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LINEAR SYSTEM AND RANDOM NOISE PROCESSES

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Abstract

Early investigators in the field of communications first realized that the presence of unwanted random noise was an important factor following the discovery that the maximum gain of an amplifier was limited by the discrete nature of currents in electron tubes. Called shot effect, this was first explained by W. Schottky and later by many other investigators. Much research on this problem during the second and third decades of the twentieth century finally led to the rigorous formulation of the phenomenon by B. J. Thompson and others in 1940. Concurrently, the problem of spontaneous thermal noise effects in conductors was studied and formulated. By 1940, the situation was developed to an extent that the application of mathematical statistics to explain and solve broader noise problems in systems was inevitable. About this time, the basic contributions of *N*. Wiener led to an understanding of the optimum linear filtration of signals imbedded in random noise. His work influenced the entire course of development of theory on the optimization of filters designed to abstract a signal out of its noisy environment.

Introduction

Probability theory a branch of mathematics concerted with the analysis of random phenomena. The actual outcome considered to be determined by change. The word probability relative frequencies coins, cards, dice and wheels provides examples. It was developed in the early 19th century as the

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study of population, economics and moral action.

Reliability characteristics, such as probability of survival, mean time to failure, availability, mean down time frequency of failure are some of the measure of system effectiveness. If a component is put in to operation at some specified time and observed until it fails, then the time to fails or life length say $T(\geq 0)$ is a continuous random variable with some probability density function (pdf) $f(t)$. We introduce the concept to maintain ability and availability of the system.

Definition 1.1. Classification of Systems. Mathematically a "system" is a functional relationship between the input $X(t_0)$ and output $Y(t)$.

The input output relationship can be written as

$$Y(t_0) = f[x(t); -\infty < t < \infty]; -\infty < t_0 < \infty$$

Based on the properties of the functional relationship gives in the above equation.

Definition 1.2. Linear System and Non-Linear System. A system is said to be linear if superposition applies that is if

$$y_1(t) = f\ [x_1(t)]$$

$$y_2(t) = f\ [x_2(t)]$$

Then for a linear system,

$$f\ [a_1 x_1(t) + a_2 x_2(t)] = a_1 Y_1(t) + a_2 Y_2(t)$$

Definition 1.3. Time Invariant and Time Varying System. A system is time invariant if a time shift in the input result in a corresponding time shift in the output so that, if

$$Y(t) = f\,[X(t)]$$

Then

$$Y(t,\ t_0) = f\ [X(t - t_0)]; -\infty < t,\ t_0 < \infty.$$

Any system not meeting the requirement stated above is called a time varying system.

Theorem 2.1. ***Time Invariant System Transfer Function.*** *If $X(w)$, $Y(w)$ and $H(w)$ are the respective Fourier Transforms of $x(t)$, $y(t)$ and $h(t)$, then*

$$Y(w) = \int_{-\infty}^{\infty} y(t) \cdot e^{-jwt}.dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} X(\in) \cdot h\,(t- \in) \cdot d \in\right] \cdot e^{-jwt} \cdot dt$$

$$= \int_{-\infty}^{\infty} X(\in)\left[\int_{-\infty}^{\infty} h(t- \in) \cdot e^{-jw(t-\in)} dt\right] \cdot e^{-jwe}.d \in$$

$$= \int_{-\infty}^{\infty} X(\in)\left[H(w)\right] \cdot e^{-jw\in}.d \in$$

$$= H(w)\int_{-\infty}^{\infty} X(\in) \cdot e^{-jw\in},\, d \in = H(w) \cdot X(w)$$

$$Y(w) = X(w) \cdot H(w).$$

Here $H(w)$ is called "Transfer Function" of the system.

Theorem 2.2. ***System Response-Convolution.*** *The response of a network to a random signal $X(t)$ is given by the convolution integral.*

$$Y(t) = \int_{-\infty}^{\infty} X(\in) \cdot h(t- \in) \cdot d \in \quad (1)$$

$$Y(t) = \int_{-\infty}^{\infty} h(\in) \cdot X(t- \in) \cdot d \in \quad (2)$$

Where $h(t)$ is the impulse response of the network.

We may view equation (2) as an operation on an ensemble member $X(t)$ of the random process $X(t)$ that produces an ensemble member of a new process $Y(t)$. Thus we may think of (2) as defining the process $Y(t)$ in terms of the process $X(t)$.

$$Y(t) = \int_{-\infty}^{\infty} h(\in) \cdot X(t- \in) \cdot d \in \quad (3)$$

Thus we shall accept the system to have $X(t)$ as its input and $Y(t)$ as its output .

Theorem 2.3. ***Mean and Mean-Squared of System Response.*** *Applying equation* (3) *to find the mean value of the system's response* (*assuming* $X(t)$ *to be wide-sense stationary*),

$$E[Y(t)] = E[\int_{-\infty}^{\infty} h(\in) \cdot X(t- \in) \cdot d \in]$$

$$= \int_{-\infty}^{\infty} h(\in) \cdot E[X(t- \in)] \cdot d \in$$

$$= \overline{X}\int_{-\infty}^{\infty} f(\in) \cdot d \in = \overline{Y} \text{ (constant)} \tag{4}$$

Equation (4) indicates that the mean value of $Y(t)$ equals the mean value of $X(t)$ times the area under the impulse response if $X(t)$ is wide sense stationary. Also,

$$E[Y^2(t)] = E[\int_{-\infty}^{\infty} h(\in_1) \cdot X(t- \in_1) \cdot d \in_1$$

$$\int_{-\infty}^{\infty} h(\in_2) \cdot X(t- \in_2) \cdot d \in_2]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t- \in_1)X(t- \in_2)]$$

$$h(\in_1)h(\in_2) \cdot d \in_1 \cdot d \in_2 \tag{5}$$

If we assume that the input is wide sense stationary then,

$$E[X(t- \in_1)X(t- \in_2)] = R_{XX}(\in_1 - \in_2) \tag{6}$$

Using (6) in (5),

$$\bar{y} = E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\in_1 - \in_2)h(\in_1)h(\in_2) \cdot d \in_1 \cdot d \in_2$$

Theorem 2.4. ***Autocorrelaion Function of Response.*** *Let* $X(t)$ *be wide sense stationary*, *the autocorrelation function of* $Y(t)$ *is given by*

$$R_{YY}(t,\, t+\tau) = E[Y(t)\cdot Y(t+\tau)]$$

$$= E[\int_{-\infty}^{\infty} h(t_1)\cdot X(t-\in_1)\cdot d\in_1$$

$$\int_{-\infty}^{\infty} h(t_2)\cdot X(t+\tau-\in_2)\cdot d\in_2]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\in_1)\cdot X(t+\tau-\in_2)]$$

$$h(\in_1)h(\in_2)\cdot d\in_1\cdot d\in_2$$

$$R_{YY}(t,\, t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\in_1-\in_2)\cdot h(\in_1)\cdot h(\in_2)\cdot d\in_1\cdot d\in_2$$

[since $x(t)$ is assumed to be wide-sense stationary] (7)

From equation (7) it is clear that

(i) $Y(t)$ is wide sense stationary, if $X(t)$ is wide sense stationary because $R_{YY}(\tau)$ does not depend on t and $E[Y(t)]$ is constant.

(ii) $R_{YY}(\tau)$ is a two-fold convolution of the input autocorrelation function with the network's impulse response, that is

$$R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau).$$

Problem 3.1. Show that a narrow-band noise process can be expressed as in-phase and quadrature components of it.

Solution. Let $n_+(t)$ and $n(t)$ denote the pre-envelope and complex envelope of the narrow band noise $n(t)$ respectively.

We assume that the power spectrum of $n(t)$ is centred around the frequency f_c

$$n_+(t) = n(t) + j\ n\ (t) \qquad (1)$$

$$n(t) = n_+(t)\exp(-j\,2\,\pi\,f_c\,t). \qquad (2)$$

Where $n(t)$ is the Hilbert transform of $n(t)$.

The complex envelope $n(t)$ may itself be expressed as

$$n(t) = n_c(t) + j\, n_S(t). \tag{3}$$

Thus combining (1), (2) and (3),

$$n_c(t) = n(t)\cos(2\pi f_c t) + n(t)\cdot \sin(2\pi f_c t). \tag{4}$$

and

$$n_S(t) = n(t)\cos(2\pi f_c t) - n(t)\cdot \sin(2\pi f_c t). \tag{5}$$

Eliminating $n(t)$ between (4) and (5)

$$n(t) = n_c(t)\cdot \cos(2\pi f_c t) - \sin(2\pi f_c t)$$

Problem 3.2. Obtain the transfer function of the pre-whitener whose input is

$$S_{n\,i}(w) = N_o\left[\frac{w^2+2}{w^2+4}\right]$$

and the output is $S_{n\,o}(w) = N_o$

Solution.

$$S_{n\,o}(w) = |H(w)|^2 \cdot S_{n\,i}(w)$$

$$N_0 = |H(w)|^2 \cdot N_o\left[\frac{w^2+2}{w^2+4}\right]$$

$$|H(w)|^2 = \left[\frac{w^2+2}{w^2+4}\right]$$

$$H(w)\cdot H^*(w) = \left[\frac{w^2-j2}{w^2-j\sqrt{2}}\right]\cdot\left[\frac{w+j2}{w+j\sqrt{2}}\right]$$

Pre whitener Transfer Function $H(w) = \left[\frac{w-j2}{w-j\sqrt{2}}\right]$.

Problem 3.3. A circuit has an impulse response given by

$h(t) = \frac{1}{T}, \; 0 \le t \le T$

$= 0, \;$ elsewhere.

Evaluate $S_{YY}(w)$ in terms of $S_{XX}(w)$.

Solution.

$$H(w) = \frac{1}{T}\int_0^T \exp(-jw\tau)\, d\tau$$

$$H(w) = \exp\left(-j\frac{wT}{2}\right)\frac{\sin\left[\frac{wT}{2}\right]}{\left[\frac{wT}{2}\right]}$$

Therefore $S_{YY}(w) = |H(w)|^2 S_{XX}(w)$

$$= H(w) \cdot H^*(w) S_{XX}(w)$$

$$= \frac{\sin^2\left[\frac{wT}{2}\right]}{\left[\frac{wT}{2}\right]} S_{XX}(w).$$

Conclusion

The paper concludes that there is a system to develop reliability of system. If a system is put into operation at specified time, but it is impossible to construct a good quality system from poor quality system element. It the beam develop a break it attains a state of failure. Maintainability provides a measure of the reliability of a system where a certain amount of failure can be allowed, while the availability provides a prediction of probability that the complex system will be ready for use at any moment in time.

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