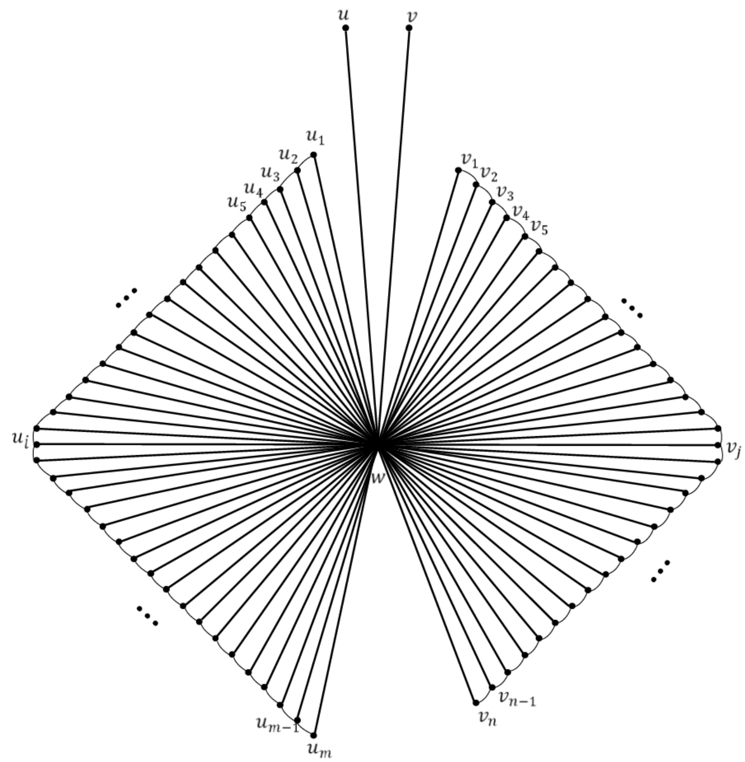

Figure 3.1 (a)**.** Butterfly Graph $BF(m, n)$.

Define $f : V(G) \to \{0, 1, 2\}$ as below:

Case (i) $m \equiv 0(\text{mod}\, 3)$ and $n = m - 1$

Let $m = 3t, t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 2(\text{mod}\, 3)$.

Let $n = 3s + 2, s \geq 0$.

$f(u) = f(v) = 0, f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 1 \\ 2, & 2t \leq i \leq 3t \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 2 \\ 2, & 2s + 3 \leq j \leq 3s + 2 \end{cases}$$

Here $k \equiv 2(\bmod 3) \Rightarrow k = 3x + 2$ and $l \equiv 1(\bmod 3) \Rightarrow l = 3y + 1$.

So $v_f(0) = v_f(1) = x + 1, v_f(2) = x, e_f(0) = y + 1, e_f(1) = e_f(2) = y$.

Case (ii) $m \equiv 1(\bmod 3)$ and $n = m - 1$

Let $m = 3t + 1, t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 0(\bmod 3)$.

Let $n = 3s, s \geq 1$.

$f(u) = f(v) = 0, f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 1 \\ 2, & 2t \leq i \leq 3t + 1 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 1 \\ 2, & 2s + 2 \leq j \leq 3s \end{cases}$$

Here $k \equiv 1(\bmod 3) \Rightarrow k = 3x + 1$ and $l \equiv 2(\bmod 3) \Rightarrow l = 3y + 2$.

So $v_f(0) = v_f(2) = x, v_f(1) = x + 1, e_f(0) = y, e_f(1) = e_f(2) = y + 1$.

Case (iii) $m \equiv 2(\bmod 3)$ and $n = m - 2$

Let $m = 3t + 2, t \geq 1$.

Here $n = m - 2 \Rightarrow n \equiv 0(\bmod 3)$.

Let $n = 3s, s \geq 1$.

$f(u) = f(v) = 0, f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t \\ 2, & 2t + 1 \leq i \leq 3t + 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 1 \\ 2, & 2s + 2 \leq j \leq 3s \end{cases}$$

Here $k \equiv 2(\bmod 3) \Rightarrow k = 3x + 2$ and $l \equiv 1(\bmod 3) \Rightarrow l = 3y + 1$.

So $v_f(0) = v_f(1) = x + 1, v_f(2) = x, e_f(0) = y + 1, e_f(1) = e_f(2) = y$.

We observe from all the above cases that $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1 \ \forall i, j \in \{0, 1, 2\}$ and hence $BF(m, n)$ is a GMC graph.

Illustration 3.1.

GMC Labeling of Butterfly graph $BF(34, 33)$ is shown in figure 3.1 (b)

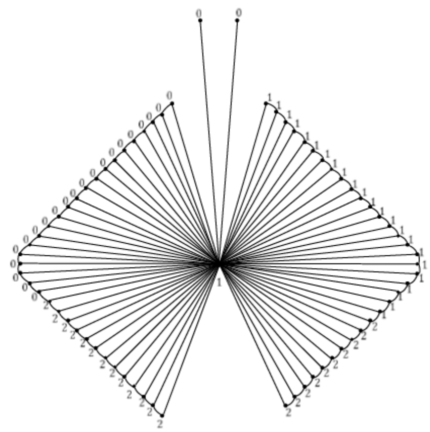

Figure 3.1 (b).

Here $v_f(0) = v_f(2) = 23$, $v_f(1) = 24$ and $e_f(0) = 44$, $e_f(1) = e_f(2) = 45$.

Therefore $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1 \; \forall \, i, j \in \{0, 1, 2\}$.

Theorem 3.2. *The graph formed by fusing two pendent vertices u and v of Butterfly graph $BF(m, n)$, $(m \geq n)$ is a GMC graph if $m \equiv 0 (\mathrm{mod}\, 3)$ for $n = m - 1$, $m \equiv 1 (\mathrm{mod}\, 3)$ for $n = m$ and $m \equiv 2 (\mathrm{mod}\, 3)$ for $n = m - 2$.*

Proof. Let G_f be the graph formed by fusing two pendent vertices u and v as one vertex u' in a Butterfly graph $BF(m, n)$. Then the vertex set $V(G_f) = \{u', w, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(G_f) =$ $\{u'w, wu_i, wv_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq m - 1\} \cup$ $\{v_j v_{j+1} : 1 \leq j \leq n - 1\}$.

Let $|V(G)| = k$ and $|E(G)| = l$. Then $k = m + n + 2$ and $l = 2m + 2n - 1$. The graph formed by fusing two pendent vertices u and v as one vertex u' in the Butterfly graph $BF(m, n)$ is as in figure 3.2 (a)

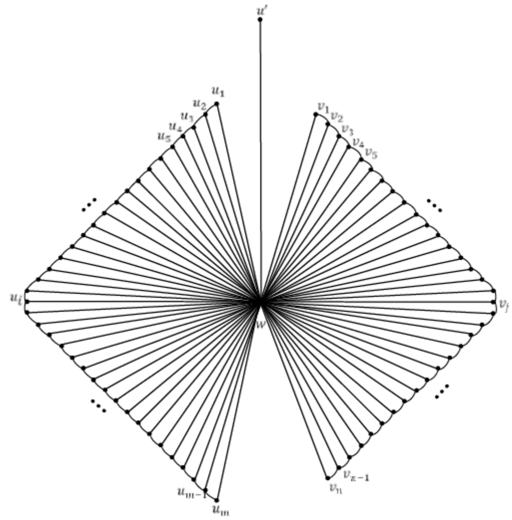

Figure 3.2 (a) Fusion of pendent vertices u and v of Butterfly Graph $BF(m, n)$.

Define $f : V(G) \to \{0, 1, 2\}$ as below:

Case (i) $m \equiv 0(\text{mod}\,3)$ and $n = m - 1$

Let $m = 3t,\ t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 2(\text{mod}\,3)$.

Let $n = 3s + 2,\ s \geq 0$.

$f(u') = 0,\ f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 1 \\ 2, & 2t \leq i \leq 3t \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 2 \\ 2, & 2s + 3 \leq j \leq 3s + 2 \end{cases}$$

Here $k \equiv 1(\text{mod}\,3) \Rightarrow k = 3x + 1$ and $l \equiv 0(\text{mod}\,3) \Rightarrow l = 3y$.

So $v_f(0) = v_f(2) = x,\ v_f(2) = x + 1,\ e_f(0) = e_f(1) = e_f(2) = y$.

Case (ii) $m \equiv 1(\text{mod}\,3)$ and $n = m$

Let $m = 3t + 1,\ t \geq 1$.

Here $n = m \Rightarrow n \equiv 1(\text{mod}\,3)$.

Let $n = 3s + 1,\ s \geq 1.$

$f(u') = 0,\ f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t \\ 2, & 2t + 1 \leq i \leq 3t + 1 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 1 \\ 2, & 2s + 2 \leq j \leq 3s + 1 \end{cases}$$

Here $k \equiv 1(\text{mod}\, 3) \Rightarrow k = 3x + 1$ and $l \equiv 0(\text{mod}\, 3) \Rightarrow l = 3y.$

So $v_f(0) = v_f(2) = x,\ v_f(1) = x + 1,\ e_f(0) = y,\ e_f(1) = e_f(2) = y.$

Case (iii) $m \equiv 2(\text{mod}\, 3)$ and $n = m - 2.$

Let $m = 3t + 2,\ t \geq 1.$

Here $n = m - 2 \Rightarrow n \equiv 0(\text{mod}\, 3).$

Let $n = 3s,\ s \geq 1.$

$f(u') = 0,\ f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t \\ 2, & 2t + 1 \leq i \leq 3t + 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 1 \\ 2, & 2s + 2 \leq j \leq 3s \end{cases}$$

Here $k \equiv 0(\text{mod}\, 3) \Rightarrow k = 3x + 1$ and $l \equiv 0(\text{mod}\, 3) \Rightarrow l = 3y.$

So $v_f(0) = v_f(2) = x,\ v_f(1) = x + 1,\ e_f(0) = e_f(1) = e_f(2) = y.$

We observe from all the above cases that $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ for all $i,\ j \in \{0, 1, 2\}$ and hence the graph formed by fusing two pendent vertices u and v as one vertex u' in the Butterfly graph $BF(m,\ n)$ is a GMC graph.

Illustration 3.2.

Fusion of pendent vertices u and v of Butterfly Graph $BF(34,\ 34)$ and its GMC Labeling is shown in figure 3.2 (b)

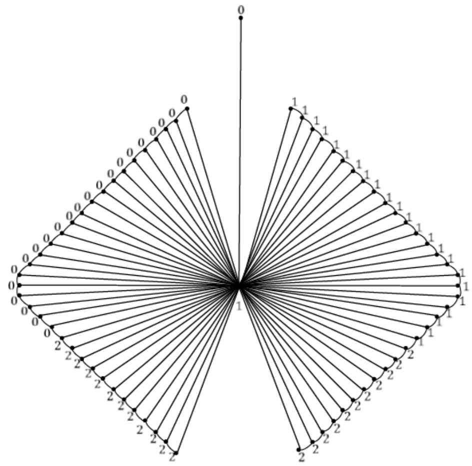

Figure 3.2 (b).

Here $v_f(0) = v_f(2) = 23$, $v_f(1) = 24$ and $e_f(0) = e_f(1) = e_f(2) = 45$.

Therefore $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j \in \{0, 1, 2\}$.

Theorem 3.3. *The graph formed by the duplication of pendent vertex u or v of Butterfly graph* $BF(m, n)$, $(m \geq n)$ *is a GMC graph if* $m \equiv 0, 1 (\bmod 3)$ *for* $n = m - 1$ *and if* $m \equiv 2 (\bmod 3)$ *for* $n = m - 2$.

Proof. Let G_d denote the graph formed by the duplication of pendent vertex u or v of Butterfly graph $BF(m, n)$, $(m \geq n)$. Then the vertex set $V(G_d) = \{u, u', v, w, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(G_f) = \{uw, u'w, vw, wu_i, wv_j, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq m-1\} \cup \{v_j v_{j+1} : 1 \leq j \leq n-1\}$.

Let $|V(G)| = k$ and $|E(G)| = l$. Then $k = m + n + 4$ and $l = 2m + 2n + 1$. The duplication of pendent vertex u of Butterfly graph $BF(m, n)$ is as in figure 3.3 (a)

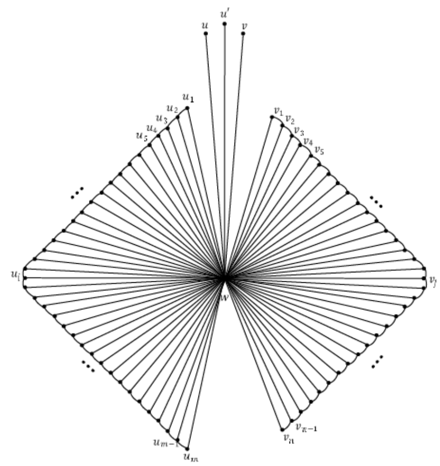

Figure 3.3 (a) Duplication of pendent vertex u of Butterfly Graph $BF(m, n)$.

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

Case (i) $m \equiv 0 (\text{mod} 3)$ and $n = m - 1$

Let $m = 3t, t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 2 (\text{mod} 3)$.

Let $n = 3s + 2, s \geq 0$.

$f(u) = f(v) = 0, f(w) = 1, f(u') = 2$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 1 \\ 2, & 2t \leq i \leq 3t \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 2 \\ 2, & 2s + 3 \leq j \leq 3s + 2 \end{cases}$$

Here $k \equiv 0 (\text{mod} 3) \Rightarrow k = 3x$ and $l \equiv 2 (\text{mod} 3) \Rightarrow l = 3y + 2$.

So $v_f(0) = v_f(1) = v_f(2) = x$, $e_f(0) = e_f(2) = y + 1$, $e_f(1) = y$.

Case (ii) $m \equiv 1 (\text{mod} 3)$ and $n = m - 1$

Let $m = 3t + 1, t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 0 (\text{mod} 3)$.

Let $n = 3s, s \geq 1$.

$f(\mathrm{u}) = f(v) = f(u') = 0,\ f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \le i \le 2t - 1 \\ 2, & 2t \le i \le 3t + 1 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \le j \le 2s + 1 \\ 2, & 2s + 2 \le j \le 3s \end{cases}$$

Here $k \equiv 2(\mathrm{mod}\,3) \Rightarrow k = 3x + 2$ and $l \equiv 0(\mathrm{mod}\,3) \Rightarrow l = 3y$.

So $v_f(0) = v_f(1) = x + 1,\ v_f(2) = x,\ e_f(0) = e_f(1) = e_f(2) = y$.

Case (iii) $m \equiv 2(\mathrm{mod}\,3)$ and $n = m - 2$.

Let $m = 3t + 2,\ t \ge 1$.

Here $n = m - 2 \Rightarrow n \equiv 0(\mathrm{mod}\,3)$.

Let $n = 3s,\ s \ge 1$.

$f(u) = f(v) = 0,\ f(w) = 1,\ f(u') = 2$

$$f(u_i) = \begin{cases} 0, & 1 \le i \le 2t \\ 2, & 2t + 1 \le i \le 3t + 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \le j \le 2s + 1 \\ 2, & 2s + 2 \le j \le 3s \end{cases}$$

Here $k \equiv 2(\mathrm{mod}\,3) \Rightarrow k = 3x$ and $l \equiv 2(\mathrm{mod}\,3) \Rightarrow l = 3y + 2$.

So $v_f(0) = v_f(1) = v_f(2) = x,\ e_f(0) = e_f(2) = y + 1,\ e_f(1) = y$.

We observe from all the above cases that $|\, v_f(i) - v_f(j)\,| \le 1$ and $|\, e_f(i) - e_f(j)\,| \le 1$ for all $i,\ j \in \{0, 1, 2\}$ and hence the duplication of pendent vertex u or v at the apex of Butterfly graph $BF(m,\ n)$ is a GMC graph.

Illustration 3.3.

Duplication of pendent vertices u of Butterfly Graph $BF(34,\ 33)$ and its GMC Labeling is shown in figure 3.3 (b)

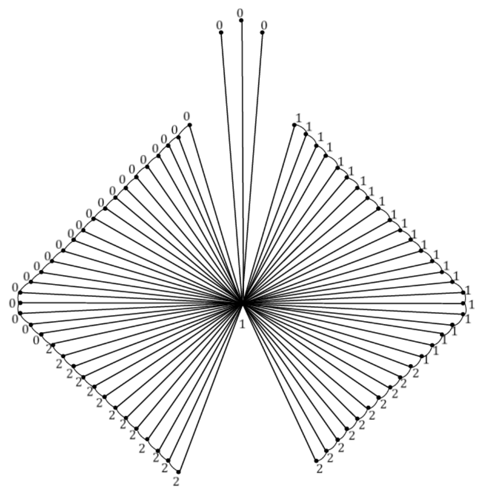

Figure 3.3 (b).

Here $v_f(0) = v_f(1) = 24$, $v_f(2) = 23$ and $e_f(0) = e_f(1) = e_f(2) = 45$.

Therefore $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j \in \{0, 1, 2\}$.

Theorem 3.4. *The graph formed by duplication of vertex v_1 or v_n of Butterfly graph $BF(m, n)$, $(m \geq n)$ is a GMC graph if $m \equiv 0(\text{mod}\, 3)$ for $n = m$, $m \equiv 1(\text{mod}\, 3)$ for $n = m - 1$ and $m \equiv 2(\text{mod}\, 3)$ for $n = m - 1$.*

Proof. Let G_d denote the graph formed by the duplication of a vertex v_1 of a Butterfly graph $BF(m, n)$, $(m \geq n)$. Then the vertex set $V(G_f) = \{u, v, w, v_1', u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(G_f) = \{uw, vw, wu_1, wv_1', v_1', v_2, wu_i, wv_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq m - 1\} \cup \{v_j v_{j+1} : 1 \leq j \leq n - 1\}$.

Let $|V(G)| = k$ and $|E(G)| = l$. Then $k = m + n + 4$ and $l = 2m + 2n + 2$. The graph formed by the duplication of vertex v_1 of a Butterfly graph $BF(m, n)$ is as in figure 3.4 (a)

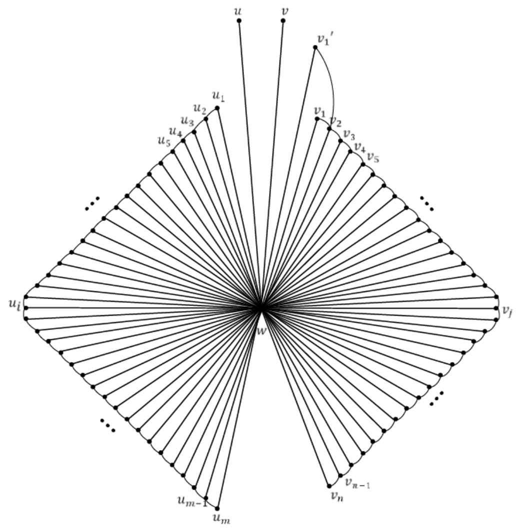

Figure 3.4 (a)**.** Duplication of vertex v_1 of Butterfly Graph $BF(m, n)$.

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

Case (i) $m \equiv 0 (\bmod 3)$ and $n = m$

Let $m = 3t, t \geq 1$.

Here $n = m \Rightarrow n \equiv 0 (\bmod 3)$.

Let $n = 3s, s \geq 0$.

$f(u) = f(v) = f(v_1') = 0, f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 2 \\ 2, & 2t - 1 \leq i \leq 3t \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 1 \\ 2, & 2s + 2 \leq j \leq 3s \end{cases}$$

Here $k \equiv 1 (\bmod 3) \Rightarrow k = 3x + 1$ and $l \equiv 2 (\bmod 3) \Rightarrow l = 3y + 2$.

So $v_f(0) = v_f(2) = x, v_f(1) = x + 1, e_f(0) = y, e_f(1) = e_f(2) = y + 1$.

Case (ii) $m \equiv 1 (\bmod 3)$ and $n = m - 1$.

Let $m = 3t + 1, t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 0 (\bmod 3)$.

Let $n = 3s,\ s \geq 1$.

$f(u) = f(v) = f(v_1') = 0,\ f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 1 \\ 2, & 2t \leq i \leq 3t + 1 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 1 \\ 2, & 2s + 2 \leq j \leq 3s \end{cases}$$

Here $k \equiv 2(\text{mod}\, 3) \Rightarrow k = 3x + 2$ and $l \equiv 1(\text{mod}\, 3) \Rightarrow l = 3y + 1$.

So $v_f(0) = v_f(1) = x + 1,\ v_f(2) = x,\ e_f(0) = y + 1,\ e_f(1) = e_f(2) = y$.

Case (iii) $m \equiv 2(\text{mod}\, 3)$ and $n = m - 1$

Let $m = 3t + 2,\ t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 1(\text{mod}\, 3)$.

Let $n = 3s + 1,\ s \geq 1$.

$f(u) = f(v) = f(v_1') = 0,\ f(w) = 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 1 \\ 2, & 2t \leq i \leq 3t + 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 2 \\ 2, & 2s + 3 \leq j \leq 3s + 1 \end{cases}$$

Here $k \equiv 1(\text{mod}\, 3) \Rightarrow k = 3x + 1$ and $l \equiv 2(\text{mod}\, 3) \Rightarrow l = 3y + 2$.

So $v_f(0) = v_f(2) = x,\ v_f(1) = x + 1,\ e_f(0) = y,\ e_f(1) = e_f(2) = y + 1$.

We observe from all the above cases that $|\, v_f(i) - v_f(j)\, | \leq 1$ and $|\, e_f(i) - e_f(j)\, | \leq 1$ for all $i,\ j \in \{0, 1, 2\}$ and hence the graph formed by the duplication of vertex v_1 of a Butterfly graph $BF(m,\ n)$ is a GMC graph.

Illustration 3.4.

Duplication of vertex v_1 of Butterfly Graph $BF(30,\ 30)$ and its GMC Labeling is shown in figure 3.4 (b)

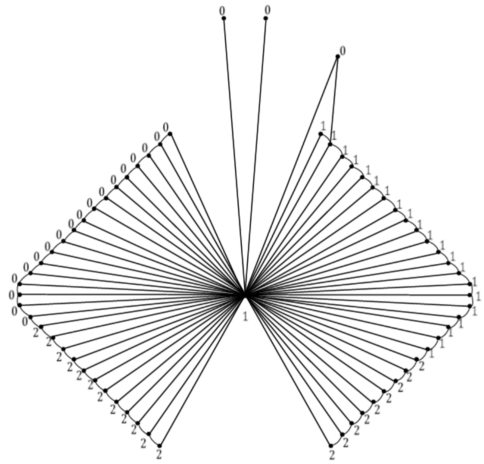

Figure 3.4 (b).

Here $v_f(0) = v_f(2) = 21$, $v_f(1) = 22$ and $e_f(0) = 40$, $e_f(1) = e_f(2) = 41$.

Therefore $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ for all $i, j \in \{0, 1, 2\}$.

Theorem 3.5. *The graph formed by the duplication of a vertex* $v_i(i = 2$ *or* 3 *or…or* $n-1)$ *of Butterfly graph* $BF(m, n), (m \geq n)$ *is a GMC graph if* $m \equiv 0(\bmod 3)$ *for* $n = m$, $m \equiv 1(\bmod 3)$ *for* $n = m-2$ *and* $m \equiv 2(\bmod 3)$ *for* $n = m-1$.

Proof. Let G_d denote the graph formed by duplication of a vertex $v_i(i = 2$ or 3 or…or $n-1)$ of a Butterfly graph $BF(m, n), (m \geq n)$. Then the vertex set $V(G_d) = \{u, v, w, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{v'_i : i = 2$ or 3 $n-1\}$ and edge set $E(G_f) = \{uw, vw, wu_i, awv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ $\cup \{u_i u_{i+1} : 1 \leq i \leq m-1\} \cup \{v_j v_{j+1} : 1 \leq j \leq n-1\} \cup \{wv'_i : i = 2$ or 3 or … or $n-1\} \cup \{v_{i-1}v'_i, v'_i v_{i+1} : i = 2$ or 3 or … or $n-1\}$.

Let $|V(G)| = k$ and $|E(G)| = l$. Then $k = m + n + 4$ and $l = 2m + 2n + 3$. The graph formed by the duplication of vertex $v_i(i = 2$ or 3 or…or $n-1)$ of a Butterfly graph $BF(m, n)$ is as in figure 3.5 (a)

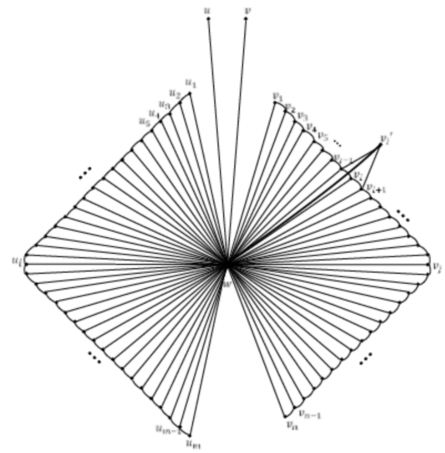

Figure 3.5 (a). Duplication of vertex $v_i(i = 2$ or 3 or…or $n-1)$ of Butterfly Graph $BF(m, n)$.

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

Case (i) $m \equiv 0(\bmod 3)$ and $n = m$

Let $m = 3t, t \geq 1$.

Here $n = m \Rightarrow n \equiv 0(\bmod 3)$.

Let $n = 3s, s \geq 1$.

$f(u) = f(v) = 0, f(w) = 1$

$f(v'_j) = 0, i = 2 \text{ or} \ldots \text{or } n-1$

$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t-2 \\ 2, & 2t-1 \leq -1i \leq 3t \end{cases}$ and $f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s+1 \\ 2, & 2s+2 \leq j \leq 3s \end{cases}$

Here $k \equiv 1(\bmod 3) \Rightarrow k = 3x+1$ and $l \equiv 0(\bmod 3) \Rightarrow l = 3y$.

So $v_f(0) = v_f(2) = x, v_f(1) = x+1, e_f(0) = y, e_f(1) = e_f(2) = y$.

Case (ii) $m \equiv 1(\bmod 3)$ and $n = m-2$.

Let $m = 3t+1, t \geq 2$.

Here $n = m - 2 \Rightarrow n \equiv 2(\text{mod}\, 3)$.

Let $n = 3s + 2s \geq 1$.

$f(u) = f(v) = 0, f(w) = 1$

$f(v_i') = 0, i = 2$ or 3 or … or $n - 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 2 \\ 2, & 2t - 1 \leq i \leq 3t + 1 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 3 \\ 2, & 2s + 4 \leq j \leq 3s \end{cases}$$

Here $k \equiv 1(\text{mod}\, 3) \Rightarrow k = 3x + 1$ and $l \equiv 0(\text{mod}\, 3) \Rightarrow l = 3y$.

So $v_f(0) = v_f(2) = x, v_f(1) = x + 1, e_f(0) = e_f(1) = e_f(2) = y$.

Case (iii) $m \equiv 2(\text{mod}\, 3)$ and $n = m - 1$

Let $m = 3t + 2, t \geq 1$.

Here $n = m - 1 \Rightarrow n \equiv 1(\text{mod}\, 3)$.

Let $n = 3s + 1, s \geq 1$.

$f(u) = f(v) = 0, f(w) = 1$

$f(v_i') = 0, i = 2$ or 3 or … or $n - 1$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq 2t - 1 \\ 2, & 2t \leq i \leq 3t + 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & 1 \leq j \leq 2s + 2 \\ 2, & 2s + 3 \leq j \leq 3s + 1 \end{cases}$$

Here $k \equiv 1(\text{mod}\, 3) \Rightarrow k = 3x + 1$ and $l \equiv 0(\text{mod}\, 3) \Rightarrow l = 3y$.

So $v_f(0) = v_f(2) = x, v_f(1) = x + 1, e_f(0) = y, e_f(1) = e_f(2) = y$.

We observe from all the above cases that $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ for all $i, j \in \{0, 1, 2\}$ and hence the graph formed by the duplication of vertex $v_i(i = 2$ or 3 or…or $n - 1)$ of a Butterfly graph $BF(m, n)$ is a GMC graph.

Illustration 3.5.

Duplication of vertex v_5 of Butterfly Graph $BF(15, 15)$ and its GMC Labeling is shown in figure 3.5 (b)

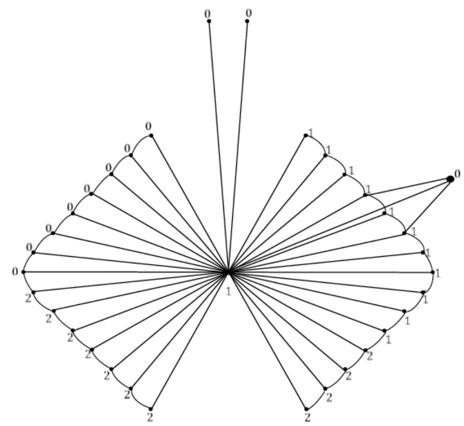

Figure 3.5 (b).

Here $v_f(0) = v_f(2) = 11$, $v_f(1) = 12$ and $e_f(0) = e_f(1) = e_f(2) = 21$.

Therefore $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ for all $i, j \in \{0, 1, 2\}$.

4. Conclusion

In this research paper we investigated the GMC labeling of Butterfly graph and also we established the GMC labeling of some graph operations on Butterfly graph such as Fusion and Duplication. Similar results on various graphs is an open area of research.

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