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ON COMPLEX DYNAMICS OF SOME FOURTH ORDER METHODS FOR COMPUTING MULTIPLE ROOTS

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Abstract

Numerous iterative methods are available in literature for computing multiple roots of nonlinear equations. These methods are categorized by their order, informational efficiency and efficiency index. Another important criterion for comparing the methods is to study their complex dynamics using the graphical tool, namely basin of attraction. In this paper, we consider several methods of order four and characterize their basins of attraction by applying them on different polynomial functions.

1. Introduction

A root α is said to be of multiplicity μ of a nonlinear equation $f(t) = 0,$ if $f^{(m)}(\alpha) = 0,\ m = 0, 1, 2, \cdots, \mu - 1$ and $f^{(\mu)}(\alpha) \neq 0.$

A plethora of higher order iterative methods, independent or based on the Newton's method (see [11])

$$t_{k+1} = t_k - \mu \frac{f(t_k)}{f'(t_k)}, \tag{1}$$

have been derived and analyzed in literature [1-5, 7-10, 12-15, 128, 20]. The methods are categorized by their order of convergence (say, q), and the number of function and derivative information (say, p) required per step. To check the effectiveness of such methods, there are two efficiency measures

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(see [16]) defined as $I = \frac{q}{p}$ (informational efficiency) and $E = q^{\frac{1}{p}}$ (efficiency index). Besides these the other measure, introduced recently, is to study complex dynamics of iterative methods. Kung and Traub [6] in 1974 introduced the concept of optimality criteria of convergence order. According to their hypothesis multipoint methods without memory requiring $p+1$ function evaluations may attain order of convergence at most 2^p. The iterative schemes satisfying this criteria are usually called optimal methods (see, for example, [6]). An optimal method of order $q = 2$ is the well known Newton's method defined by (1).

In this paper, we study the complex dynamics of some existing optimal fourth methods for computing multiple roots. For example, we consider the fourth order methods proposed by Li-Liao-Cheng [7], Li-Cheng-Neta [8], Sharma-Sharma [13], Zhou-Chen-Song [20], Soleymani-Babajee-Lotfi [15] and Kansal-Kanwar-Bhatia [5]. Rest of the paper is organized as follows. In Section 2, the optimal fourth order methods are introduced. In Section 3, the complex dynamics in the form of basins of attraction is studied. Section 4 contains the concluding remarks.

2. Methods for Relative Examination

Let us tabulate the optimal fourth-order methods that are to be examined. These are the methods by Li-Liao-Cheng [7], Li-Cheng-Neta [8], Sharma-Sharma [13], Zhou-Chen-Song [20], Soleymani-Babajee-Lotfi [15] and Kansal-Kanwar-Bhatia [5].

Li-Liao-Cheng method (LLC):

$$v_k = t_k - \frac{2\mu}{\mu+2}\frac{f(t_k)}{f'(t_k)},$$

$$t_{k+1} = t_k - \frac{\mu(\mu-2)\left(\frac{\mu}{\mu+2}\right)^{-\mu} f'(v_k) - \mu^2 f'(t_k)}{f'(t_k) - \left(\frac{\mu}{\mu+2}\right)^{-\mu} f'(v_k)} \frac{f'(t_k)}{2f'(t_k)}.$$

Li-Cheng-Neta method (LCN):

$$v_k = t_k - \frac{2\mu}{\mu+2}\frac{f(t_k)}{f'(t_k)},$$

$$t_{k+1} = t_k - \alpha_1 \frac{f(t_k)}{2f'(v_k)} - \frac{f(t_k)}{\alpha_2 f'(t_k) + \alpha_3 f'(v_k)},$$

where

$$\alpha_1 = -\frac{1}{2}\frac{(\frac{\mu}{\mu+2})^{\mu}\mu(\mu^4+4\mu^3-16\mu-16)}{\mu^3-4\mu+8},$$

$$\alpha_2 = -\frac{1}{2}\frac{(\mu^3-4\mu+8)^2}{\mu(\mu^4+4\mu^3-4\mu^2-16\mu-16)(\mu^2-4\mu+4)},$$

$$\alpha_3 = -\frac{\mu^2(\mu^3-4\mu+8)}{(\frac{\mu}{\mu+2})^{\mu}(\mu^4+4\mu^3-4\mu^2-16\mu+16)(\mu^2+2\mu-4)}.$$

Sharma-Sharma method (SS):

$$v_k = t_k - \frac{2\mu}{\mu+2}\frac{f(t_k)}{f'(t_k)},$$

$$t_{k+1} = t_k - \frac{\mu}{8}[(\mu^3-4\mu+8) - (\mu+2)^2(\frac{\mu}{\mu+2})^{\mu}\frac{f'(t_k)}{f'(v_k)}$$

$$\times(2(\mu-1) - (\mu+2)(\frac{\mu}{\mu+2})^{\mu}\frac{f'(t_k)}{f'(v_k)})]\frac{f'(t_k)}{f'(t_k)}.$$

Zhou-Chen-song method (ZCS):

$$v_k = t_k - \frac{2\mu}{\mu+2}\frac{f(t_k)}{f'(t_k)},$$

$$t_{k+1} = t_k - \frac{\mu}{8}[\mu^3(\frac{\mu+2}{\mu})^{2\mu}(\frac{f'(t_k)}{f'(v_k)})^2 - 2\mu^2(\mu+3)(\frac{\mu+2}{\mu})^{\mu}\frac{f'(v_k)}{f'(t_k)}$$

$$+ (\mu^3+6\mu^2+8\mu+8)]\frac{f'(t_k)}{f'(t_k)}.$$

Soleymani-Babajee-Lotfi method (SBL):

$$v_k = t_k - \frac{2\mu}{\mu+2}\frac{f(t_k)}{f'(t_k)},$$

$$u_{k+1} = t_k - \frac{f'(u_k)f(t_k)}{q_1(f'(u_k))^2 + q_2 f'(u_k)f'(t_k) + q_3(f'(u_k))^2},$$

where

$$q_1 = \frac{1}{16}\mu^{3-\mu}(2+\mu)^{\mu}$$

$$q_2 = \frac{8-\mu(2+\mu)(\mu^2-2)}{8\mu},$$

$$q_3 = \frac{1}{16}(\mu+2)\mu^{\mu-1}(2+\mu)^{3-\mu}.$$

Kansal-Kanwar-Bhatia method (KKB):

$$v_k = t_k - \frac{2\mu}{\mu+2}\frac{f(t_k)}{f'(t_k)},$$

$$u_{k+1} = t_k - \frac{\mu}{4}f(t_k)(1 + \frac{\mu^4 p^{-2\mu}(p^{-1\mu} - \frac{f'(v_k)}{f'(t_k)})^2(p^{\mu}-1)}{8(2p^{\mu} + \mu(p^{\mu}-1))})$$

$$\times(\frac{4-2\mu+\mu^2(p^{-\mu}-1)}{f'(t_k)}) - \frac{p^{-\mu}(2p^{\mu}+\mu(p^{\mu}-1))^2}{f'(t_k)-f'(v_k)}),$$

where $p = \frac{\mu}{\mu+2}$.

3. Basins of Attraction

In this section, we present complex geometry of the above considered methods with a tool, namely basin of attraction, by applying the methods to some complex polynomials $F(z)$. Basin of attraction of the root is an useful geometrical tool for comparing convergence regions of the iterative methods [17, 19]. To start with, let us recall some basic ideas concerning with this graphical tool.

Let $R : \mathbb{C} \to \mathbb{C}$ be a rational mapping on the Riemann sphere. We define orbit of a point $z_0 \in \mathbb{C}$ is defined as the set $\{z_0, R(z_0), R^2(z_0), \cdots, R^n(z_0), \cdots\}$. A point $z_0 \in \mathbb{C}$ is a fixed point of the rational function R if it satisfies the equation $R(z_0) = z_0$. A point z_0 is said to be periodic with period $m > 1$ if $R^m(z_0) = z_0$, where m is the smallest such integer. A point z_0 is called attracting if $|R'(z_0)| < 1$, repelling if $|R'(z_0)| > 1$ and neutral if $|R'(z_0)| = 1$. Moreover, if $|R'(z_0)| = 0$, the fixed point is super attracting. Let z_f^* be an attracting fixed point of the rational function R. The basin of attraction of the fixed point z_f^* is defined as

$$A(z_f^*) = \{z_0 \in \mathbb{C} : R^n(z_0) \to z_f^*,\ n \to \infty\}.$$

The set of points whose orbits tend to an attracting fixed point z_f^* is called the Fatou set. The complementary set, called the Julia set, is the closure of the set consisting of repelling fixed points, which establishes the borders between the basins of attraction of the roots. Attraction basins allow us to assess those starting points which converge to the concerned root of a polynomial when we apply an iterative method, so we can visualize which points are good options as starting points and which are not.

We select z_0 as the initial point belonging to D, where D is a rectangular region in $\mathbb{C}$ containing all the roots of the equation $f(z) = 0$. An iterative method beginning at a point $z_0 \in D$ can converge to the zero of the function $F(z)$ or diverge. In order to assess the basins we consider 10-3 as the stopping criterion for convergence up to maximum of 25 iterations. If this tolerance is not achieved in the required iterations, the procedure is dismissed with the result showing the divergence of the iteration function started from z_0. While drawing the basins the following criterion is adopted: A color is allotted to every initial guess z_0 in the attraction basin of a zero. If the iterative formula begins at the point z_0 converges then it forms the basins of attraction with that assigned color and if the formula fails to converge in the required number of iteration then it is painted with black color.

To view the complex geometry, the above considered methods are applied to the following five problems:

Example 1. Firstly, consider the function $F_1(z) = (z^2 - 1)^2$ having two roots $\{-1, 1\}$ with multiplicity $\mu = 2$. Basins of attraction assessed by the methods are shown in Figure 1 ((i)-(vi)). A color is allotted to each basin of attraction of the concerned root. In particular, we assign red and green colors corresponding to the roots −1 and 1.

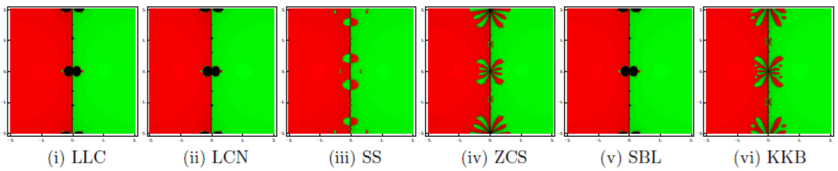

Figure 1. Basins of attraction for methods in polynomial $F_1(z)$.

Example 2. In the next trial, let us consider the polynomial $F_2(z) = (z^3 + z)^2$ having three roots $\{0, -i, i\}$ with multiplicity $\mu = 2$. The result for the basins of attraction are depicted in Figure 2 ((i)-(vi)). To distinguish the basins, the red, green and blue colors have been allotted to the roots i, $-i$ and 0, respectively.

Example 3. Now, we consider the polynomial $F_3(z) = (z^4 - \frac{9}{4}z^2 + \frac{1}{2})^3$ having four roots $\{\frac{1}{2}, -\frac{1}{2}\sqrt{2}, -\sqrt{2}\}$ with multiplicity $\mu = 3$. The graphics of basins of attraction obtained by the methods are shown in Figure 3 ((i)-(vi)). A color is allotted to each basin of attraction of a root. The colors chosen are red, green, blue and yellow corresponding to roots $\sqrt{2}, -\sqrt{2}, -\frac{1}{2}.$ and $\frac{1}{2}$.

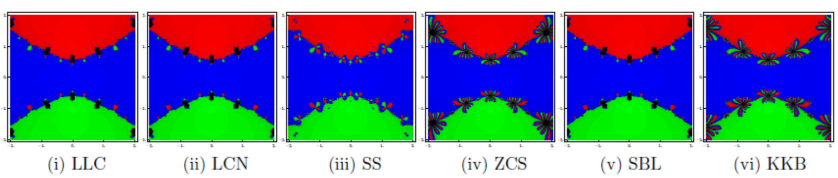

Figure 2. Basins of attraction for methods in polynomial $F_2(z)$.

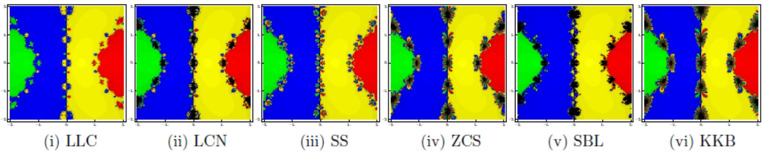

Figure 3. Basins of attraction for methods in polynomial $F_3(z)$.

Example 4. Consider the fifth degree polynomial $F_4(z) = (z^5 - z)^4$ which has five roots $\{0, 1, -1, i, -i\}$ of multiplicity $\mu = 4$. The corresponding basins of attraction analyzed by the methods are shown in Figure 4 ((i)-(vi)). The basins so generated are distinguished by the colors red, purple, blue, yellow and green corresponding to basins of the roots $i, -i, -1, 1$ and 0.

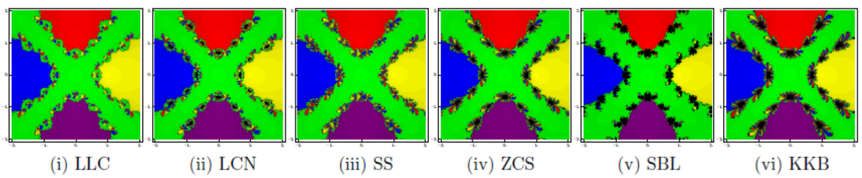

Figure 4. Basins of attraction for methods in polynomial $F_4(z)$.

Example 5. Lastly, consider the polynomial $F_5(z) = (z^6 + \frac{9}{4}z^4 - z^2 - \frac{9}{4})^5$ having six roots $\{1, -1, i - i, \frac{3}{2}i, -\frac{3}{2}i\}$ with multiplicity $\mu = 4$. The corresponding basins of attraction are given in Figure 5 ((i)-(vi)). The colors, assigned to characterize the basins of attraction, are sky-blue, green, yellow, blue, red and purple for the roots $1, -1, i, -i\frac{3}{2}i$ and $-\frac{3}{2}i$, respectively.

Along with basins of attraction we also provide some other useful information of the performance of methods in Tables 1-5, which include:

- OC: Order of convergence.

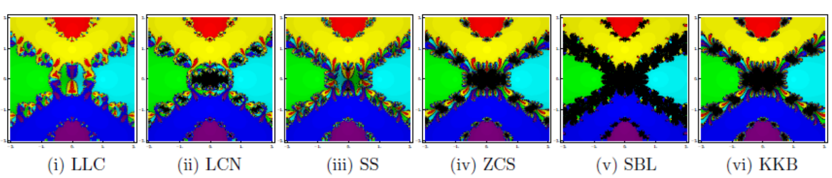

Figure 5. Basins of attraction for methods in polynomial $F_5(z)$.

- E-Time: Elapsed CPU time (in seconds) consumed by a method to draw the basins.

- IP: Mean of iterations, measured in iterations/point.

- NC: Nonconvergent points, as a percentage of the total number of starting points evaluated.

- IC: Mean of iterations, measured in iterations/(point–nonconvergent points).

From the above graphics we can observe that, in general, the methods by Li-Liao-Cheng (LLC) and Sharma-Sharma (SS) perform better, since in the examples either they have no divergent points or very few divergent points. This feature can also be verified by the numerical results shown in the Tables 1 – 5. In example 1, Li-Liao-Cheng (LLC), Li-Cheng-Neta (LCN) and Soleymani-Babajee-Lotfi (SBL) methods show divergent nature at some points as indicated by black spots in the pictures. In example 2 all methods have some divergent points except SS. In examples 3, 4 and 5, the methods by Li-Liao-Cheng (LLC) and Sharma-Sharma (SS) perform better than rest of the methods. The elapsed CPU time, to generate the graphics, is small in LLC than rest of the methods, which indicates less complexity of this method.

Table 1. Performance of methods for example 1.

Methods	LLC	LCN	SS	ZCS	SBL	KKB
OC	4.00	4.00	4.00	4.00	4.00	4.00
E-Time	44.34	61.62	46.21	58.59	50.79	61.56
IP	3.06	3.06	2.84	3.50	3.06	3.35
NC	1.98	1.98	0.40	0.51	1.98	0.45
IC	2.62	2.62	2.75	3.39	2.62	3.25

Table 2. Performance of methods for example 2.

Methods	LLC	LCN	SS	ZCS	SBL	KKB
OC	4.00	4.00	4.00	4.00	4.00	4.00
E-Time	62.37	82.12	65.19	91.85	72.48	95.44
IP	3.74	3.74	3.49	4.85	3.74	4.64
NC	2.13	2.13	0.00	1.33	2.13	0.81
IC	3.28	3.28	3.49	4.57	3.28	4.47

Table 3. Performance of methods for example 3.

Methods	LLC	LCN	SS	ZCS	SBL	KKB
OC	4.00	4.00	4.00	4.00	4.00	4.00
E-Time	73.69	112.21	87.91	116.66	117.30	123.96
IP	3.57	4.61	3.91	4.96	5.13	5.02
NC	0.40	5.21	0.40	2.69	7.42	2.87
IC	3.49	3.49	3.83	4.41	3.54	4.43

Table 4. Performance of methods for example 4.

Methods	LLC	LCN	SS	ZCS	SBL	KKB
OC	4.00	4.00	4.00	4.00	4.00	4.00
E-Time	69.67	99.53	85.71	102.71	111.62	120.68
IP	4.02	4.51	4.54	5.25	5.83	5.70
NC	0.00	1.96	0.15	2.75	8.90	3.98
IC	4.02	4.10	4.51	4.69	3.96	4.90

Table 5. Performance of methods for example 5.

Methods	LLC	LCN	SS	ZCS	SBL	KKB
OC	4.00	4.00	4.00	4.00	4.00	4.00
E-Time	115.39	161.99	162.51	207.81	269.07	232.63
IP	4.84	6.12	6.45	7.55	10.60	8.02
NC	0.08	7.24	1.21	10.65	31.68	13.15
IC	4.84	4.65	6.03	5.48	3.93	5.45

4. Conclusions

We have analyzed the basins of attraction of multiple zeros by employing some existing optimal fourth order techniques for computing multiple zeros of functions. These graphics are very useful tool to observe the behavior and suitability of any method. If we choose an initial guess t_0 in a domain where different basins of the roots meet each other, it is uncertain to predict which root is going to be reached by the iterative method that begins at t_0. Thus, it is not advisable to start the methods with the initial guess lying in such a domain. Also, black zones and the zones with different colors (i.e. borders) are not suitable to choose the initial guess t_0 when we want to achieve a particular root. The most attractive graphics appear when we have very intricate frontiers of the basins. Such graphics belong to the cases where the iteration is failing with respect to the initial guess. We conclude the paper with the remark that among all the considered methods, the methods of Sharma-Sharma and Li-Liao-Cheng show the excellent convergence behavior.

References

[1] R. Behl, F. Zafar, A. S. Alshormani, M. U. D. Junjua and N. Yasmin, An optimal eighth-order scheme for multiple zeros of univariate functions, Int. J. Comput. Meth. (2018) http://doi.org/10.1142/S0219876218430028.

[2] C. Dong, A family of multipoint iterative functions for finding multiple roots of equations, Int. J. Comput. Math. 21 (1987), 363-367.

[3] Y. H. Geum, Y. I. Kim and B. Neta, A class of two-point sixth-order multiple-zero finders of modified double-Newton type and their dynamics, Appl. Math. Comput. 270 (2015), 387-400.

[4] E. Hansen and M. Patrick, A family of root finding methods, Numer. Math. 27 (1977), 257-269.

[5] M. Kansal, V. Kanwar and S. Bhatia, On some optimal multiple root-finding methods and their dynamics, Appl. Appl. Math. 10 (2015), 349-367.

[6] H. T. Kung and J. F. Traub, Optimal order of one-point and multipoint iteration, J. Assoc. Comput. Mach. 21 (1974), 643-651.

[7] S. Li, X. Liao and L. Cheng, A new fourth-order iterative method for finding multiple roots of nonlinear equations, Appl. Math. Comput. 215 (2009), 1288-1292.

[8] S. G. Li, L. Z. Cheng and B. Neta, Some fourth-order nonlinear solvers with closed formulae for multiple roots, Comput Math. Appl. 59 (2010), 126-135.

[9] B. Neta, New third order nonlinear solvers for multiple roots, App. Math. Comput. 202 (2008), 162-170.

[10] N. Osada, An optimal multiple root-finding method of order three, J. Comput. Appl. Math. 51 (1994), 131-133.

[11] E. Schroder, Uber unendlich viele Algorithmen zur Auflosung der Gleichungen, Math. Ann. 2 (1870), 317-365.

[12] M. Sharifi, D. K. R. Babajee and F. Soleymani, Finding the solution of nonlinear equations by a class of optimal methods, Comput. Math. Appl. 63 (2012), 764-774.

[13] J. R. Sharma and R. Sharma, Modified Jarratt method for computing multiple roots, Appl. Math. Comput. 217 (2010), 878-881.

[14] J. R. Sharma, S. Kumar and L. Jantschi, On a class of optimal fourth order multiple root

solvers without using derivatives, Symmetry 11 (2019) doi:10.3390/sym11121452.

[15] F. Soleymani, D. K. R. Babajee and T. Lotfi, On a numerical technique for finding multiple zeros and its dynamics, J. Egypt. Math. Soc. 21 (2013), 346-353.

[16] J. F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[17] J. L. Varona, Graphic and numerical comparison between iterative methods, Math. Intell. 24 (2002), 37-46.

[18] H. D. Victory and B. Neta, A higher order method for multiple zeros of nonlinear functions, Int. J. Comput. Math. 12 (1983), 329-335.

[19] E. R. Vrscay and W. J. Gilbert, Extraneous fixed points, basin boundaries and chaotic dynamics for Schroder and Konig rational iteration functions, Numer. Math. 52 (1988), 1-16.

[20] X. Zhou, X. Chen and Y. Song, Constructing higher-order methods for obtaining the multiple roots of nonlinear equations, J. Comput. Appl. Math. 235 (2011), 4199-4206.