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STUDY ON OPERATORS IN 2-FUZZY 2-INNER PRODUCT SPACE

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Abstract

The study on operators in 2-fuzzy 2-inner product space is introduced in this paper. Notions such as self-adjoint fuzzy operator, normal fuzzy operator and unitary operator are coined and some properties of such fuzzy operators are discussed.

1. Introduction

In 1965, Zadeh [13] introduced the idea of fuzzy sets, that established a new revolutionary field in mathematics. Katsaras [6] introduced the concept of a fuzzy norm on a linear space in 1984. Chen and Mordeson [2], Bag and Samanta [1], and others have provided several definitions of fuzzy normed spaces. Somasundaram and Thangaraj Beaula [10] coined the notion of 2-fuzzy 2-normed linear spaces, and Thangaraj Beaula and Gifta [12] further developed some standard results. C. R. Diminnie, S. Gahler and A. White [3] introduced the idea of 2-inner product space. Further definitions of fuzzy inner product space [4, 7] and fuzzy normed linear space [5, 8, 9] were given

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by various authors. In [11], Vijayabalaji and Thilaigovindan proposed fuzzy n-inner product space as a generalization of then-inner product space. This paper introduces the study on operators in 2-fuzzy 2-inner product space is introduced in this paper. Various operators such as self-adjoint fuzzy operator, normal fuzzy operator and unitary operator are coined in this generalized fuzzy setting and their properties are discussed.

2. Preliminaries

Definition 2.1. A fuzzy set is defined as $\widetilde{A} = \{(x, \mu_A(x)) : x \in X\}$, with a membership function $\mu_A(x) : X \to [0, 1]$, where $\mu_A(x)$ denotes the degree of membership of the element x to the set A.

Definition 2.2. Let X be a non empty set and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$ then $f = \{(x, \mu)/x \in X$ and $\mu \in (0, 1]\}$. Clearly f is bounded function for $|f(x)| \leq 1$. Let K be the space of real numbers then $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by $f + g = \{(x, \mu) + (y, \eta)\} = \{(x + y),$ $(\mu, \eta)/(x, \mu) \in f$ and $(y, \eta) \in g\}$ and $kf = \{(kf, \mu)/(x, \mu) \in f\}$ where $k \in K$.

The linear space $F(X)$ is said to be normed space if for every $f \in F(X)$ there is associated a non-negative real number $\| f \|$ called the norm of f in such a way,

(i) $\| f \| = 0$ if and only if $f = 0$.

For, $\| f \| = 0 \Leftrightarrow \{\| (x, \mu) \| / (x, \mu) \in f\} = 0$

$\Leftrightarrow x = 0, \mu \in (0, 1] \Leftrightarrow f = 0$

(ii) $\| kf \| = |k| \| f \|, k \in K$.

For

$$\| kf \| = \{\| k(x, \mu) \| /(x, \mu)f,\ k \in K\}$$

$$= \{| k | \| x, \mu \| /(x, \mu) \in f\} = | k | \| f \|$$

(iii) $\| f + g \| < \| f \| + \| g \|$ for every $f, g \in F(X)$.

For,

$$\| f + g \| = \{\| (x, \mu) + (y, \eta) \| / x,\ y \in X,\ \mu, \eta \in (0, 1]\}$$

$$= \{\| (x + y), (\mu \wedge \eta) \| / x,\ y \in X,\ \mu, \eta \in (0, 1]\}$$

$$\leq \{\| (x, \mu \wedge \eta) \| + \| (y, \mu \wedge \eta) \| / (x, \mu) \in f \text{ and } (y, \eta) \in g\}$$

$$= \| f \| + \| g \|$$

Then $(F(X), \| \cdot \|)$ is a normed linear space.

Definition 2.3. A 2-fuzzy set on X is a fuzzy set on $F(X)$.

Definition 2.4. Let $F(X)$ be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R(R,$ the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $F(X)$) if and only if,

(N_1) for all $t \in R$ with $t \leq 0,\ N(f_1, f_2, t) = 0$.

(N_2) for all $t \in R$ with $t \leq 0,\ N(f_1, f_2, t) = 1$ if and only if f_1 and f_2 are linearly dependent.

(N_3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.

(N_4) for all $t \in R$, with $t \leq 0,\ N(f_1, cf_2, t) = N(f_1, f_2, t / | c |)$ if $c \neq 0,\ c \in K$ (field).

(N_5) for all $s, t \in R,\ N(f_1, f_2 + f_3, s + t) \geq \min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$.

(N_6) $N(f_1, f_2, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

(N_7) $\lim_{t \to \infty} N(f_1, f_2, t) = 1$.

Then $(F(X), N)$ is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Definition 2.5. A 2-fuzzy 2-normed linear space (X, N) is said to be complete if every Cauchy sequence in X converge to some point in X.

Definition 2.6. Let $F(X)$ be a linear space over the complex field $\mathbb{C}$.

Define a fuzzy subset μ as a mapping from $F(X) \times F(X) \times F(X) \times \mathbb{C} \to [0, 1]$ such that $f_1 \in F(X)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfying the following conditions

(I_1) For $f, g, h \in F(X)$ and $s, t \in \mathbb{C}$

$$\mu(f + g, h, f_1, |t| + |s|) \geq \min\{\mu(f, h, f_1, |t|), \mu(g, h, f_1, |s|)\}$$

(I_2) For $s, t \in \mathbb{C}$, $\mu(f, g, h, |st|) \geq \min\{\mu(f, f, h, |s|^2), \mu(g, g, h, |t|^2)\}$.

(I_3) For $t \in C$, $\mu(f, g, h, |t|) = \mu(g, f, h, |t|)$.

(I_4) For $\alpha_1, \alpha_2 \in \mathbb{C}$ with $\alpha_1 \neq 0, \alpha_2 \neq 0, \mu(\alpha_1 f, \alpha_2 g, h, t) = \mu\left(f, g, h, \frac{t}{|\alpha_1 \alpha_2|}\right)$.

(I_5) $\mu(f, f, h, t) = 0 \forall t \in \mathbb{C}/R^+$

$\mu(f, f, h, t) = 1 \; \forall t > 0$ if and only if f, h are linearly dependent.

(I_6) $\mu(f, g, h, t)$ is invariant under any permutation.

(I_7) $\forall t > 0$, $\mu(f, f, h, t) = \mu(g, g, h, t)$

(I_8) $\mu(f, g, h, t)$ is monotonic non-decreasing function of $\mathbb{C}$ and

$\lim_{t \to \infty} \mu(f, g, h, t) = 1$.

Then μ is said to be the 2-fuzzy 2-inner product on $F(X)$ and the pair (X, μ) is called 2-fuzzy 2-inner product space.

3. Adjoint Fuzzy Operator in 2-Fuzzy 2-Hilbert Space

Definition 3.1. Let (X, μ) be the 2-fuzzy 2-inner product space. A linear functional T defined on $F(X)$ is said to be continuous if f_n converges to f implies sequence $\{Tf_n\}$ converges to Tf, for any $\{f_n\}, f \in F(X)$. For a given $t > 0$ and $0 < r < 1$ there exist a positive number $n_0 \in N$ such that

$$\mu(f_n - f, f_n - f, h, t) > 1 - r$$

For $h \in F(X)$ and for every $n \geq n_0$, where $0 < t' \leq 1$ and $r \in (0, 1)$.

Then for a given $t' > 0$ and $0 < r' < 1$ there exist a positive number

$n_0 \in N$ such that

$$\mu(Tf_n - Tf, Tf_n - Tf, h, t') > 1 - r'$$

For $h \in F(X)$ and for every $n \geq n_0$, where $0 < t' \leq 1$ and $r' \in (0, 1)$.

Theorem 3.2. *Let* (X, μ) *be a 2-fuzzy 2-inner product space and* $\inf\{t : \mu(f, g, h, t) \geq \alpha\} < \infty$ *for all* $f, g \in F(X)$ *then*

$$\inf\{t + s : \mu(f + g, f_1, h, t + s) \geq \alpha\} = \inf\{t : \mu(f, f_1, h, t) \geq \alpha\}$$
$$+ \inf\{s : \mu(g, f_1, h, s) \geq \alpha\}.$$

Proof. Let us consider

$$\inf\{t : \mu(f, f_1, h, t) \geq \alpha\} + \inf\{s : \mu(g, f_1, h, s) \geq \alpha\} = \inf\{t + s : \mu(f, f_1, h, t) \geq \alpha, \mu(f, f_1, h, s) \geq \alpha\} = \inf\{t + s : \inf\{\mu(f, f_1, h, t) \geq \alpha, \mu(g, f_1, h, s) \geq \alpha\}\} \geq \inf\{t + s : \mu(f + g, f_1, h, t + s) \geq \alpha\} \tag{1}$$

Conversely, for any $\delta > 0$

Assume that,

$$k = \inf\{1 - \left(1 - \mu\left(f, f_1, h, \inf\{t : \mu(f, f_1, h, t) \geq \alpha\} - \frac{\delta}{2}\right)\right)$$
$$\left(1 - \mu\left(g, f_1, h, \inf\{s : \mu(g, f_1, h, s) \geq \alpha\} - \frac{\delta}{2}\right)\right)\}$$
$$= \inf\{\left(1 - \mu\left(-f, f_1, h, -\inf\{t : \mu(f, f_1, h, s) \geq \alpha\} + \frac{\delta}{2}\right)\right)$$
$$\mu\left(-g, f_1, h, -\inf\{s : \mu(g, f_1, h, s) \geq \alpha\} + \frac{\delta}{2}\right)\}$$
$$\geq 1 - \mu(-f, -g, f_1, h, -\inf\{t : \mu(f, f_1, h, s) \geq \alpha\} - \inf\{s : \mu(g, f_1, h, s) \geq \alpha\} + \delta) = \mu(f + g, f_1, h, \inf\{t : \mu(f, f_1, h, t) \geq \alpha\} + \inf\{s : \mu(g, f_1, h, s) \geq \alpha\} - \delta)$$

By the definition of infimum $\mu\left(f, f_1, h, \inf\{t : \mu(f, f_1, h, t) \geq \alpha\} - \frac{\delta}{2}\right) < \alpha$

Hence $1 - \mu\left(g, f_1, h, \inf\{t : \mu(g, f_1, h, s) \geq \alpha\} - \frac{\delta}{2}\right) < 1 - \alpha$

Similarly $1 - \mu\left(g, f_1, h, \inf\{s : \mu(g, f_1, h, s) \geq \alpha\} - \frac{\delta}{2}\right) < 1 - \alpha$

Therefore $\inf\left\{\left(1 - \mu(f, f_1, h, \inf\{t : \mu(f, f_1, h, t) \geq \alpha\})) - \frac{\delta}{2}\right),\right.$

$\left.\left(1 - \mu\left(g, f_1, h, \inf\{s : \mu(g, f_1, h, s) \geq \alpha\} - \frac{\delta}{2}\right)\right)\right\} > 1 - \alpha$

(i.e) $1 - k > 1 - \alpha$ which implies that $k < \alpha$

As a result

$$\mu(f + g, f_1, h, \inf\{t : \mu(f, f_1, h, t) \geq \alpha\} + \inf\{s : \mu(g, f_1, h, s) \geq \alpha\} - \delta)$$

$\leq k < \alpha$ (i.e.) $\inf\{t + s : \mu(f + g, f_1, h, t + s) \geq \alpha\} \geq \inf\{t : \mu(f, f_1, h, t) \geq \alpha\}$

$$+ \inf\{s : \mu(g, f_1, h, t) \geq \alpha\} \quad (2)$$

From (1) and (2),

$$\inf\{t + s : \mu(f + g, f_1, h, t + s) \geq \alpha\} = \inf\{t : \mu(f, f_1, h, t) \geq \alpha\}$$

$$+ \inf\{s : \mu(g, f_1, h, t) \geq \alpha\}$$

Theorem 3.3. *Let* (X, μ) *be a* 2*-fuzzy* 2*-Hilbert space and* T *be a continuous linear functional then there exists a unique* T^* *a continuous linear functional on* $F(X)$ *such that*

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(f, T^*g, h, t) \geq \alpha\}$$

for every $f, g, h \in F(X)$.

Proof. Choose $g \in F(X)$, define: $G_g : F(X) \to \mathbb{R}$, by $G_g(f) = \inf\{t : \mu(f, g, h, t) \geq \alpha\}$ for every $f \in F(X)$ such that,

$$G_g(f + l) = G_g(f) + G_g(l)$$

$$G_g(kf) = kG_g(f)$$

for every $f, g, l \in F(X)$, k a scalar in $\mathbb{R}$.

Also, there exists $l_g \in F(X)$ such that

$G_g(f) = \inf\{t : \mu(f, g, h, t) \geq \alpha\}$

Define $T^* : F(X) \to \mathbb{R}$ such that $T_g^* = l_g$, for every $g \in F(X)$

Let $g, l \in F(X)$ and k, s are scalar, then

$$\inf\{t : \mu(f, T^*(kg + sl), h, t) \geq \alpha\} = \inf\{t : \mu(Tf, kg, sl, h, t) \geq \alpha\}$$

By the theorem (3.2) and (I_4)

$$\inf\{t : \mu(f, T^*(kg + sl), h, t) \geq \alpha\} = \inf\{t : \mu(Tf, kg, h, t) \geq \alpha\}$$

$$+ \inf\{t : \mu(Tf, sl, h, t) \geq \alpha\}$$

$$= k. \inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = s. \inf\{t : \mu(Tf, l, h, t) \geq \alpha\}$$

$$= k. \inf\{t : \mu(f, T^*g, h, t) \geq \alpha\} = s. \inf\{t : \mu(f, T^*l, h, t) \geq \alpha\}$$

Uniqueness of T^*: Let T_1^*, T_2^* be two adjoint fuzzy operators for $T \in F(X)$,

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(f, T_1^*g, h, t) \geq \alpha\}$$

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(f, T_2^*g, h, t) \geq \alpha\}$$

for every $f, g \in F(X)$

It implies that,

$$\inf\{t : \mu(f, T_1^*g, h, t) \geq \alpha\} = \inf\{t : \mu(f, T_2^*g, h, t) \geq \alpha\}$$

and hence T^* is unique.

Definition 3.4. Let (X, μ) be a 2-fuzzy 2-Hilbert space with $\inf\{t : \mu(f, g, h, t) \geq \alpha\}$ for every $f, g \in F(X)$ and let T be a continuous linear functional, then T is selfadjoint fuzzy operator, if

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(T^{*}f, g, h, t) \geq \alpha\}$$

Where T^{*} is adjoint fuzzy operator of T.

Theorem 3.5. *Let (X, μ) be a 2-fuzzy 2-Hilbert space with $\inf\{t : \mu(f, g, h, t) \geq \alpha\}$ and let T be a continuous linear functional, then T is self-adjoint fuzzy operator.*

Proof. Since $F(X)$ is set of all fuzzy sets on X a non empty set and $\inf\{t : \mu(f, g, h, t) \geq \alpha\}$ for every $f, g \in F(X)$, then $\inf\{t : \mu(Tf, g, h, t) \geq \alpha\}$ is real for all $f \in F(X)$.

Now

$$\begin{aligned} \inf\{t : \mu(Tf, g, h, t) \geq \alpha\} &= \inf\{\bar{t} : \mu(Tf, g, h, t) \geq \alpha\} \\ &= \inf\{t : \mu(f, Tg, h, t) \geq \alpha\} \\ &= \inf\{t : \mu(T^{*}f, g, h, t) \geq \alpha\} \end{aligned}$$

Therefore,

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf \inf\{t : \mu(T^{*}f, g, h, t) \geq \alpha\}$$

T is a self-adjoint fuzzy operator.

Theorem 3.6. *Let (X, μ) be a 2-fuzzy 2-Hilbert space with $\inf\{t : \mu(f, g, h, t) \geq \alpha\}$ for every $f, g \in F(X)$ and let T^{*} be the adjoint fuzzy operator of* T *is a continuous linear functional then*

(i) $\inf\{t : \mu(f, T^{**}g, h, t) \geq \alpha\} = \inf\{t : \mu(f, Tg, h, t) \geq \alpha\}$

(ii) $\inf\{t : \mu(f, (kT)^{*}g, h, t) \geq \alpha\} = \inf\{t : \mu(f, kT^{*}g, h, t) \geq \alpha\}$

(iii) $\inf\{t + s : \mu(f, (kT + sD)^{*}g, h, t) \geq \alpha\} = \inf\{t + s : \mu(f(kT^{*} + sD^{*}), g, h, t) \geq \alpha$

(iv) $\inf\{t : \mu(f, (TD)^{*}g, h, t) \geq \alpha\} = \inf\{t : \mu(f, (D^{*}T^{*})g, h, t) \geq \alpha\}$

Proof.

(i) $\inf\{t : \mu(f, T^{**}g, h, t) \geq \alpha\} = \inf\{t : \mu(T^{*}f, g, h, t) \geq \alpha\}$

$$= \inf\{t : \mu(f, Tg, h, t) \geq \alpha\}$$

(ii) $\inf\{t : \mu(f, (kT)^{*}g, h, t) \geq \alpha\} = \inf\{t : \mu(kTf, g, h, t) \geq \alpha\}$

$$= \inf\{t : \mu(kTf, g, h, t) \geq \alpha\}$$

$$= \inf\left\{t : \mu\left(Tf, g, h, \frac{t}{|k|}\right) \geq \alpha\right\}$$

$$= \inf\left\{t : \mu\left(f, T^{*}g, h, \frac{t}{|k|}\right) \geq \alpha\right\}$$

$$= \inf\left\{t : \mu\left(f, kT^{*}g, h, \frac{t}{|k|}\right) \geq \alpha\right\}$$

(iii) $\inf\{t+s : \mu(f, (kT+sD)^{*}g, h, t) \geq \alpha\} = \inf\{t+s : \mu(f(kT+sD), g, h, t) \geq \alpha\}$

$$\geq \inf\{t+s : \inf\{\mu(kTf, g, h, t) \geq \alpha, \mu(sDf, g, h, t) \geq \alpha\}$$

$$= \inf\{t+s : \inf\{\mu(kf, T^{*}g, h, t) \geq \alpha, \mu(sf, D^{*}g, h, t) \geq \alpha\}\}$$

$$= \inf\left\{t+s : \inf\left\{\mu\left(kf, T^{*}g, h, \frac{t}{|k|}\right) \geq \alpha, \mu\left(sf, D^{*}g, h, \frac{t}{|s|}\right) \geq \alpha\right\}\right\}$$

$$= \inf\{t+s : \inf\{\mu(f, kT^{*}g, h, t) \geq \alpha, \mu(f, sD^{*}g, h, t) \geq \alpha\}\}$$

$$= \inf\{t+s : \inf\{\mu(f(kT^{*} + sD^{*})g, h, t) \geq \alpha\}\}$$

Repeating as in theorem (3.2), the above equality is proved.

(iv) $\inf\{t : \mu(f, (TD)^{*}g, h, t) \geq \alpha\} = \inf\{t : \mu(fTD, g, h, t^{2}) \geq \alpha\}$

$$= \inf\{t : \mu(Df, T^{*}g, h, t) \geq \alpha\}$$

$$= \inf\{t : \mu(f, (D^{*}T^{*})g, h, t) \geq \alpha\}$$

4. Normal Fuzzy Operator in 2-Fuzzy 2-Inner Product Space

Definition 4.1. Let (X, μ) be the 2-fuzzy 2-inner product space. An operator N is said to be normal fuzzy operator if it commutes with its adjoint (i.e.) $\inf\{t : \mu(NN^*f, g, h, t) \geq \alpha\} = \inf\{t : \mu(N^*Nf, g, h, t) \geq \alpha\}$ for every $f, g, h \in F(X)$.

Theorem 4.2. *If N is a normal fuzzy operator and self-adjoint fuzzy operator on $F(X)$ then*

$$\inf\{t : \mu(N^*Nf, f, h, t) \geq \alpha\} = \inf\{t^2 : \mu(Nf, Nf, h, t^2) \geq \alpha\}$$

Proof. If N is a normal fuzzy operator then,

$$\inf\{t : \mu(N^*Nf, g, h, t) \geq \alpha\} = \inf\{t : \mu(NN^*f, g, h, t^2) \geq \alpha\} \tag{3}$$

Taking $g = f$, (3) becomes

$$\inf\{t : \mu(N^*Nf, f, h, t) \geq \alpha\} = \inf\{t : \mu(NN^*f, f, h, t^2) \geq \alpha\}$$

$$= \inf\{t : \mu(Nf, Nf, h, t^2) \geq \alpha\}$$

Definition 4.3. Let (X, μ) be the 2-fuzzy 2-inner product space. An operator T is said to be unitary fuzzy operator if

$$\inf\{t : \mu(T^*Tf, f, h, t) \geq \alpha\} = \inf\{t : \mu(TT^*f, f, h, t) \geq \alpha\}$$

$$= \inf\{t : \mu(f, g, h, t) \geq \alpha\}$$

Theorem 4.4. *If T is a fuzzy operator on a 2-fuzzy 2-Hilbert space (X, μ) then the following conditions are equivalent to one another*

(i) $\inf\{t : \mu(T^*Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(f, g, h, t) \geq \alpha\}$

(ii) $\inf\{t : \mu(Tf, Tg, h, t) \geq \alpha\} = \inf\{t : \mu(f, g, h, t) \geq \alpha\}$ *for every* $f, g, h \in F(X)$

(iii) $\inf\{t^2 : \mu(Tf, Tg, h, t^2) \geq \alpha\} = \inf\{t^2 : \mu(f, g, h, t^2) \geq \alpha\}$ *for every* $f \in F(X)$.

Proof. $(i) \Rightarrow (ii)$

Given $\inf\{t : \mu(T^* Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(f, g, h, t) \geq \alpha\}$

Consider $\inf\{t : \mu(Tf, Tg, h, t) \geq \alpha\} = \inf\{t : \mu(f, T^* Tg, h, t) \geq \alpha\}$

$$= \inf\{t : \mu(f, g, h, t) \geq \alpha\}$$

(ii)$\Rightarrow$(iii)

Given $\inf\{t : \mu(Tf, Tg, h, t) \geq \alpha\} = \inf\{t : \mu(f, g, h, t) \geq \alpha\}$ (4)

By taking $g = f$, (4) becomes

$$\inf\{t^2 : \mu(Tf, Tf, h, t^2) \geq \alpha\} = \inf\{t^2 : \mu(f, f, h, t^2) \geq \alpha\}$$

(iii)$\Rightarrow$(iv)

Given $\inf\{t^2 : \mu(Tf, Tf, h, t^2) \geq \alpha\} = \inf\{t^2 : \mu(f, f, h, t^2) \geq \alpha\}$

Consider

$$\inf\{t : \mu(T^* Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(Tf, Tg, h, t) \geq \alpha\}$$

$$= \inf\{t : \mu(f, g, h, t) \geq \alpha\}$$

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