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ON APPROXIMATION OF CONTINUOUS FUNCTION IN THE HÖLDER METRIC BY $(C, 2)[F, d_n]$ MEANS OF ITS FOURIER SERIES

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Abstract

In this paper we will present a new estimate for the $[F, d_n]$ matrix method. So we studied "on approximation of function in the Hölder metric by $(C, 1)[F, d_n]$ means of Fourier series" has been determined by Rathore Shrivastava and Mishra [15] and again in 2022, Rathore, Shrivastava and Mishra [17] determined a theorem on "approximation of function in the Hölder metric by $(C, 2)(E, q)$ product summability of Fourier series". Further we generalize a theorem "on approximation of continuous function in the Hölder metric by $(C, 2)[F, d_n]$ means of its Fourier series".

1. Introduction

Chandra [1] was first to extend the result of Prossdorf's [12]. In 1983, Mohapatra and Chandra [11] found the degree of approximation in the Hölder metric using matrix transform. In this direction we studied on approximation of f belong to many classes also Hölder metric by Cesaro, Norlund, Euler mean has been discussed by several researchers like respectively Das, Ghosh and Ray [2], Lal and Kushwaha [7], Rathore and Shrivastava ([13], [14]), Kushwaha [6], Rathore, Shrivastava and Mishra [16] etc. In 2021, Rathore, Shrivastava and Mishra [15] have been determined "on

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approximation of function in the Hölder metric by $(C, 1)[F, d_n]$ product summability of Fourier series". Recently Rathore, Shrivastava and Mishra [17] determined a theorem on "approximation of function in the Hölder metric by $(C, 2)(E, q)$ product summability of Fourier series". We extend the result on approximation of function in the Hölder metric by $(C, 2)[F, d_n]$ mean of its Fourier series, has been proved.

2. Definition and Notation

Let $f(x)$ be periodic with period -2π and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

with n^{th} partial sum $S_n(f; x)$.

Let $C_{2\pi}$ denote the Banach Spaces of all $2\pi-$ periodic continuous function defined on $[-\pi, \pi]$ under "sup" norm. For $0 < \alpha \leq 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha = \{f \in C_{2\pi} : | f(x) - f(y) | \leq K | x - y |^\alpha\} \tag{2.2}$$

The space H_α is a Banach space (see Prössdorf's [12]) with the norm $\| \cdot \|_\alpha$ defined by

$$\| f \|_\alpha = \| f \|_c + \underset{x, y}{Sup}\, \Delta^\alpha [f(x, y)], \tag{2.3}$$

where

$$\| f \|_c = \underset{-\pi \leq x \leq \pi}{Sup} | f(x) |. \tag{2.4}$$

and

$$\Delta^\alpha \{f(x, y)\} = | f(x) - f(y) | | x - y |^{-\alpha}, \ (x \neq y). \tag{2.5}$$

We shall use the convention that $\Delta^0 f(x, y) = 0$ the metric induced by norm (2.3) on the H_α is called the Hölder metric. If can be seen that

$\| f \|_{\beta} \leq (2\pi)^{\alpha-\beta} \| f \|_{\alpha}$ for $0 \leq \beta < \alpha \leq 1$. Thus $\{(H_{\alpha}, \| \cdot \|_{\alpha})\}$ is a family of Banach spaces which decreases as α increase.

Let $d_1, d_2, \ldots, d_n,$ be a fixed sequence of positive number and x be a real number. The element P_{nk} of $[F, d_n]$ matrix are defined by the relations

$$\prod_{j=1}^{n} \frac{x + d_j}{1 + d_j} = \sum_{k=0}^{\infty} P_{nk} x^k \tag{2.6}$$

and

$$P_{00} = 1. \tag{2.7}$$

Let

$$\sigma_x(f, x) = \sum_{k=0}^{\infty} P_{nk}\, S_k(f; x). \tag{2.8}$$

Denote the $[F, d_n]$ mean of the Fourier series of $F \in L[-\pi, \pi]$ at x, where $S_k(f; x)$ is the k^{th} partial sum of (2.1).

The $[F, d_n]$ method was introduced by Jakimovsky [4] as generalization of both the Euler E_r method and Stirling-Karamata-Lototsky method. When $d_n = \frac{(n-1)}{c}$, $n = 1, 2, 3, \ldots$ and c, a positive integer, the $[F, d_n]$ matrix reduces to the matrix corresponding to the Stirling-Karamata-Lototsky method defined by Karamata [5]. The Euler $E_r (0 < r < 1)$ are obtained with $d_n = \frac{(1-r)}{r}$, $n = 1, 2, 3, \ldots.$ Lorch and Newman [8] studied the Lebesgue constant for this method. Several fundamental properties of $[F, d_n]$ matrix have been discussed in Meir and Miracle [9, 10]

$$S_n = 2 \sum_{k=1}^{n} \frac{d_k}{(1 + d_k)^2} \tag{2.9}$$

and

$$U_n = 1 + 2 \sum_{k=1}^{n} \frac{1}{(1 + d_k)}. \tag{2.10}$$

The $[F, d_n]$ matrix is regular by Jakimovsky [4] if $U_n \to \infty$ as $n \to \infty$ we shall consider only regular matrices and indeed assume that d_n is bounded away from zero for large n.

The $\frac{d_n}{(1+d_n)^2}$ is bounded away from zero and $S_n \to \infty$ as $n \to \infty$ $n+1 = [U_n]$ be the integral part of U_n.

The series $\sum_{k=0}^{\infty} u_k$ is said to be $(C, 2)$ summable to S is defined as (see Hardy [3])

$$t_n^{(C,\,2)}(f : x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) S_k \to S \text{ as } n \to \infty \tag{2.11}$$

Let $\{t_n^{(C,\,2)}\}$ denote the sequence of $(C, 2)$ mean of $\{S_n\}$ if

$$t_n^{(C,\,2)[F,\,d_n]}(f : x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) t_k^{F(\alpha,\,q)} \to S \text{ as } n \to \infty \tag{2.12}$$

where $t_n^{(C,\,2)[F,\,d_n]}$ denote the sequence of $(C, 2)[F, d_n]$ product mean of the sequence S_n, the series $\sum_{k=0}^{\infty} u_k$ is said to be summable to the number S by $(C, 2)[F, d_n]$ method.

"The degree of approximation $E_n(f)$ be given by

$$E_n(f) = \min \| T_n - f \|_p, \tag{2.13}$$

where $T_n(x)$ is a trigonometric polynomial of degree n" by (Zygmund [19]).

We shall use following notation:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x) \tag{2.14}$$

$$\text{and } \Phi(t) = \Phi_x(t) - \Phi_y(t). \tag{2.15}$$

3. Some Theorems

Chandra [1] obtained the following result:

Theorem 3.1. "*Let* $0 \leq \beta < \alpha \leq 1$ *and let* $f \in H_\alpha$. *Then*

$$\| E_n^q(f) - f \|_\beta = O\{n^{\beta-\alpha} \log n\}. \tag{3.1}$$

where $E_n^q(f, x)$ *denotes* (E, q) *transform of* $S_n(f; x)$".

Rathore, Shrivastava and Mishra [17] obtained the following result.

"On approximation of continuous function in the Hölder metric by product summability $(C, 2)(E, q)$ mean of its Fourier series" has been established.

Theorem 3.2. "*If* $0 \leq \beta < \alpha \leq 1$ *and* $f \in H_\alpha$ *then*

$$\| C_n^2 E_n^q - f(x) \|_\beta = O[(n+1)^{\beta-\alpha} \log(n+1)]. \tag{3.2}$$

Where $C_n^2 E_n^q$ *is the product summability* $(C, 2)(E, q)$ *mean of* $S_n(f; x)$".

4. Main Theorem

We prove the following theorem

"On approximation of function in the Hölder metric by mean $(C, 2)[F, d_n]$ of its Fourier series" has been established.

Theorem. "*If* $0 \leq \beta < \alpha \leq 1$ *and* $f \in H_\alpha$ *then*

$$\| t_n^{(C, 2)[F, d_n]} - f(x) \|_\beta = O[(n+1)^{\beta-\alpha} \log(n+1)]. \tag{4.1}$$

where $t_n^{(C, 2)[F, d_n]}$ *is the product summability mean* $(C, 2)[F, d_n]$ *of* $S_n(f; x)$".

5. Lemmas

We shall use the following lemmas:

Lemma 5.1. *Let* $\Phi_x(t)$ *be defined in* (2.15) *then for* $f \in H_\alpha$, *we have*

$$| \Phi_x(t) - \Phi_y(t) | \le 4k | x - y |^\alpha$$

$$| \Phi_x(t) - \Phi_y(t) | \le 4k | t |^\alpha. \tag{5.1}$$

It is easy to verify.

Lemma 5.2.

$$\prod_{k=1}^{n} \frac{\exp(it) + d_k}{1 + d_k} = \exp\{(U_n - 1)it/2 - S_n t^2/4\} + O(S_n t^3). \tag{5.2}$$

This is due to Lorch and Newman [8].

Lemma 5.3. *Let* $K_n(t) = \dfrac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \dfrac{\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)}{\sin t/2}$

then $| K_n(t) | = O(n+1)$, *for* $0 \le t \le \dfrac{\pi}{(n+1)}$.

Proof. Apply $\sin nt \le n \sin t$ for $0 \le t \le \dfrac{\pi}{n+1}$ then

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \exp\left(\frac{-S_n t^2}{4}\right) \frac{U_n \sin t/2}{\sin t/2}$$

$$= O(U_n) \frac{1}{(n+2)\pi} - \sum_{k=0}^{n} k$$

$$= \frac{2}{(n+2)\pi} - \frac{n(n+1)}{2}$$

$$= O(n+1). \tag{5.3}$$

Lemma 5.4. *Let* $K_n(t) = \dfrac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \dfrac{\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)}{\sin t/2}$

then $| K_n(t) | = O\left(\frac{1}{t}\right)$, *for* $\dfrac{\pi}{(n+1)} \le t \le \pi$.

Proof. Using $\sin\frac{t}{2} \le \left(\frac{t}{\pi}\right)$ and $\left|\sin\frac{U_n t}{2}\right| \le 1$ for $\frac{\pi}{n+1} \le t \le \pi$

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\exp\left(\frac{-S_n t^2}{4}\right)\frac{1}{t/\pi}$$

$$= \frac{2}{(n+2)\pi} - \sum_{k=0}^{n} k$$

$$= \frac{2}{(n+2)t} - \frac{n(n+1)}{2t}$$

$$= O\left(\frac{1}{t}\right). \tag{5.4}$$

Lemma 5.5. *Let* $K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\frac{O(S_n t^3)}{\sin t/2}$

then $= O(n+1)$, *for* $0 \le t \le \frac{\pi}{n+1}$.

Proof. Using $\left|\sin\frac{t}{2}\right| \le \frac{t}{\pi}$ for $0 \le t \le \frac{\pi}{n+1}$

$$\text{then } K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\frac{O(S_n t^3)}{t/2}$$

$$= \frac{O(S_n)}{(n+1)(n+2)} 2\sum_{k=0}^{n}(n-k+1)t^2$$

$$= \left[\frac{2}{(n+2)\pi} - \sum_{k=0}^{n} k\right]t^2$$

$$= \left[\frac{2}{(n+2)t} - \frac{n(n+1)}{2t}\right]\frac{\pi^2}{(n+1)^2}$$

$$= O(n+1). \tag{5.5}$$

Lemma 5.6. *Let* $K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\frac{O(S_n t^3)}{\sin t/2}$

then $= O(n+1)$, *for* $\frac{\pi}{(n+1)} \le t \le \pi$.

Proof. Using $\sin \frac{t}{2} \geq \frac{t}{\pi}$ for $\frac{\pi}{(n+1)} \leq t \leq \pi$

$$\text{then } K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} O(S_n) \frac{t^3}{t/\pi}$$

$$= \left[\frac{2}{(n+2)\pi} - \sum_{k=0}^{n} k\right] t^2$$

$$= O(n+1). \tag{5.6}$$

6. Proof of the Main Theorem

Following Titchmarsh [18], the k^{th} partial sum $S_k(f; x)$ of the Fourier series (2.1) is given by

$$S_k(f; x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \frac{1}{\sin t/2} \phi_x(t) \sin\left(k + \frac{1}{2}\right) t dt. \tag{6.1}$$

The $[F, d_n]$ transform $t_n^{[F, d_n]}$ of $S_k(f; x)$ is given by

$$t_n^{[F, d_n]} - f(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{t}{2}} \sum_{k=0}^{\infty} P_{nk} \sin\left(k + \frac{1}{2}\right) t dt. \tag{6.2}$$

The $(C, 2)[F, d_n]$ transform of $S_k(f; x)$ by $t_n^{(C, 2)[F, d_n]}$ we have

$$t_n^{(C, 2)[F, d_n]} - f(x) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{t}{2}} \sum_{k=0}^{\infty} P_{nk} \sin\left(k + \frac{1}{2}\right) t dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{t}{2}} I_m\left\{\sum_{k=0}^{\infty} P_{nk} \exp\left(i\left(k + \frac{1}{2}\right)\right)\right\} dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{t}{2}} I_m\left\{\exp\left(\frac{it}{2}\right) \sum_{k=0}^{\infty} P_{nk} \exp(ikt)\right\} dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}\, I_m\left\{\exp\left(\frac{it}{2}\right)\prod_{k=1}^{n}\frac{\exp(it)+d_k}{1+d_k}\right\}dt. \quad (6.3)$$

Writing $I_n(x) = t_n^{(C,\,2)[F,\,d_n]} - f(x)$, we have

$$|\, I_n(x)\,| = |\, t_n^{(C,\,2)[F,\,d_n]} - f(x)\,|$$

$$\leq \left|\frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}\, I_m\left\{\exp\left(\frac{it}{2}\right)\prod_{k=1}^{n}\frac{\exp(it)+d_k}{1+d_k}\right\}dt\right|,$$

$$|\, I_n(x) - I_n(y)\,|$$

$$= \left|\frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\int_0^{\pi}\frac{\phi_x(t)-\phi_y(t)}{\sin\frac{t}{2}}\, I_m\left\{\exp\left(\frac{it}{2}\right)\prod_{k=1}^{n}\frac{\exp(it)+d_k}{1+d_k}\right\}dt\right|$$

$$\leq \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\int_0^{\pi}\frac{|\,\phi_x(t)-\phi_y(t)\,|}{\left|\sin\frac{t}{2}\right|}\, I_m\left\{\exp\left(\frac{it}{2}\right)\prod_{k=1}^{n}\frac{\exp(it)+d_k}{1+d_k}\right\}dt$$

using Lemma 5.2

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}$$

$$\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\left\{I_m\left\{\exp\left(\frac{it}{2}\right)\left\{\exp\left\{\frac{(U_n-1)it}{2}-\frac{S_nt^2}{4}\right\}\right\}+O(S_nt^3)\right\}\right\}dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\left\{\sin\left(\frac{U_nt}{2}\right)\exp\left(\frac{-S_nt^2}{4}\right)+O(S_nt^3)\right\}dt$$

$$\leq \left|\frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\sin\left(\frac{U_nt}{2}\right)\exp\left(\frac{-S_nt^2}{4}\right)dt\right|$$

$$+\left|\frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\{O(S_n t^3)\}dt\right|$$

$$=|I_1|+|I_2| \tag{6.5}$$

Then $|I_1| = \dfrac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\displaystyle\int_0^{\pi}\frac{|\phi(x)|}{\sin\frac{t}{2}}\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)dt.$

Applying Lemma 5.3

$$|I_1| = \int_0^{\pi}|\phi(x)|K_n(x)dt$$

$$= \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}|\phi(x)|K_n(t)dt,\right)$$

$$|I_1| = |I_{1.1}| + |I_{1.2}|. \tag{6.6}$$

Now

$$|I_{1.1}| = \int_0^{\frac{\pi}{n+1}}|\phi(x)|K_n(x)dt$$

$$= O(n+1)\int_0^{\frac{\pi}{n+1}}|t|^{\alpha}dt \text{ using Lemma 5.1 and 5.3}$$

$$= O(n+1)^{-\alpha} \tag{6.7}$$

Now

$$|I_{1.2}| = \int_{\frac{\pi}{n+1}}^{\pi}|\phi(x)|K_n(x)dt$$

$$= \int_{\frac{\pi}{n+1}}^{\pi}\frac{|t|^{\alpha}}{t}dt \text{ using Lemma 5.1 and 5.4}$$

$$= \int_{\frac{\pi}{n+1}}^{\pi} |t|^{\alpha-1} dt$$

$$= O(n+1)^{-\alpha}. \tag{6.8}$$

Then

$$|I_2| = \left| \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} \{O(S_n t^3)\} \right| dt$$

$$= \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} |\phi(t)| K_n(t) dt \right) \text{ using Lemma 5.5}$$

$$= |I_{2.1}| + |I_{2.2}|. \tag{6.9}$$

Now

$$|I_{2.1}| = \int_0^{\frac{\pi}{n+1}} |\phi(x)| K_n(t) dt$$

$$= O(n+1) \int_0^{\frac{\pi}{n+1}} |t|^{\alpha} dt \text{ using Lemmas 5.1 and 5.5}$$

$$= O(n+1)^{-\alpha}. \tag{6.10}$$

Similarly

$$|I_{2.2}| = O(n+1)^{-\alpha}. \text{ Using Lemma 5.6} \tag{6.11}$$

Now combining (6.10) and (6.11), we have

$$|I_2| = O(n+1)^{-\alpha}. \tag{6.12}$$

Now using $|\Phi(t)| = |\Phi_x(t) - \Phi_y(t)|$

$$= O(|x-y|^{\alpha}), \tag{6.13}$$

we obtain

$$I_1 = \frac{O(|x-y|^{\alpha})}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\int_0^{\pi}\frac{1}{\sin\frac{t}{2}}\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)dt$$

$$= O(U_n)\frac{(|x-y|^{\alpha})}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\int_0^{\pi}\frac{\sin\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\exp\left(\frac{-S_n t^2}{4}\right)dt$$

$$+ O\frac{(|x-y|^{\alpha})}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\int_0^{\pi}\frac{1}{t}\exp\left(\frac{-S_n t^2}{4}\right)dt$$

$$= O(|x-y|^{\alpha}) + O(|x-y|^{\alpha})\log(n+1)$$

$$= O(|x-y|^{\alpha}) + \log(n+1) \tag{6.14}$$

and

$$I_2 = \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\left(\int_0^{\frac{\pi}{(n+1)}} + \int_{\frac{\pi}{(n+1)}}^{\pi}\right)\frac{O(|x-y|^{\alpha})}{\sin\frac{t}{2}}\{O(S_n t^3)\}dt$$

$$= \frac{O(|x-y|^{\alpha})}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\left\{\left(\int_0^{\frac{\pi}{(n+1)}}\frac{O(S_n t^3)}{\frac{t}{2}}dt + \int_{\frac{\pi}{(n+1)}}^{\pi}\frac{O(S_n t^3)}{\frac{t}{\pi}}\right)dt\right\}$$

$$= O(|x-y|^{\alpha}). \tag{6.15}$$

Now for $k = 1, 2,$ and for $0 \le \beta < \alpha \le 1$,

we observe that

$$|I_k| = |I_k|^{1-\beta/\alpha}|I_k|^{\beta/\alpha} \tag{6.16}$$

By using (6.12) and (6.15) respectively above identity (6.16) for $k = 2$ gives

$$I_2 = O\{|x-y|^{\beta}(n+1)^{\beta-\alpha}\}. \tag{6.17}$$

Again using (6.7), (6.8), (6.14) and identity (6.16) for $k = 1$, we obtain

$$| I_1 | = O\{| x - y |^{\beta}(n+1)^{\beta-\alpha} \log(n+1)\}. \tag{6.18}$$

Thus from (6.17) and (6.18), we get

$$\sup_{\substack{(x, y) \\ x \neq y}} | \Delta^{\beta} I_n(x, y) | = \sup_{\substack{(x, y) \\ x \neq y}} \frac{| I_n(x) - I_n(y) |}{(x-y)^{\beta}}$$

$$= O\{(n+1)^{\beta-\alpha} \log(n+1)\}. \tag{6.19}$$

Now using the fact that $f \in H_{\alpha} \Rightarrow \phi_x(t) = O(t^{\alpha})$ and proceeding as above, we obtain

$$\| I_n \|_c = \underset{-\pi \leq x \leq \pi}{Sup} \| t_n^{(C, 2)[F, d_n]} - f(x) \|$$

$$= O\{(n+1)^{-\alpha} \log(n+1)\}. \tag{6.20}$$

Combining the results (6.18) and (6.19) and using (6.20), we derive

$$\| t_n^{(C, 2)[F, d_n]} - f(x) \|_{\beta} = O\{(n+1)^{\beta-\alpha} \log(n+1)\}.$$

Hence proof of the main theorem is completed.

7. Application

We can derive the following corollary.

Corollary 7.1. *If* $f \in Lip\alpha$, *when* $0 < \alpha \leq 1$. *Then for* $n > 1$

$$\| t_n^{(C, 2)[F, d_n]} - f(x) \| = O(n)^{-\alpha} \log n.$$

If we put $\beta = 0$ then Theorem 3.2 is particular case of main Theorem.

8. Conclusion

We would like to mention that from our result some newer method of summability like $F(a, q)$, $(f, d_n)(e, c)$ and Norlund mean can be used to sum infinite series. Further the result of our theorem is more general rather than the result of any other previous proved theorems. Also our results play an important role in application in pure and applied mathematics.

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