Advances and Applications in Mathematical Sciences
Volume 21, Issue 12, October 2022, Pages 6703-6710
© 2022 Mili Publications, India

TRIBONACCI *r*-GRACEFUL LABELING OF SOME TREE RELATED GRAPHS

K. SUNITHA and M. SHERIBA

Assistant Professor
Department of Mathematics
Scott Christian College (Autonomous)
Nagercoil-629003, India
E-mail: ksunithasam@gmail.com

Part Time Research Scholar
Department of Mathematics
Scott Christian College (Autonomous)
Nagercoil-629003, India
Affiliated to Manonmaniam Sundaranar University
Tirunelveli-627012, India
E-mail: sheribajerin@gmail.com

Abstract

Let r be any natural number. An injective function $\phi : V(G) \to \{0, 1, 2, \ldots, T_{q+r-1}\}$, where T_{q+r-1} is the $(q+r-1)^{th}$ Tribonacci number in the Tribonacci sequence is said to be Tribonacci r-graceful if the induced edge labeling $\phi^{*} : E(G) \to \{T_r, T_{r+1}, \ldots, T_{q+r-1}\}$ such that $\phi^{*}(uv) = |\phi(u) - \phi(v)|$ is bijective. If a graph G admits Tribonacci r-graceful labeling, then G is called a Tribonaccir-graceful graph. A graph G is said to be Tribonacci arbitrarily graceful if it is Tribonacci r-graceful for all r. In this paper we investigate the Twig graph T_n, the regular caterpillar graph $P_m \Theta nK_1$, the Bistar graph $B_{m,n}$ and the Subdivision of the bistar graph $S(B_{m,n})$ are Tribonacci r-graceful.

2020 Mathematics Subject Classification: 05C78.

Keywords: Tribonacci sequence, Tribonacci r-graceful labeling, r-graceful graph, Twig graph, regular caterpillar graph, Bistar graph and Subdivision of the bistar graph.

Received February 3, 2022; Accepted March 2, 2022

1. Motivation and Main Results

Graphs considered throughout this paper are finite, simple, undirected and nontrivial. Labeling of graph is the assignment of values to vertices and edges or both subject to certain conditions. In 1967, Rosa [6] introduced the concept of graceful labeling. In 1982, Slater [4] introduced the concept of k-graceful labeling of graphs. Let G be a simple graph with p vertices and q edges. Let k be any natural number. Define an injective mapping $\phi : V(G) \to \{0, 1, 2, ..., q + k - 1\}$ that induces bijective mapping $\phi^* : E(G) \to \{k, k + 1, ..., q + k - 1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$ for all $uv \in E(G)$ and $u, v \in V(G)$, then ϕ is called k-graceful labeling while ϕ^* is called an induced edge k-graceful labeling and the graph G is called k-graceful graph. Graphs that are k-graceful for all k are sometimes called arbitrarily graceful. In this sequel, we introduce a Tribonacci r-graceful labeling. We follow D. B. West [10] and J. A. Gallian [2], for standard terminology and notations.

Definition 1.1. Let r be any natural number. An injective function $\phi : V(G) \to \{0, 1, 2, ..., T_{q+r-1}\}$, where T_{q+r-1} is the $(q + r - 1)^{th}$ Tribonacci number in the Tribonacci sequence is said to be Tribonacci r-graceful if the induced edge labeling $\phi^* : E(G) \to \{T_r, T_{r+1}, ..., T_{q+r-1}\}$ such that $\phi^*(uv) = |\phi(u) - \phi(v)|$ is bijective. If a graph G admits Tribonacci r-graceful labeling, then G is called a Tribonacci r-graceful graph. A graph G is said to be Tribonacci arbitrarily graceful if it is Tribonacci r-graceful for all r.

Definition 1.2 [8]**.** A Twig graph T_n is obtained from P_n by attaching exactly two pendent edges to each internal vertices of P_n.

Definition 1.3 [4]**.** A regular caterpillar graph $P_m \Theta nK_1$ is obtained from the path P_m by joining nK_1 vertices to each vertices of the path P_m.

Definition 1.4 [9]**.** Bistar $B_{m,n}$ is the graph obtained from K_2 by attaching m pendent edges to one end of K_2 and n pendent edges to the other end of K_2.

Definition 1.5 [10]**.** Let u, v be the central vertices of the bistar graph $B_{m,n}$. Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be the vertices joined with u and v respectively. The subdivision of bistar graph $S(B_{m,n})$ is obtained from the bistar graph $B_{m,n}$ by subdividing the edges uu_i $(1 \leq i \leq m)$ and vv_i $(1 \leq i \leq n)$ with new vertices $u_i^1 (1 \leq i \leq m)$ and $v_i^1 (1 \leq i \leq n)$ respectively.

Our main result can be stated as the following theorems.

Theorem 1. *The Twig graph T_n is Tribonacci r-graceful for all $n \geq 3, r \geq 2$.*

Proof. Let $v_i, 1 \leq i \leq n$ be the vertices of the path P_n and let $u_i^j, 1 \leq i \leq n, j = 1, 2$ be the vertices which are attached to the internal vertices of the path P_n. The resultant graph is T_n with $V(T_n) = \{\{v_i / 1 \leq i \leq n\} \cup \{u_i^1, u_i^2 / 2 \leq i \leq n-1\}\}$ and $E(T_n) = \{\{v_i u_i^1, v_i u_i^2 / 2 \leq i \leq n-1\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\}\}$ such that $|V(T_n)| = p = 3n - 4$ and $|E(T_n)| = q = 3n - 5$.

Define a function $\phi : V(T_n) \rightarrow \{0, 1, 2, \ldots, T_{q+r-1}\}$ by

$$\phi(v_1) = 0,\ \phi(v_i) = \phi(v_{i-1}) + (-1)^i T_{q+r-1-3(i-2)},\ 2 \leq i \leq n,\ r \geq 2$$

$$\phi(u_i^j) = \phi(v_i) - T_{q+r-1-(3i-j-3)},\ 2 \leq i \leq n,\ j = 1, 2,\ r \geq 2$$

Thus ϕ admits Tribonacci r-graceful labeling.

Hence the Twig graph T_n is Tribonacci r-graceful for all $n \geq 3,\ r \geq 2$.

Example 1. The Tribonacci 3-graceful labeling of T_5 is given in Figure 1

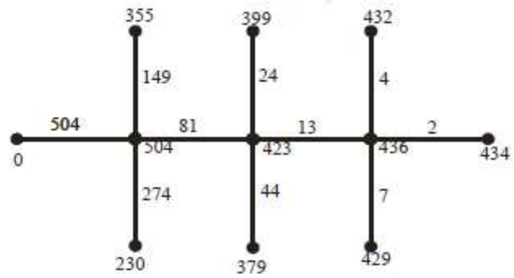

Figure 1.

Theorem 2. *The regular caterpillar graph* $P_m \Theta\, nK_1$ *is Tribonacci r-graceful for all* $m,\ r \geq 2,\ n \geq 1$.

Proof. Let $v_i, 1 \leq i \leq m$ be the vertices of the path P_m and let $v_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$ be the vertices attached to each vertices of the path P_m. The resultant graph is $P_m \;\Theta\; nK_1$ whose vertex set is $V[P_m \;\Theta\; nK_1] = \{\{v_i / 1 \leq i \leq m\} \cup \{v_{ij}/1 \leq i \leq m, 1 \leq j \leq n\}\}$ and edge set is $E[P_m \;\Theta\; nK_1] = \{\{v_i v_{i+1}/1 \leq i \leq m-1\} \cup \{v_i v_{ij}/1 \leq i \leq m, 1 \leq j \leq n\}\}$ such that $|\, V[P_m \Theta\; nK_1]\,| = p = m + mn$ and $|\, E[P_m \Theta nK_1]\,| = q = m + mn - 1$.

Define a function $\phi : V[P_m \Theta\; nK_1] \to \{0, 1, 2, \ldots, T_{q+r-1}\}$ by

$$\phi(v_1) = 0,\ \phi(v_i) = \phi(v_{i-1}) + (-1)^i T_{q+r-1-(i-2)},\ 2 \leq i \leq m,\ r \geq 2$$

$$\phi(v_{1j}) = T_{q+r-1-j-(m-2)},\ 1 \leq j \leq n,\ r \geq 2$$

$$\phi(v_{ij}) = \phi(v_i) - T_{q+r-1-(i-1)n-j-(m-2)},\ 2 \leq i \leq m,\ 1 \leq j \leq n,\ r \geq 2.$$

Thus ϕ admits Tribonacci r-graceful labeling.

Hence the regular caterpillar graph $P_m \,\Theta\, nK_1$ are Tribonacci r-graceful for all $m,\ r \geq 2,\ n \geq 1$.

Example 2. The Tribonacci 2-graceful labeling of $P_4 \,\Theta\, 3K_1$ is given in Figure 2

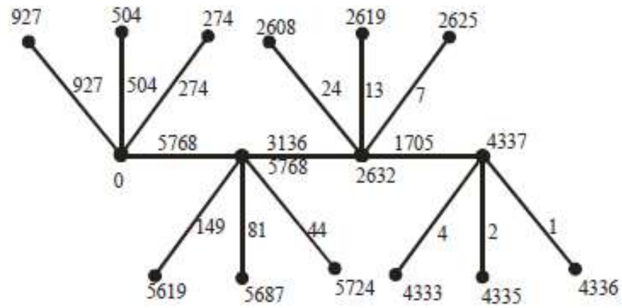

Figure 2.

Theorem 3. *The Bistar graph* $B_{m,n}$ *is Tribonacci r-graceful for all* $m, n \geq 2,\ r \geq 2$.

Proof. Let u, v be the vertices of K_2 and let $u_i, 1 \leq i \leq m$ be the m vertices attached to one end of K_2 and $v_j, 1 \leq j \leq n$ be the n vertices attached to the other end of K_2. The resultant graph is $B_{m,n}$ whose vertex set is $V(B_{m,n}) = \{u_i, v_j, u, v / 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set is $E(B_{m,n}) = \{\{uu_i / 1 \leq i \leq m\} \cup \{vv_j / 1 \leq j \leq n\} \cup \{uv\}\}$ such that $|V(B_{m,n})| = p = m + n + 2$ and $|E(B_{m,n})| = q = m + n + 1$.

Define a function $\phi : V(B_{m,n}) \to \{0, 1, 2, \ldots, T_{q+r-1}\}$ by

$\phi(u) = 0,\ \phi(v) = T_r,\ \phi(v_j) = \phi(v) + T_{r+j},\ 1 \leq j \leq n,\ r \geq 2$

$\phi(u_i) = T_{q+r-1-(i-1)},\ 1 \leq i \leq m,\ r \geq 2$

Thus ϕ admits Tribonacci r-graceful labeling.

Hence $B_{m,n}$ is Tribonacci r-graceful for all $m, n \geq 2$ and $r \geq 2$.

Example 3. The Tribonacci 3-graceful labeling of Bistar graph $B_{5,5}$ is given in Figure 3

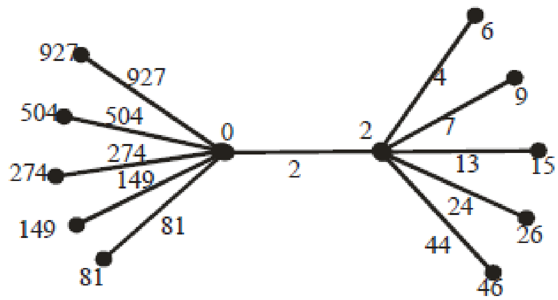

Figure 3.

Theorem 4. *The subdivision of the bistar graph* $S(B_{m,n})$ *is arbitrarily graceful for all* $m, n \geq 1$.

Proof. Let u, v be the central vertices of the bistar graph $B_{m,n}$ and let $u_i, 1 \leq i \leq m$ and $v_i, 1 \leq i \leq n$ be the vertices joined with u and v respectively. Let $u_i^1, 1 \leq i \leq m$ and $v_i^1, 1 \leq i \leq n$ be the new vertices obtained by subdividing the edges $uu_i, 1 \leq i \leq m$ and $vv_i, 1 \leq i \leq n$ respectively. The resulting graph is $S(B_{m,n})$ whose vertex set is $V[S(B_{m,n})] = \{\{u_i, u_i^1 / 1 \leq i \leq m\} \cup \{v_i, v_i^1 / 1 \leq i \leq n\} \cup \{u, v\}\}$ and edge set is $E[S(B_{m,n})] = \{\{u_i^1 u_i, uu_i^1 / 1 \leq i \leq m\} \cup \{uv\} \cup \{vv_i^1, v_i^1 v_i / 1 \leq i \leq n\}\}$ such that $|V[S(B_{m,n})]| = p = 2(m+n+1)$ and $|E[S(B_{m,n})]| = q = 2(m+n)+1$.

Case 1. *For* $m, n \geq 1$ and $r = 1$.

Define a function $\phi : V[S(B_{m,n})] \to \{0, 1, 2, \ldots, T_{q+r-1}\}$ by

$\phi(u) = 0, \phi(v) = T_{q+r-1}, r \geq 2, \phi(u_i^1) = T_{r+i}, 1 \leq i \leq m, r \geq 2$

$\phi(u_i) = \phi(u_i^1) + T_{2m+r+1-i}, 1 \leq i \leq m, r \geq 2, \phi(v_1^1) = \phi(v) - T_r, r \geq 2$

$\phi(v_i^1) = \phi(v) - T_{q+r-i}, 2 \leq i \leq n, r \geq 2$

$\phi(v_i) = \phi(v_i^1) - T_{2m+i+1}, 1 \leq i \leq n$

Case 2. For $m, n \geq 1$ and $r \geq 2$.

Define a function $\phi : V[S(B_{m,n})] \to \{0, 1, 2, \ldots, T_{q+r-1}\}$ by

$\phi(u) = 0$, $\phi(v) = T_{q+r-1}$, $r \geq 2$, $\phi(u_i^1) = T_{r+i-1}$, $1 \leq i \leq m$, $r \geq 2$

$\phi(u_i) = \phi(u_i^1) + T_{2m+r-i}$, $1 \leq i \leq m$, $r \geq 2$, $\phi(v_i^1) = \phi(v)$

$-T_{q+r-1-i}$, $1 \leq i \leq n$, $r \geq 2$

$\phi(v_i) = \phi(v_i^1) - T_{2m+i+2}$, $1 \leq i \leq n$

Thus ϕ admits Tribonacci r-graceful labeling for all.

Hence the subdivision of the bistar graph $S(B_{m,n})$ is arbitrarily graceful for all $m, n \geq 1$.

Example 4. The Tribonacci 1-graceful labeling of $S(B_{2,3})$ is given in Figure 4

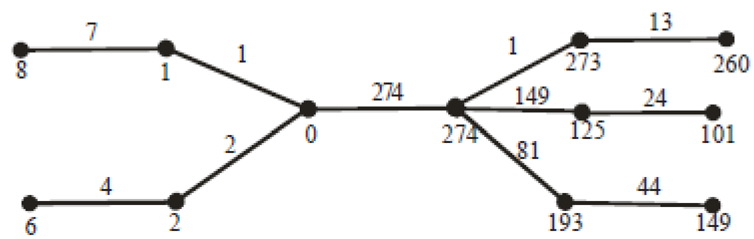

Figure 4.

The Tribonacci 3-graceful labeling of $S(B_{3,3})$ is given in Figure 5.

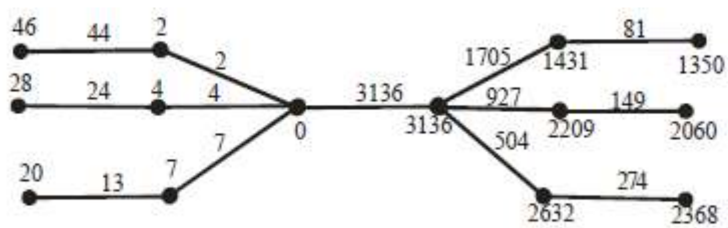

Figure 5.

2. Remark

Finally we list some remarks on our main results and closely related things.

Remark 2.1. The Tribonacci sequence is obtained as follows:

$T_0 = 0, T_1 = T_2 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}\ \forall\ n \geq 3$

i.e., {0, 1, 1, 2, 4, 7, 13, 24, 44, 81, ...} is the Tribonacci sequence

Conclusion

In this paper, we find that the Twig graph T_n, the regular caterpillar graph $P_m \Theta nK_1$, the Bistar graph $B_{m,n}$, the Subdivision of the bistar graph $S(B_{m,n})$ are Tribonacci r-graceful. In future, we investigate that Tribonacci r-odd graceful labeling of some graphs.

References

[1] W. David Bange and Anthoney E. Barakaukas, Fibonacci Graceful Graphs, University of Wisconsin-La Crosse, La Crosse, WI 53601.

[2] J. A. Gallian, A Dynamic Survey of Graph Labeling, Article number #DS6, Electronic Journal of Combinatorics 1 (2013).

[3] N. Murugesan and R. Uma, Super vertex Gracefulness of Some Special Graphs, IQSR Journal of Mathematics ver. 11(3) (2015), 07-15.

[4] P. J. Slater, On k-graceful graphs, In: Proc. of the 13th South Eastern Conference on Combinatorics, Graph Theory and Computing (1982), 53-57.

[5] P. Prathan and Kamesh Kumar, On k-Graceful Labeling of some graphs, Journal of Applied Mathematics 34(1-2) (2016), 09-17.

[6] A. Rosa, On certain valuations of the vertices of a graph, in: Theorie des Graphes, Jonrnees Internationales d'etucies, Rome 1966 (Dunod, Paris, 1967) 349-355.

[7] Teena Liza John and Mathew T. K. Varkey, P^* Graceful graphs, International Journal of Mathmatics and Combinatorics 4 (2012), (97-102).

[8] K. Thirugnanasambandan and G. Chitra, Fibonacci Mean Anti-Magic Labeling of graphs, International Journal of Computer Applications 134(15) (2016), (0975-8887).

[9] R. Uma and D. Amuthavalli, Fibonacci graceful labelling of some star related graphs, International Journal of Computer Applications 134(15) (2016), (0975-8887).

[10] D. B. West, Introduction to Graph Theory, Prentice-Hall of India, New Delhi, (2001).