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INDEPENDENCE POLYNOMIAL AND *Z* COUNTING POLYNOMIAL OF A FIBONACCI TREE

K. U. SREEJA, P. B. VINODKUMAR and P. B. RAMKUMAR

Research scholar
Department of Mathematics
Rajagiri School of Engineering
and Technology, Kakkanad, India
E-mail: sreejaku@gmail.com

Department of Mathematics
Faculty of Rajagiri School of Engineering
and Technology, Kakkanad, India

Department of Mathematics
Faculty of Rajagiri School of Engineering
and Technology, Kakkanad, India

Abstract

The concept of independence polynomial and *Z* counting polynomial has been independently proposed in many areas such as statistical mechanics, quantum theory, and so on. Although their information content is diametrically opposed, both polynomials have numerous interesting graph theoretical interpretations. Independence polynomials are used almost everywhere in science and mathematics, but they are an NP-complete problem for an arbitrary graph. We investigate the independence polynomial and *Z* counting polynomial of Fibonacci trees, which are acyclic graphs. Hosoya proposed the *Z* index in 1971, and it has found widespread application in chemical graph theoretical concepts. It is therefore worthwhile to investigate the structural properties of these polynomials in terms of size, isomorphism, *Z* index, and independence number.

1. Introduction

Polynomials such as the characteristic polynomial, the Independence polynomial, and the *Z* counting polynomial are widely studied [1] and contain

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a great deal of information about a graph. A comprehensive analysis of the mathematical properties of these polynomials is noted. It is used in a variety of fields, including quantum theory, statistical mechanics, and information chemistry [2].

When Ho soya introduced the Z counting polynomial, it was unknown in the field of chemistry, but it has since become a research topic of interest, particularly in mathematical chemistry. The study of independence polynomial and Z counting polynomial is important for all connected graphs of various types, such as monocyclic, non-monocyclic, and acyclic graphs. The graphical construction of the Z counting polynomial and independence polynomial is impractical for larger graphs. So the idea of recurrence relation introduced and it reduce the given graph in to smaller graphs from that we can easily computed corresponding polynomials.

Definition 1.1 (3)**.** An independent set in a graph G is a vertex subset $S \subseteq V(G)$ that contains no edge of G. The independence number of a graph is the maximum size of an independent set of vertices i.e. $\alpha(G)$.

The independence polynomial was introduced by I. Gutman and F. Harary. Even if independence polynomials exist almost everywhere in combinatorics, determining the independence polynomial of a graph is an NP-complete problem [3].

Definition 1.2 (3)**.** Let s_k denote the number of independent sets of size k, which are induced sub graphs of G, then $I(G, z) = \sum_{k=0}^{\alpha(G)} s_k z^k$ where $\alpha(G)$ is the independence number of G.

We can use the following result to calculate the independence polynomial of a graph.

Theorem 1.3 (3)**.** *Let G be a simple graph. Let $v \in V(G)$ and $N[v]$ be the closed neighborhood of v. Then* $I(G;z) = I(G-v;z) + zI(G-N[v];z)$.

In the following section, we will look at a special acyclic graph known as the Fibonacci tree. We will thoroughly examine its independence polynomial and Z counting polynomial.

2. Fibonacci Tree

Definition 2.1 (4)**.** Fibonacci tree of order n has the Fibonacci trees of orders $n-1$ and $n-2$ as left and right sub trees. It is denoted by r_n where $n = 0, 1, 2, 3, \ldots$.

Fibonacci tree of order 1 is a single vertex and other Fibonacci trees of different orders are given below [4].

Fibonacci tree of order 2 denoted by r_2 is given in figure 1.

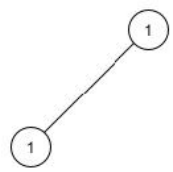


Figure 1. r_2.

Figure 2. represents a Fibonacci tree of order 3 denoted by r_3, with left r_2 and right r_1 as sub trees.

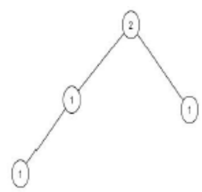


Figure 2. r_3

and so on.

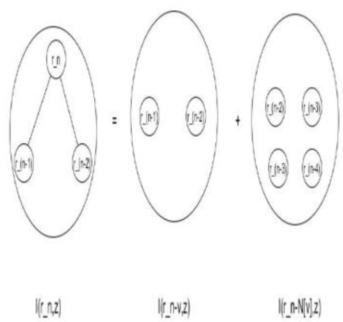
Fibonacci trees are the Fibonacci computation's recursive call structure. The formulas for $I(G, z)$ are extremely difficult to calculate, but there is a visual representation method in the Fibonacci family.

2.1. Diagrammatic Representation of Independence Polynomial of a Fibonacci Tree

To compute the independence polynomial of a graph, we use a visual aid in the form of a rooted tree of sub graphs, which we proved in our paper [5]. We have a node at the root of the tree that is the original graph, and we are attempting to calculate its independence polynomial. On the next level of the tree, we introduce two nodes. Put the $G-v$ sub graph in the first node and the $G-N[v]$ sub graph in the second. The closed neighborhood of v is denoted by $N[v]$. Finally, we can decompose the independence polynomial of a graph vertex by vertex for graphs with disjoint vertices using the recurrence relation $I(G;x) = I(G-v;x) + xI(G-N[v];x)$.

Theorem 2.2. *The following formula gives the independence polynomial of a Fibonacci tree of order n.*

$$I(r_n, z) = I(r_{n-2}, z)[I(r_{n-1}, z) + zI(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)] \text{ where } n \geq 4.$$



Proof. According to the Fibonacci tree structure, r_n has a single vertex in the middle, with r_{n-1} and r_{n-2} as left and right sub trees. Theorem 1.3 states that $I(r_n, z) = I(r_n - v) + zI(r_n - N[v])$. Let G_1 denote the graph of r_{n-1}, G_2 the graph of r_{n-2}, G_3 the graph of r_{n-3} and G_4 the graph of $r_{n-4} \cdot r_n - v$ is made up of two disjoint vertex graphs, G_1 and G_2. Then, applying the independence polynomial theorem of vertex disjoint graphs, $I(r_n - v, z) = I(G_1, z) \cdot I(G_2, z)$. Similarly, $r_n - N[v]$ is made up of four vertex disjoint graphs. G_2, G_3, G_3, G_4 with G_3 repeating twice due to the definitions of r_{n-1} and r_{n-2}. Using the recurrence theorem once more,

$$I(r_n - N[v], z) = I(G_2, z)I(G_3, z)I(G_3, z)I(G_4, z).$$

When these are substituted, the relationship

$I(r_n, z) = I(G_1, z) \cdot I(G_2, z) + zI(G_2, z)I(G_3, z)I(G_3, z)I(G_4, z)$ is obtained. Hence $I(r_n, z) = I(r_{n-1}, z)I(r_{n-2}, z) + zI(r_{n-2}, z)I(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)$.

2.2. *Z* counting polynomial

Polya pioneered the concept of a counting polynomial in chemistry in 1936. Despite the fact that the spectra of the characteristic polynomial of graphs were extensively studied, the subject received little attention from chemists for several decades. The study of the entropy of chain hydrocarbon molecules reveals yet another connection between counting polynomials and thermodynamic partition functions. [6] In this context, Gutman suggested the Z counting polynomial, also described as the acyclic polynomial, for the same intention. Apart from this problem, these polynomials have significant mathematical significance because they have a relationship with Z-counting and orthogonal polynomials [6].

Definition 2.3 [6]**.** The Z counting polynomial $Z(G, z)$, of a simple connected graph G is defined as $Z(G, z) = \sum_{k=0}^{\left[\frac{V}{2}\right]} p(G, k)z^k$ where the coefficient $p(G, k)$ is the number of independent sets of k edges of G and V is the number of vertices in G.

The Z index is a topological index associated with the Z counting polynomial. It was introduced in 1971 by Haruo Hosoya. It is widely used in chemical graph theoretical concepts for quantitative structure-property relationships.

Definition 2.4 [6]**.** The Z index of $G, Z = Z(G)$ is defined by the expression $Z = \sum_{k=0}^{\left[\frac{V}{2}\right]} p(G, k)$.

The Z index appears to have no edge contributions, but we can achieve this for a single connected acyclic graph (tree) by superimposing all of the tree's independent sets of edges.

The Z index equals the value of the Z counting polynomial when $z = 1 : Z(G) = Z(G; z = 1)$. [6] By replacing vertices with edges, we can calculate the recurrence relation of a Z counting polynomial in the same way that we can calculate the recurrence relation of an independence polynomial.

Theorem 2.5. *To compute the Z counting polynomial, the recurrence relation is given by* $Z(G, z) = Z(G - e; z) + zZ(G - \{e\}; z)$ *where* $G - e$ *and* $G - \{e\}$

denote spanning sub graphs of G obtained by erasing an edge e, and the edge e and all edges adjacent to e respectively.

Definition 2.6 [7]**.** The Z counting polynomial of a disconnected graph D with components $D_i (i = 1, \ldots, n)$, that is $D = \bigcup_{i=1}^{n} D_i$.

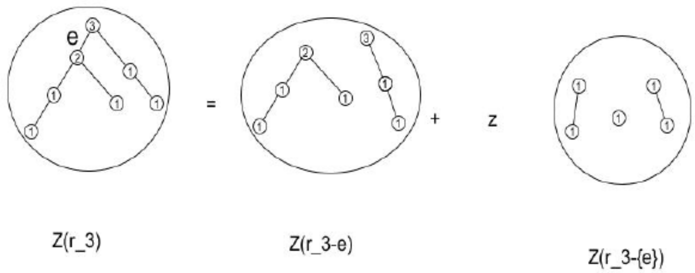
The Z counting polynomial of the Fibonacci tree of order 2 is given by $Z(r_2, z) = 1 + z$ using this definition and the recurrence relation.

The Z counting polynomial can be calculated using definition, but it is computationally impractical for larger graphs. The Z counting polynomial can be computed much more efficiently and rapidly using a diagrammatic representation in the form of a rooted tree of sub graphs.

2.3. Diagrammatic Representation of Z Counting Polynomial of a Fibonacci Tree

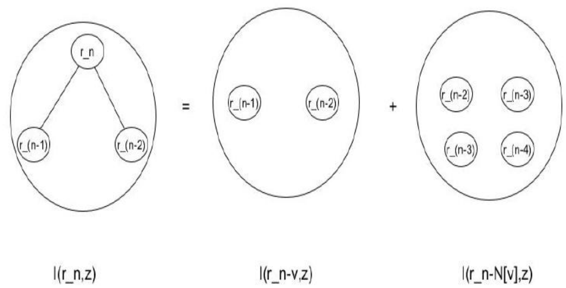
Theorem 2.7. *The Z counting polynomial of Fibonacci tree of order* n, r_n *is given by* $Z(r_n, z) = Z(r_{n-1}, z) Z(G_1, z) + zZ(r_{n-2}, z) Z(r_{n-3}, z) Z(r_{n-2}, z)$ *where* $Z(G_1, z) = Z(r_{n-2}, z) + zZ(r_{n-3}, z)$ *where* $n \geq 3$.

Proof. This result will be demonstrated first for $n = 3$.



$Z(r_3, z) = Z(r_2, z) Z(G_1, z) + zZ(r_1, z) Z(r_0, z) Z(r_1, z)$ where $Z \leq (G_1, z)$ $= Z(r_1, z) + zZ(r_0, z)$. So $Z(r_3, z) = (1 + z)(1 + z) + z = 1 + 3z + z^2$ upon simplification. As a result, the result holds true for $n = 3$.

In general, we can easily demonstrate this result using the definition of the Fibonacci tree.



According to the Fibonacci tree structure, r_n has a single vertex in the middle, with left and right sub trees of r_{n-1} and r_{n-2}. As a result, when calculating the Z counting polynomial, we will select the required edge as the edge connecting to this single vertex in the middle from the left side. Using the theorem 2.9, we will compute the Z counting polynomials of $(r_n - e)$ and $(r_n - \{e\})$ is the required edge defined above. The Z counting polynomial of $(r_n - e)$ yields two disjoint graphs: the r_{n-1} and the graph of G_1, which is the r_{n-2} graph with an edge joined near r_n (right). Using the main theorem 2.9 again, $Z(r_n - e, z) = Z(r_{n-1}, z)Z(G_1, z)$ where $Z(G_1, z)$ can be calculated using the repeated action of theorem 2.9, we will have $Z(G_1, z) = Z(r_{n-2}) + zZ(r_{n-3})$. Similarly, we can compute the Z counting polynomial of a graph of $(r_n - \{e\})$, which is the graph of r_n from which the neighborhoods of the selected edge e (left of r_n) are deleted, yielding the disjoint union of graphs r_{n-2}, r_{n-3} and r_{n-2}. By substituting these values, we get the result.

Cyclicity is a structural property that is important in combinatorics. The size of the corresponding graphs changes as the number of cycles changes.

Arocha demonstrates that $I(C_n, z) = F_{n-1}(z) + 2zF_{n-2}(z)$ where $F_n(z), n \geq 0$ are the Fibonacci polynomials defined recursively by $F_0(z) = 0, F_1(z) = 1, F_n(z) = zF_{n-1}(z)zF_{n-1}(z) + F_{n-2}(z)$ where $n \geq 2$ ([9]). As a result, we get the following result.

Corollary 2.8. *In terms of Fibonacci polynomials, the Z counting polynomial of* C_n *is given as* $Z(C_n, z) = F_{n-1}(z) + zF_{n-2}(z)$.

Theorem 2.9 [8]**.** *For a connected graph* $G, I(G, z) = Z(G, z)$ *holds if and only if* $G = C_n$.

Corollary 2.10. *If a graph is isomorphic to the cyclic graph* $C_n (n \geq 3)$, *their independence polynomial and Z counting polynomial are identical*.

Proof. The assertion is derived from the definitions of $I(G, z)$ and $Z(G, z)$. We have $I(G, z) = I(H, z)$ if G and H are isomorphic. The theorem also states that $I(G, z) = Z(G, z)$ if and only if $G = C_n$. As a result, if a graph G is isomorphic to a graph C_n, then $I(G, z) = I(C_n, z)$. However, for a cyclic graph of order n, $I(C_n, z) = Z(C_n, z)$. As a result, $Z(C_n, z) = I(G, z) = Z(G, z)$ where G is isomorphic to C_n. Hence the result.

3. The Graph Characteristics of the Independence Polynomial and the *Z* Counting Polynomial

Changes in the order of the Fibonacci tree result in changes in the independence number and *Z* index. In the Tables, they are shown along with their independence polynomial and *Z* counting polynomial.

Table 1.

Independence polynomial of $r_n, n = 1, 2, 3, 4$	Z polynomial of $r_n, n = 1, 2, 3, 4$
$1+2z$	$1+z$
$1+4z+3z^2$	$1+3z+z^2$
$1+7z+15z^2+11z^3+2z^4$	$1+6z+9z^2+3z^3$
$1+12z+55z^2+123z^3+142z^4+81z^5+18z^6$	$1+11z+42z^2+62z^3+47z^4+11z^5+z^6$

Table 2.

Graph	Independence number	Z index
r_1	1	2
r_2	2	5
r_3	4	19
r_4	6	175

As the order of the Fibonacci tree increases, the value of the Z index increases. The combinatorial possibilities for the Z index appear to be greater than the independence number of the corresponding graph's independence polynomial.

Remark 3.1. A one-to-one correspondence is established between sets of independent edges and sets of independent vertices using these various polynomials. It's interesting to look at the independence polynomials and Z count polynomials of general graphs. More attention will be paid to the structural changes of the general graph in terms of its independence number and the Z index in the future.

4. Conclusion

Thus, we have demonstrated that graphs that are cyclic and isomorphic to cyclic graphs are only graphs whose independence and Z-counting polynomials coincide. The mathematical structure of two different counting polynomials, independence polynomial and Z counting polynomial, as well as their recursion relation, aids in the analysis of physicochemical and mathematical phenomena.

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