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SOME FIXED POINT RESULTS IN MODULAR-LIKE METRIC SPACES AND PARTIAL MODULAR-LIKE METRIC SPACES WITH ITS NON-ARCHIMEDEAN VERSION

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Abstract

In this paper, the notion of partial modular-like metric space from partial modular metric spaces with some properties and examples, and observed that the restriction of partial modular-like metric space to equivalent to dislocated (modular-like) metric space is self distance axiom. All the results are also study in non-Archimedean modular sense. Some fixed point results via C-class function are introduced with suitable examples to validate the results. As an application for the existence and uniqueness of solutions for a system of Volterra integral equations is given.

1. Introduction

In [13], Matthews introduced partial metric space which is a generalization of metric space as self-distance is nonzero. In [3], Amini-Harandi introduced metric like space as generalization of partial metric space. In [6], Chistyakov introduced modular metric space which generalizes metric space. For some nonlinear contraction fixed point theorem in modular spaces is not possible, to remove this difficulty, in ([15], [16]) Paknazar et al. introduced non-Archimedean modular metric spaces by changing its triangular property. In [8], Hosseinzadeh and Paryaneh, introduced partial

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modular metric space which generalizes modular metric space. Recently, in [18], Rasham et al. introduced modular-like metric spaces. In [21], Taki-Eddine and Aliouche, introduced convex partial metric spaces which generalizes convex metric spaces. In [10], Karapinar and Salimi, analyzed that metric-like space and dislocated metric spaces are exactly same. Since then, many researchers developed fixed point theory in these generalized spaces (see [9], [12], [14], [17], [19]).

Using altering distance function ([11]), Alber and Guerre-Delabriere ([2]), introduced weakly contractive mapping in Hilbert space, and in [20], Rhoades this mapping in metric spaces. Later, in [7], Dutta and Choudhury generalized this mapping and the results related to Banach fixed point. In [4], Ansari introduced C-class function.

In this paper, first part is about the development of modular metric spaces, modular-like metric spaces, partial modular metric spaces and generalized contractive condition. The second part includes the concept of modular-like metric spaces as a generalization of partial modular metric spaces and partial modular-like metric spaces, some basic definitions, properties and examples. The third part includes some fixed point results via C-class function and examples in partial modular-like metric spaces and modular-like metric spaces. The fourth part is on non-Archimedean modular sense. The last part includes an application for the existence and uniqueness of solutions for a system of Volterra integral equation.

2. Preliminaries

Throughout this paper $\mathbb{R}$, denotes set of real numbers, and $\mathbb{R}^+$, denotes set of positive numbers.

In [18], Rasham et al., defined modular-like metric spaces which is exactly same as dislocated modular metric spaces (see [10]).

Definition 2.1. A function $\Theta : (0, +\infty) \times X \times X \to [0, +\infty)$ is called a modular-like metric on X if for all $x, y, z \in X$, it satisfies:

(i) if $\Theta_\lambda(x, y) = 0$, for all $\lambda > 0$ then $x = y$,

(ii) $\Theta_\lambda(x, y) = \Theta_\lambda(y, x)$ for all $\lambda > 0$ and

(iii) $\Theta_{\lambda+\mu}(x, y) \leq \Theta_\lambda(x, z) + \Theta_\mu(z, y)$ for all $\lambda, \mu > 0$.

The pair (X, Θ_λ) is called modular-like metric space or dislocated modular metric space.

If (i) is replaced by "if $\Theta_\lambda(x, y) = 0$ if and only if $x = y$, for all $\lambda > 0$" then it is a modular metric space, (X, ω_λ). The space will be regular modular-like metric space if (i) is replaced by "if $\Theta_\lambda(x, y) = 0$ for some, $\lambda > 0$ then $x = y$". The space satisfies all conditions of modular metric space except positiveness of $\omega_\lambda(x, x)$ for each $x \in X$.

The space will be non-Archimedean modular-like metric space if (iii) is replaced by "$\Theta_{\max\{\lambda, \mu\}}(x, y) \leq \Theta_\lambda(x, z) + \Theta_\mu(y, z)$" for all $x, y \in X$, and $\lambda, \mu > 0$.

The space will be convex modular-like metric space if (iii) is replaced by "$\Theta_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu}\Theta_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\Theta_\mu(y, z)$" for all $x, y \in X$, and $\lambda, \mu > 0$. For a fixed $x_0 \in X$, the set $X_\Theta = X_\Theta(x_0) = \{x \in X : \Theta_\lambda(x, x_0) \to c$ as $\lambda \to \infty\}, c \geq 0$ is said to be modular space, the set $X_\Theta^* = X_\Theta^*(x_0)$ $= \{x \in X : \exists\lambda = \lambda(x) > 0, \Theta_\lambda(x, x_0) < \infty$ is said to be convex modular-like space.

For $x_0 \in X$, and $\epsilon > 0, > 0$, $\overline{B_{\Theta_\lambda}(x_0, \epsilon)} = \{x \in X : |\Theta_\lambda(x, x_0) - \Theta_\lambda(x, x)|$ $\leq \epsilon\}$ is a closed ball in X_Θ.

Definition 2.2 [8]**.** A function $p : (0, +\infty) \times X \times X \to [0, +\infty)$ is called a partial modular metric on X if the following conditions holds:

$(P_1)\, x = y$ if and only if $p_\lambda(x, y) = p_\lambda(x, x) = p_\lambda(y, y)$ for $\lambda > 0$;

$(P_2)\, p_\lambda(x, x) \leq p_\lambda(x, y)$ for all $\lambda > 0$ and for all $x, y \in X$, (self distance axiom)

$(P_3)\, p_\lambda(x, y) = p_\lambda(y, x)$ for all $\lambda > 0$ and for all $x, y \in X$,

$(P_4)\, p_{\lambda+\mu}(x, y) = p_\lambda(x, z) + p_\mu(y, z)$

$$-\left[\frac{p_\lambda(x, x) + p_\lambda(z, z) + p_\mu(z, z) + p_\mu(y, y)}{2}\right]$$

for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Then the pair (X, p_λ) is called partial modular metric space.

From P_1 and P_2, $p_\lambda(x, y) = 0$ implies $x = y$. From P_4, for all $\lambda, \mu > 0$ and $x, y, z \in X$, we infer that $p_{\lambda+\mu}(x, y) = p_\lambda(x, z) + p_\mu(y, z)$ (see [12]). With these conditions except P_2, the space is equivalent to modular-like metric space. But with P_2, the space is called a partial modular-like metric space. The space will be non-Archimedean partial modular-like metric space if P_4 is replaced by "$p_{\max\{\lambda, \mu\}}(x, y) \leq p_\lambda(x, z) + p_\mu(y, z)$" for all $x, y \in X$, and $\lambda, \mu > 0$.

Each modular-like metric space generates a topology τ_ω, whose base is the family of open balls, for $x_0 \in X$, and $\epsilon, \lambda > 0$, $B_{p_\lambda}(x_0, \epsilon) = \{x \in X : | p_\lambda(x, x_0) - p_\lambda(x, x) | < \epsilon\}$.

For a fixed $x_0 \in X$, the set $X_{p_\lambda} = X_{p_\lambda}(x_0) = \{x \in X : p_\lambda(x, x_0) \to c$ as $\lambda \to \infty\}$, $c \geq 0$ is said to be partial modular-like metric space.

Example 2.3. Let $X = \mathbb{R}^+$, where $p_\lambda(x, y) = \omega_\lambda^p(x, y)$, it satisfies all the properties P_1, P_2, P_3 and P_4. Hence it is a partial modular metric.

Example 2.4. Let $X \neq \phi$, and p_λ be a partial modular-like metric. Define $p_\lambda(x, y) = \frac{1}{\lambda}\max(x, y)$ for all $\lambda > 0$. Then it is a

(i) Partial modular-like metric space,

(ii) Non-Archimedean partial modular-like metric space,

(iii) Modular-like metric space with modular Θ_λ,

(iv) Non-Archimedean modular-like metric space with modular Θ_λ.

Example 2.5. Let $X \neq \phi, \Theta_\lambda$ be a partial modular-like metric. Define

$\Theta_\lambda(x, y) = \begin{cases} e^{-\lambda}\{\max(x, y) - \min(x, y)\}, \forall x \neq y \\ e^{-\lambda}\max(x, y), \forall x = y \end{cases}$, for all $\lambda > 0$ and $x, y \in X$.

Triangular inequality: for $x = y$,

$$e^{-\lambda}\max(x, y) \leq e^{-\lambda}\max(x, z) + e^{-\lambda}\max(z, y)$$

Now, check for $x \neq y$, if $x > z > y, y > z > x$ then equality occurs, otherwise for $z > x > y, z > y > x$ and $y > x > z, x > y > z$, we have

$$e^{-\lambda}\{\max(x, y) - \max(x, y)\} \leq e^{-\lambda}\{\max(x, z) - \min(x, z)\} + e^{-\lambda}\{\max(z, y) - \min(z, y)\}.$$

Then the space is a modular-like metric space but not a partial modular metric space, as well as modular-like metric space because self distance axiom may not be satisfied, $\Theta_\lambda(x, x) \not\leq \Theta_\lambda(x, y)$.

Remarks. (i) Every partial modular-like metric is a modular-like metric but converse is not true.

(ii) Every non-Archimedean partial modular-like metric is a non-Archimedean modular-like metric but converse is not true.

Definition 2.6 [21]. Let (X, p) be a partial metric space and $I = [0, 1]$ be the closed unit interval. A function $\Omega : X \times X \times (0, +\infty) \to X$ is called a convex structure on X if it satisfies:

$$p(z, \Omega(x, y, \lambda)) \leq (1 - \lambda)p(x, z) + \lambda p(z, y) \text{ for all } x, y, z \in X \text{ and } \lambda \in I.$$

A partial metric space (X, p) with this convex structure is called convex partial metric space.

Definition 2.7. Partial modular-like metric spaces is a convex if we replace condition (P_4) by $p_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu} p_\lambda(x, z) + \frac{\mu}{\lambda + \mu} p_\mu(y, z)$ for all $x, y, z \in X$ and $\mu, \lambda > 0$. The set $X^*_{p_\lambda} = X^*_{p_\lambda}(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0, p_\lambda(x, x_0) < \infty\}$ is said to be partial convex modular-like metric space.

Example 2.8. Let (X, Θ_λ) be a modular-like metric space, and p_λ be a partial modular-like metric. Define $p_\lambda(x, y) = \begin{cases} \Theta_\lambda(x, y) + c, \; c > 0 \text{ for } x \neq y \\ \Theta_\lambda(x, y), \; c = 0 \text{ for } x = y \end{cases}$, for all $x, y \in X$ and $\lambda > 0$.

(i) If $\Theta_\lambda(x, x) \leq \Theta_\lambda(x, y)$, then (X, p_λ) is partial modular-like metric space.

(ii) If (X, Θ_λ) is a convex modular-like metric space and $\Theta_\lambda(x, x) \leq \Theta_\lambda(x, y)$ then (X, p_λ) is a convex partial modular-like metric space.

In both cases P_1, P_2 and P_3 are satisfied. Now we will check for P_4.

P_4: For (i); $p_\lambda(x, y) = \Theta_\lambda(x, y) + c \leq \Theta_\lambda(x, z) + c + \Theta_\lambda(z, y) + c = p_\lambda(x, z) + p_\lambda(z, y)$.

For (ii);
$$p_\lambda(x, y) = \Theta_\lambda(x, y) + c \leq \frac{\lambda}{\lambda + \mu}\Theta_\lambda(x, z) + \frac{\mu}{\lambda + \mu}\Theta_\lambda(z, y)$$
$$+ \left(\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu}\right)c = \frac{\lambda}{\lambda + \mu} p_\lambda(x, z) + \frac{\mu}{\lambda + \mu} p_\mu(y, z).$$

Lemma 2.9. (see [8]). (i) *Let* (X, Θ_λ) *be a modular-like metric space. Then* $p : (0, +\infty) \times X \times X \to [0, +\infty)$ *is a partial modular-like metric on X, where* $p_\lambda(x, y) = \Theta_\lambda(x, y)$ *with* $\Theta_\lambda(x, x) \leq \Theta_\lambda(x, y)$, *for all* $\lambda > 0$ *and* $x, y \in X$.

(ii) *Let* (X, p_λ) *be a partial modular metric space, and* $\Theta : (0, +\infty) \times X \times X \to [0, +\infty)$ *be a modular-like metric space on X. If* $\Theta^p_\lambda(x, y) = \begin{cases} 2p_\lambda(x, y) - p_\lambda(x, x) - p_\lambda(y, y), \; x \neq y \\ b, \; x = y \end{cases}$, *for all* $x, y \in X, \lambda > 0$ *such that* $b \geq 0$, *then* (X, Θ^p_λ) *is a modular-like metric space.*

Definition 2.10 (see [8], [18]). Let (X, p_λ) be a partial modular-like metric space. Then,

(i) A sequence $\{x_n\}$ in X is convergences to a point $x \in X$ if and only if for $\lambda > 0$, $\lim_{n \to \infty} p_\lambda(x_n, x) = p_\lambda(x, x)$.

(ii) A sequence $\{x_n\}$ in X is Cauchy if for $\lambda > 0$, $\lim_{m, n \to \infty} p_\lambda(x_m, x_n)$ $= \lim_{m \to \infty} p_\lambda(x_m, x_m) = \lim_{n \to \infty} p_\lambda(x_n, x_n)$.

(iii) X is complete if any Cauchy sequence in X is convergent to a point $x \epsilon X$.

(iv) $\{x_n\}$ is a Cauchy sequence in (X, p_λ) if and only if it is a Cauchy sequence in the modular-like metric space (X, Θ_λ).

(v) (X, p) is complete if and only if (X, Θ_λ) is complete.

Lemma 2.11 (see [3], [10]). *Let* (X, Θ_λ) *be a modular-like metric space.*

(i) *If* $\Theta_\lambda(x, y) = 0$ *then* $\Theta_\lambda(x, x) = \Theta_\lambda(y, y) = 0$.

(ii) *If* $\{x_n\}$ *is a sequence with* $\lim_{n \to \infty} \Theta_\lambda(x_{n+1}, x_n) = 0$ *then*

$$0 = \lim_{n \to \infty} \Theta_\lambda(x_{n+1}, x_{n+1}) = \lim_{n \to \infty} \Theta_\lambda(x_n, x_n).$$

(iii) *If* $x \neq y$ *then* $\Theta_\lambda(x, y) > 0$.

(iv) $\Theta_{2\lambda}(x, x) \leq \frac{2}{n} \sum_{i=1}^{n} \Theta_\lambda(x, x_i); 1 \leq i \leq n$

(*In non-Archimedean modular* $\Theta_\lambda(x, x) \leq \frac{2}{n} \sum_{i=1}^{n} \Theta_\lambda(x, x_i); 1 \leq i \leq n$

(v) *If* $x_n \to x$, *as* $n \to \infty$ *then* $\lim_{n \to \infty} \Theta_\lambda(x_n, y) = \Theta_\lambda(x, y)$.

Note. These results are also true for partial modular-like metric spaces. All the results are also true in the sense of non-Archimedean modular.

Definition 2.12 [4]. A mapping $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:

(i) $F(x, y) \leq x$,

(ii) $F(x, y) = x \Rightarrow x = 0$ or $y = 0$ for all $x, y \in [0, \infty)$.

Definition 2.13 ([4], [11]). Let Ψ be the set of altering distance function and Φ be the set of ultra altering distance function. Define $\psi, \phi : [0, \infty) \to [0, \infty)$ such that both are non-decreasing and continuous if

(i) $\psi(t) = 0$ if and only if $t = 0$, then ψ is called altering distance function. Moreover if

(ii) $\phi(t) > 0, t > 0$ and $\phi(0) \geq 0$, then ϕ is ultra altering distance function

Definition 2.14 [4]**.** A tripled (ψ, ϕ, F) where $\psi \in \Psi, \phi \in \Phi$ and $F \in C$ is say to be monotone if for any $x, y \in [0, \infty), x \leq y \Rightarrow F(\psi(x), \phi(x)) \leq F(\psi(x), \phi(x))$. Strictly monotonic if '≤' changes to '<'.

Definition 2.15 [1]**.** Let X be a partial modular metric space. Let A, B self mappings of X, a point a in X is called a coincidence point of A and B; $Aa = Ba$. We shall call $q = Aa = Ba$ a point of coincidence (POC) of A and B. Moreover, A and B is said to be weakly compatible if they commute at coincidence points.

Lemma 2.16 (see [1])**.** *Let $X \neq \phi$, A and B be two self-mappings which are weakly compatible and have unique point of coincidence. Then A and B have common unique fixed point.*

3. Main Results

A. Fixed Point Results in partial modular-like metric spaces and modular-like metric spaces.

Theorem 3.1. *Let A and B be two self maps defined on a complete partial modular-like metric space (X, p_λ) satisfying the following conditions.*

I. $A(X) \subseteq B(X)$

II. $\psi(p_\lambda(Ax, Ay)) \leq F(\psi(M(x, y)), \phi(M(x, y)))$

holds for all $x, y \in X$, where $\psi \in \Psi, \phi \in \Phi$ and $F \in C$, such that (ψ, ϕ, F) is monotone and

$$M(x, y) = \max\{p_\lambda(Bx, By), p_\lambda(Bx, Ax), p_\lambda(Bx, Ay)\}.$$

If the pair (A, B) is weakly compatible, then A and B have a unique common fixed point.

Proof. Let x_0 be any point in X. Define the sequence $\{y_n\}$ such that $y_n = Ax_n = Bx_{n+1}$. By II, we have

$$\psi(p_\lambda(y_n, y_{n+1})) = \psi(p_\lambda(Bx_{n+1}, Bx_{n+2})) = \psi(p_\lambda(Ax_n, Ax_{n+1}))$$

$$\leq F(\psi(M(x_n, x_{n+1})), \phi(M(x_n, x_{n+1})) \leq \psi(M(x_n, x_{n+1}))) \tag{1}$$

and $M(x_n, x_{n+1}) = \max\{p_\lambda(Bx_n, Bx_{n+1}), p_\lambda(Bx_n, Ax_n), p_\lambda(Bx_{n+1}, Ax_{n+1})\}$

$$= \max\{p_\lambda(y_{n-1}, y_n), p_\lambda(y_{n-1}, y_n), p_\lambda(y_n, y_{n+1})\}$$

$$= \max\{p_\lambda(y_{n-1}, y_n), p_\lambda(y_n, y_{n+1})\}$$

Case I. $M(x_n, x_{n+1}) = p_\lambda(y_n, y_{n+1})$. Since ψ is non decreasing so from (1), we have $\psi(p_\lambda(y_n, y_{n+1})) \leq \psi(M(x_n, x_{n+1})) = \psi(p_\lambda(y_n, y_{n+1}))$, which is a contradiction.

Case II. $M(x_n, x_{n+1}) = p_\lambda(y_n, y_{n+1})$.

So, $\psi(p_\lambda(y_n, y_{n+1})) \leq \psi(p_\lambda(y_{n-1}, y_n)) \Rightarrow p_\lambda(y_n, y_{n+1}) \leq (y_{n-1}, y_n)$. Thus, $\{p_\lambda(y_n, y_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. Let

$$\lim_{n\to\infty} p_\lambda(y_n, y_{n+1}) = r \geq 0; r \in [0, \infty).$$

Now we show that $r = 0$. Taking $n \to \infty$ in (1); $\psi(r) \leq F(\psi(r), \phi(r))$ So, $\psi(r) = 0$ or $\phi(r) = 0$; shows $r = 0$.

Now we will show that $\{y_n\}$ is a Cauchy sequence in X; i.e., $\lim_{n\to\infty}\{y_n\} = 0$.

In the contrary, we suppose $\{y_n\}$ is not a Cauchy sequence. Since,

$$\lim_{n\to\infty} p_\lambda(y_n, y_{n+1}) = 0 \tag{2}$$

so, we can find $\epsilon > 0$ and subsequences $\{y_{m(k)}\}, \{y_{n(k)}\}$ of $\{y_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $p_\lambda(y_{m(k)}, y_{n(k)}) \geq \epsilon$, i.e., $p_\lambda(y_{m(k)}, y_{n(k)-1}) \geq \epsilon$. Now,

$$\epsilon \leq p_\lambda(y_{m(k)}, y_{n(k)}) \geq p_{\frac{\lambda}{2}}(y_{m(k)}, y_{n(k)-1}) + p_{\frac{\lambda}{2}}(y_{m(k)-1}, y_{n(k)})$$

$$\leq \epsilon + p_{\frac{\lambda}{2}}(y_{m(k)-1}, y_{n(k)})$$

Taking $k \to \infty$ and using (2) we get, $\lim_{n \to \infty} p_{\lambda}(y_{m(k)}, y_{n(k)}) = \epsilon$ (3)

$$p_{\lambda}(y_{m(k)-1}, y_{n(k)}) \leq p_{\frac{\lambda}{2}}(y_{m(k)-1}, y_{n(k)}) + p_{\frac{\lambda}{2}}(y_{m(k)}, y_{n(k)}) \tag{4}$$

Taking $k \to \infty$ and using (2) and (3) we get, $\lim_{n \to \infty} p_{\lambda}(y_{m(k)-1}, y_{n(k)}) = \epsilon$

$$p_{\lambda}(y_{m(k)-1}, y_{n(k)}) \leq p_{\frac{\lambda}{2}}(y_{m(k)-1}, y_{n(k)}) + p_{\frac{\lambda}{2}}(y_{m(k)}, y_{n(k)})$$

Taking $k \to \infty$ and using (2) and (3) we get, $\lim_{n \to \infty} p_{\lambda}(y_{m(k)-1}, y_{n(k)}) = \epsilon$

(5)

$$p_{\lambda}(y_{m(k)-1}, y_{n(k)-1}) \leq p_{\frac{\lambda}{2}}(y_{m(k)-1}, y_{n(k)}) + p_{\frac{\lambda}{2}}(y_{m(k)}, y_{n(k)-1})$$

$$\leq p_{\frac{\lambda}{4}}(y_{m(k)-1}, y_{n(k)}) \leq p_{\frac{\lambda}{4}}(y_{m(k)}, y_{n(k)}) + p_{\frac{\lambda}{2}}(y_{m(k)-1}, y_{n(k)})$$

Taking $k \to \infty$ and using (2) and (3) we get, $\lim_{n \to \infty} p_{\lambda}(y_{m(k)-1}, y_{n(k)-1}) = \epsilon$

(6)

Now, $\psi(p_{\lambda}(y_{m(k)}, y_{n(k)})) = \psi(p_{\lambda}(Ax_{m(k)}, Ax_{n(k)}))$

$= F(\psi(M(x_{m(k)}, x_{n(k)})), \phi(M(x_{m(k)}, x_{n(k)}))) \leq \psi(M(x_{m(k)}, x_{n(k)})))$

$M(x_{m(k)}, x_{n(k)}) = \max\{p_{\lambda}(Bx_{m(k)}, Bx_{n(k)}), p_{\lambda}(Bx_{m(k)}, Ax_{n(k)}),$

$p_{\lambda}(Bx_{m(k)}, Ax_{n(k)})\}$

$= \max\{p_{\lambda}(y_{m(k)-1}, y_{n(k)-1}), p_{\lambda}(y_{m(k)-1}, y_{n(k)}), p_{\lambda}(y_{m(k)-1}, y_{n(k)})\}$

Using (2), (3), (4), (5) and (6) we get, $\lim_{n \to \infty} M(x_{m(k)}, x_{n(k)}) = \epsilon$ (7)

$$\psi(p_{\lambda}(y_{m(k)}, y_{n(k)})) \leq F(\psi(M(x_{m(k)}, x_{n(k)})), \phi(M(x_{m(k)}, x_{n(k)})))$$

Taking $k \to \infty$ and using (3) and (7) we get

$\psi(\in) \leq F(\psi(\in), \phi(\in))$ so, $\psi(\in) = 0$ or $\phi(\in) = 0$, which shows $\epsilon = 0$, i.e., $\lim_{n\to\infty} p_\lambda(y_{m(k)}, y_{n(k)}) = 0$. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete so there exists $q \in X$ such that $\lim_{n\to\infty} y_n = q$.

Thus $\lim_{n\to\infty} p_\lambda(y_n, y_m) = \lim_{n\to\infty} p_\lambda(y_n, q) = p_\lambda(q, q) = 0$, and $\lim_{n\to\infty} Ax_n$ $\lim_{n\to\infty} Bx_{n+1} = q; n = 0, 1, 2, \ldots$.

Suppose there exists a point $v \in X$ such that $q = Bv$. We claim $Av = Bv$.

$$\psi(w_\lambda(Ax_n, Av)) \leq F(\psi(M(x_n, v)), \phi(M(x_n, v))) \tag{8}$$

where

$$M(x_n, v) = \max\{p_\lambda(Bx_n, Bv), p_\lambda(Bx_n, Ax_n), p_\lambda(Bv, Av)\}$$

$$= \max\{p_\lambda(y_{n-1}, Bv), p_\lambda(y_{n-1}, Ax_n), p_\lambda(Bv, Av)\}$$

$$M(x_n, v) = \max\{p_\lambda(q, q), p_\lambda(q, q), p_\lambda(q, Av)\}.$$

Taking $n \to \infty$, in (8) $Av = q = Bv$, $POC(A, B) \neq \phi$.

Since, (A, B) weakly compatible, so $Aq = Bq$. If possible let, $Aq = Bq = q'$ and $Aq = Bq = r$.

$$\psi(p_\lambda(r, q')) = \psi(p_\lambda(Aq, Qq)) \leq F(\psi(M(q, q)), \phi(M(q, q)))$$

$$M(r, v) = \max\{p_\lambda(Bq, Bq), p_\lambda(Bq, Aq), p_\lambda(Bq, Aq)\} = p_\lambda(r, q')$$

By definition of C-class function we have $r = q'$. Hence the pair (A, B) have a unique point of coincidence. Since the pair is weakly compatible so by the Lemma 2.16, A and B have common unique fixed point.

It is observed above theorem is true for modular-like metric space with all the conditions stated in the theorem. As modular-like metric spaces generalized partial modular-like metric spaces. So, above theorem will be more generalized in modular-like metric spaces.

Corollary 3.2. *Let A and B be two self maps defined on a complete* *modular-like metric space* (X, Θ_λ) *satisfying the following conditions I and II of Theorem* 3.1. *If the pair* (A, B) *is weakly compatible, then A and B have a common fixed point*.

Example 3.3. Let A and B be two self maps defined on a complete partial modular-like metric space (X, p_λ) where $X = \mathbb{R}^+$. Let $Ax = 1$, $Bx = 2x - 1$.

Let $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{1}{r} t$, $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{1}{2r} t$ and $F : [0, \infty) \to [0, \infty) \to \mathbb{R}$ by $F(S, T) = rS$, $0 < r < 1$. Then (ψ, ϕ, F) is monotone. (see [5])

1. Clearly, $A(X) = \{1\} \subseteq B(X) = \mathbb{R}^+$. The pair $POC(A, B) \neq \phi$, and weakly compatible at that point.

2. $\psi(p_\lambda(Ax, Ay)) \leq M(x, y) \Rightarrow \psi(p_\lambda(Ax, Ay)) \leq r(\frac{1}{r} M(x, y))$

$$\psi(p_\lambda(Ax, Ay)) \leq F(\psi(M(x, y)), \phi(M(x, y)))$$

$$M(x, y) = \max\{p_\lambda(Bx, By), p_\lambda(Bx, Ax), p_\lambda(By, Ay)\}.$$

It satisfies all the condition of Theorem 3.1. So $\{1\}$ is the unique fixed point of A and B.

B. Fixed Point Results in non-Archimedean partial modular-like metric spaces and non-Archimedean modular-like metric spaces.

In this part, for any $X \neq \phi$ and two self maps A and B the following nonlinear contraction is taken in the sense of non-Archimedean modular (see [10], [15], [16])

$$M(x, y) = \max\{p_1(Bx, By), p_1(Bx, Ax), p_1(By, Ay), \frac{[p_1(By, Ax) + p_1(Bx, Ay)]}{4}\},$$

$\forall x, y \in X$.

Theorem 3.4. *Let A and B be two self maps defined on a complete non-Archimedean partial modular-like metric space* (X, p_1) *satisfying the following conditions*:

I. $A(X) \subseteq B(X)$

II. $\psi(p_1(Ax, Ay)) \leq F(\psi(M(x, y)), \phi(M(x, y)))$

holds for all $x, y \in X$, *where* $\psi \in \Psi$, $\phi \in \Phi$ *and* $F \in C$, *such that* (ψ, ϕ, F) *is monotone. If the pair* (A, B) *is weakly compatible, then A and B have a unique common fixed point.*

Proof. Let x_0 be any point in *X*. Define the sequence $\{y_n\}$ such that $y_n = Ax_n = Bx_{n+1}$ By II, we have

$$\psi(p_\lambda(y_n, y_{n+1})) = \psi(p_\lambda(Bx_{n+1}, Bx_{n+2})) = \psi(p_\lambda(Ax_n, Ax_{n+1}))$$

$$\leq F(\psi(M(x_n, x_{n+1})), \phi(M(x_n, x_{n+1})) \leq \psi(M(x_n, x_{n+1})) \tag{9}$$

Since,

$$\frac{[p_1(y_n, y_n) + p_1(y_{n-1}, y_{n+1})]}{2} \leq \frac{[p_1(y_n, y_n) + p_1(y_{n-1}, y_n) + p_1(y_n, y_{n+1})]}{4}$$

and

$$M(x_n, x_{n+1}) = \max\{p_1(Bx_n, Bx_{n+1}), p_1(Bx_n, Ax_n), p_1(Bx_{n+1}, Ax_{n+1}), \frac{[p_1(Bx_{n+1}, Ax_n) + p_1(Bx_n, Ax_{n+1})]}{4}\}$$

$$= \max\{p_1(y_{n+1}, y_n), p_1(y_{n-1}, y_n), p_1(y_n, y_{n+1}), \frac{[p_1(y_n, y_n) + p_1(y_{n-1}, y_{n+1})]}{4}\}$$

$$= \max\{p_1(y_{n-1}, y_n), p_1(y_n, y_{n+1}), p_1(y_n, y_n)\}$$

Case I. $M(x_n, x_{n+1}) = p_1(y_n, y_n)$. Since ψ is non decreasing so from (9), we have $\psi(p_1(y_n, y_{n+1})) \leq \psi(M(x_n, x_{n+1})) = \psi(p_1(y_n, y_n))$, which is a contradicts property P_2.

Case II. $M(x_n, x_{n+1}) = p_\lambda(y_n, y_{n+1})$. Since ψ is non decreasing so from (9), we have $\psi(p_1(y_n, y_{n+1})) \leq \psi(M(x_n, x_{n+1})) = \psi(p_1(y_n, y_{n+1}))$, which is a contradiction.

Case III. $M(x_n, x_{n+1}) = p_\lambda(y_{n-1}, y_n)$.

So, $\psi(p_1(y_n, y_{n+1})) \leq \psi(p_1(y_{n-1}, y_n)) \Rightarrow p_1(y_n, y_{n+1}) \leq p_1(y_{n-1}, y_n)$. Thus, $\{p_1(y_n, y_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. Let

$$\lim_{n\to\infty} p_1(y_n, y_{n+1}) = r \geq 0; r \in [0, \infty).$$

Now we show that $r = 0$. Taking $n \to \infty$ in (9); $\psi(r) \leq F(\psi(r), \phi(r))$ So, $\psi(r) = 0$ or $\phi(r) = 0$, which shows $r = 0$. Now we will show that $\{y_n\}$ is a Cauchy sequence in X; i.e., $\lim_{n\to\infty}\{y_n\} = 0$. In the contrary, we suppose $\{y_n\}$ is not a Cauchy sequence. Since,

$$\lim_{n\to\infty} p_1(y_n, y_{n+1}) = 0 \tag{10}$$

so, we can find $\epsilon > 0$ and subsequences $\{y_{m(k)}\}, \{y_{n(k)}\}$ of $\{y_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $p_1(y_{m(k)}, y_{n(k)}) \geq \epsilon$, i.e., $p_1(y_{m(k)}, y_{n(k)-1}) \geq \epsilon$. Now,

$$\epsilon \leq p_1(y_{m(k)}, y_{n(k)}) \geq p_1(y_{m(k)}, y_{n(k)-1}) + p_1(y_{m(k)-1}, y_{n(k)})$$

$$\leq \epsilon + p_1(y_{m(k)-1}, y_{n(k)})$$

Taking $k \to \infty$ and using (10) we get, $\lim_{n\to\infty} p_1(y_{m(k)}, y_{n(k)}) = \epsilon$. (11)

$$p_1(y_{n(k)-1}, y_{m(k)}) \leq p_1(y_{n(k)-1}, y_{m(k)}) + p_1(y_{n(k)}, y_{m(k)})$$

Taking $k \to \infty$ and using (10) and (11) we get, $\lim_{n\to\infty} p_1(y_{m(k)-1}, y_{n(k)}) = \epsilon$.

(12)

$$p_1(y_{m(k)-1}, y_{n(k)}) \leq p_1(y_{m(k)-1}, y_{n(k)}) + p_1(y_{m(k)}, y_{n(k)}).$$

Taking $k \to \infty$ and using (10) and (11) we get, $\lim_{n\to\infty} p_1(y_{m(k)-1}, y_{n(k)}) = \epsilon$.

(13)

$$p_1(y_{m(k)-1}, y_{n(k)}) \leq p_1(y_{m(k)-1}, y_{n(k)}) + p_1(y_{m(k)}, y_{n(k)})$$

$$\leq p_1(y_{m(k)-1}, y_{n(k)}) + p_1(y_{m(k)}, y_{n(k)}) + p_1(y_{n(k)-1}, y_{m(k)})$$

Taking $k \to \infty$ and using (10) and (11) we get, $\lim_{n\to\infty} p_1(y_{m(k)-1}, y_{n(k)}) = \epsilon$.

(14)

Now,

$$\psi(p_1(y_{m(k)}, y_{n(k)})) = \psi(p_1(Ax_{m(k)}, Ax_{n(k)})) \leq \psi(M(x_{m(k)}, x_{n(k)}))$$

$M(x_{m(k)}, x_{n(k)}) =$

$$\max\{p_1(Bx_{m(k)}, Bx_{n(k)}), p_1(Bx_{m(k)}, Ax_{m(k)}), p_1(Bx_{n(k)}, Ax_{n(k)}), \frac{[p_1(Bx_{n(k)}, Ax_{m(k)}) + p_1(Bx_{m(k)}, Ax_{n(k)})]}{4}\}$$

$$= \max\{p_1(y_{m(k)-1}, y_{n(k)-1}), p_1(y_{m(k)-1}, y_{m(k)}), p_1(y_{n(k)-1}, y_{n(k)}), \frac{[p_1(y_{n(k)-1}, y_{m(k)}) + p_1(y_{m(k)-1}, y_{n(k)})]}{4}\}$$

Using (10), (11), (12), (13) and (14) we get, $\lim_{n\to\infty} M(x_{m(k)}, x_{n(k)}) = \epsilon$ (15)

$$\psi(p_1(y_{m(k)}, y_{n(k)})) \leq F(\psi(M(x_{m(k)}, x_{n(k)})), \phi(M(x_{m(k)}, x_{n(k)})))$$

Taking $k \to \infty$ and using (11) and (15) we get $\psi(\in) \leq F(\psi(\in), \phi(\in))$ so, $\psi(\in) = 0$ or $\phi(\in) = 0$, which shows $\epsilon = 0$.

Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete so there exists $q \in X$ such that $\lim_{n\to\infty} y_n = q$.

Thus $\lim_{n\to\infty} p_1(y_n, y_m) = \lim_{n\to\infty} p_1(y_n, q) = p_1(q, q) = 0$,

and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_{n+1} = q; n = 0, 1, 2, \ldots.$

Suppose there exists a point $v \in X$ such that $q = Bv$. We claim $Av = Bv$. This shows $POC(A, B) \neq \phi$. If possible, suppose there exists a point $v \in X$ such that $q' = Av = Bv$. Now we show unique point of coincidence:

$$\psi(p_1(Ax_n, Av)) \leq F(\psi(M(x_n, v)), \phi(M(x_n, v))) \tag{16}$$

$M(x_n, v)$

$$= \max\{p_1(Bx_n, Bv), p_1(Bx_n, Ax_n), p_1(Bv, Av), \frac{[p_1(Bv, Ax_n) + p_1(Bx_n, Av)]}{4}\}$$

$$= \max\{p_\lambda(y_{n-1}, Bv), p_\lambda(y_{n-1}, y_n), p_\lambda(Bv, Av), \frac{[p_1(Bv, y_n) + p_1(y_{n-1}, Av)]}{4}\}$$

Taking $n \to \infty$, in (16) $Av = q = Bv$, $POC(A, B) \neq \phi$.

Since, (A, B) weakly compatible, so $Aq = Bq$. If possible, let $Aq = Bq = q'$ and $Aq = Bq = r$.

$$\psi(p_1(r, q')) = \psi(p_1(Aq, Aq)) \leq F(\psi(M(q, q)), \phi(M(q, q)))$$

$$M(r, q') = \max\{p_1(Bq, Bq), p_1(Bq, Aq), p_1(Bq, Aq),$$

$$\frac{[p_1(Bq', Ar) + p_1(Br, Aq')}{4}\} = p_1(r, q')$$

By definition of C-class function we have $q = Av$, hence $q = Bv = Av$. Hence, we can show (A, B) have unique point of coincidence. Since the pair is weakly compatible so by the Lemma 2.16, A and B have common unique fixed point.

Corollary 3.5. *Let A and B be two self maps defined on a complete non-Archimedean modular-like metric space (X, Θ_1) satisfying the following conditions* I *and* II *of Theorem* 3.3. *If the pair (A, B) is weakly compatible, then A and B have a common fixed point.*

Example 3.6. Let A and B be two self maps defined on a complete partial modular-like metric space (X, p_1) where $X = \mathbb{R}^+$. Let $Ax = 1$, $Bx = 2x - 1$.

Let $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{1}{r}t$, $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{1}{2r}t$ and $F : [0, \infty) \to [0, \infty) \to \mathbb{R}$ by $F(S, T) = rS$, $0 < r < 1$. From Example 3.3 and Theorem 3.4we get, $\{1\}$ is the unique fixed point of A and B.

4. Application

Consider the set of Volterra type integral equations: ([14], [17])

(i) $v(t) = q(t) + \int_0^t G(t, s, v(t))ds$, where $t \in [0, k] = I \subset \mathbb{R}$ and $G : [0, k] \times [0, k] \times \mathbb{R} \to \mathbb{R}; i = \{1, 2\}$ and $q : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $C(I, \mathbb{R})$ be the set of real continuous functions defined on I and $A, B : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ self-mappings defined by

(ii) $Av(t) = q(t) + \int_0^t G(t, s, v(t))ds, \forall v \in C(I, \mathbb{R}), t \in I$. Clearly $v(t)$ is a solution of (i) if and only if it is a common fixed point of A and B.

Theorem 4.1. *Let*

I. *For any* $v \in C(I, \mathbb{R})$, *there exist* $r \in C(I, \mathbb{R})$ *such that* $Av = Br$.

II. $ABv(t) = BAv(t)$ *whenever* $Av(t) = Bv(t)$ *for all* $t \in I, v \in C(I, \mathbb{R})$.

III. *There exists a continuous function* $b : I \times I \to \mathbb{R}^+$ *such that for all* $t, s \in I$ *and* $v, z \in C(I, \mathbb{R})$

$$| G(t, s, v(G)) - G(t, s, z(s)) | \leq f(t, s) | Bv(s) - Bz(s) |.$$

IV. $\sup_{t \in [0, k]} \int_0^t f(t, s)ds \leq 1$.

Then the system (i) *of integral equations has solutions* $v^* \in C(I, \mathbb{R})$.

Proof. For $x \in X = C(I, \mathbb{R})$.

Define $\| x \|_\infty \max_{t \in [0, k]} \{| x(t) |\}$, is taken arbitrarily the modular-like metric induced by the norm is $\Theta_\lambda(x, y) = \begin{cases} \max_{t \in [0, k]} \{e^{-\lambda} | x(t) - y(t) |\}; x \neq y \\ b; x = y \end{cases}$ for all $b \geq 0, \lambda > 0$ $x, y \in X$.

From (I), it is clear that $A(C(I, \mathbb{R})) \subset B(C(I, \mathbb{R}))$.

From (II), the pair (A, B) are weakly compatible.

$$| Av(t) - Az(t) | \leq \int_0^t | G(t, s, v(s)) - G(t, s, z(s)) | ds$$

$$\leq \int_0^t f(t, s) | Bv(s) - Bz(s) | ds$$

$$\leq | Bv(s) - Bz(s) |$$

$$\Rightarrow \max_{t \in [0, k]} \{e^{-\lambda} | Av(t) - Az(t) |\} \leq \max_{t \in [0, k]} \{e^{-\lambda} | Bv(t) - Bz(s) |\}$$

$$\Rightarrow \Theta_\lambda(Av(t), Az(t)) \leq \Theta_\lambda(Bv(t), Bz(t)) \leq M(v(t), z(t))$$

Let $\psi : [0, \infty) \to [0, \infty)$ by $\psi(T) = \frac{1}{q} T; \phi : [0, \infty) \to [0, \infty)$ by $\phi(T) = \frac{1}{2q} T$ and $F : [0, \infty) \to [0, \infty) \to \mathbb{R}^+$ by $F(S, T) = qS$, $0 < r < 1$. Then (ψ, ϕ, F) is monotone.

$$\Theta_\lambda(Av(t), Az(t)) \leq F(\psi(M(v, z)), \phi(M(v, z)))$$

where

$$M(v, z) = \max \{\Theta_\lambda(Bv, Bz), \Theta_\lambda(Bv, Av), \Theta_\lambda(Az, Bz)\}$$

All the conditions of Corollary 3.2 are satisfied. Therefore A and B have a common unique fixed point $v^* \in C(I, \mathbb{R})$.

Conclusions

Modular-like metric spaces and dislocated modular metric spaces are exactly same. Partial modular-like metric spaces cannot be equivalent to dislocated modular metric spaces unless we drop small self distance axiom. Because of the restriction of self distance axiom dislocated modular metric spaces generalized partial modular metric spaces and partial modular-like metric spaces. All the results are also true in the sense of non-Archimedean modular.

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