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# ON THE GENERALIZED NON-COMMUTING GRAPHS OF DIHEDRAL SEMI-DIHEDRAL AND QUASI-DIHEDRAL GROUPS

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## Abstract

In this paper, $G$ denotes a finite group and $\Gamma$ denotes a simple graph. Two elements $x$ and $y$ commute if $xy = yx$. The non-commuting graph is a graph whose vertices are non-centrals elements of a group in which two vertices are adjacent if they do not commute. In this paper, the non-commuting graph is extended by defining the generalized non-commuting graph. The generalized non-commuting graph of dihedral groups, semi-dihedral groups and quasi-dihedral groups are also determined.

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## 1. Introduction

A graph $\Gamma$ is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively [1]. The graph is called directed if its edges are identified with ordered pair of vertices. Otherwise, $\Gamma$ is called undirected. Two vertices are adjacent if they are linked by an edge [2]. A complete graph is a graph where each ordered pair of distinct vertices are adjacent, denoted by $K_n$. The graph denoted by $K_0$ is called null if it has no vertices, while the empty graph denoted by $K_e$ is a graph that has no edges between its vertices [1, 2].

In 1975, the idea of non-commuting graphs comes from an old question of Erdős on the size of the cliques and answered in affirmative by Neumann [3].

This paper is structured as follows: Section 1 provides some fundamental concepts of graph theory. In Section 2, some previous works related to the non-commuting graph are stated. The main results are presented in Section 3.

## 2. Preliminaries

In this section, we provide some previous works related to the non-commuting graph. Neumann [3] in 1975 answered the old question of Erdős on the size of the subgraphs (cliques). The definition of a non-commuting graph is stated in the following:

**Definition 1.** [3] Let $G$ be a finite non-abelian group with the center denoted by $Z(G)$. A non-commuting graph is a graph whose vertices are non-central elements of $G$ (i.e $G \setminus Z(G)$. Two vertices $v_1$ and $v_2$ are adjacent whenever $v_1 v_2 \neq v_2 v_1$.

In [3], it is mentioned that Erdős in his paper asked if there is a finite bound on the cardinalities of cliques of $\Gamma$. This question has been answered by some researchers, first conformed of Erdős's question was by Neumann [3]. According to Neumann [3] there is a finite complete subgraph in some groups. Likewise, one of the answers of Erdős's question was given by Abdollahi et al. [4] who emphasized the existence of finite bound on the cardinalities of complete subgraph in $\Gamma$. They used the graph theoretical concepts to investigate the algebraic properties of the graph.

The non-commuting graph was generalized by Erfanian and Tolue [5] to what is called the relative non-commuting graph. The idea of this generalized graph comes from the relative commutativity degree where the vertices of this graph are the group excludes the centralizer of its subgroup. Two vertices are linked by an edge if their commutator is not equal to one.

Moghaddamfar et al. [6] conjectured that for some finite non-Abelian groups $G_1$ and $G_2$ if the non-commuting graph of $G_1$ is isomorphic to the non-commuting graph of $G_2$, then $|G_1| = |G_2|$ Besides, they obtained some graph properties for this conjecture. However, Moghaddamfar [7] later gave some counterexamples for the conjecture mentioned in [6]. Darafsheh [8] used the mentioned conjecture to the simple groups $G_1$ and for certain simple groups $G_2$ where he proved that if the non-commuting graph of $G_1$ and $G_2$ are isomorphic, then $|G_1| = |G_2|$.

In [9], it is mentioned that the commuting graph is a graph whose vertices are non-central elements of a group in which two vertices are adjacent whenever they commute. In 2001, Segev [10] obtained the commuting graph of non-solvable groups. In 2008, Iranmanesh and Jafarzadeh [11] investigated the properties of commuting graph of symmetric groups and alternating groups.

In 2011, Chelvam et al. [12] determined the commuting graphs of dihedral groups. In addition, Raza and Faizi [13] found certain properties of commuting graph of dihedral groups of order $2n$ including the chromatic number and clique number.

Recently, El-sanfaz et al. [14] extended the commuting graph by defining the generalized commuting graph. The following is the definition of the generalized commuting graph.

**Definition 2.** [15] Suppose $G$ be a finite non-abelian group and $\Omega$ be a no-empty subset of $G \times G$. The generalized commuting graph denoted by $\Gamma_{\Omega}^{G\cdot C}$ is a graph whose vertices are non-central elements of $\Omega$ in $G$. i.e.

$V(\Gamma_{\Omega}^{G \cdot C}) = \Omega - A$ where $A = \{w \in \Omega : gw = wg : g \in G\}$. Two vertices $w_1, w_2 \in \Omega$, are adjacent if $w_1, w_2 = w_2, w_1$ for all $w_1, w_2 \in \Omega$, where $\Omega = \{(a, b) \in G \times G : |a| = |b| = 2, ab = ba, a \neq b\}$.

The following theorems show some results on the generalized commuting graph of dihedral groups, semi-dihedral groups and quasi-dihedral groups that are needed in this paper.

**Theorem 1.** [15] *Let G be a dihedral group of order $2n$, where n is even. Let $\Omega$ be a set of $G \times G$. Then* $\Gamma_{\Omega}^{G \cdot C} = \bigcup_{i=1}^{\frac{n}{2}} K_3$.

**Theorem 2.** [15] *Let G be a semi-dihedral group, $G \cong \langle a, b : a^{2^n} = b^2 = 1, ab = ba^{2^{n-1}+1} \rangle$, where $n \geq 3$. Let $\Omega$ be a non-empty subset of $G \times G$. Then* $\Gamma_{\Omega}^{G \cdot C} = \bigcup_{i=1}^{2^{n-2}} K_3$.

**Theorem 3.** [15] *Let G be a quasi-dihedral group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$. Let $\Omega$ be a non-empty subset of $G \times G$. Then* $\Gamma_{\Omega}^{G \cdot C} K_3$.

Since the generalized non-commuting graph of dihedral groups, semi-dihedral groups and quasi-dihedral groups are going to be determined, thus the following results are needed and used in this paper.

**Theorem 4.** [16] *Let G be a dihedral group of order $2n$, $G \cong \langle a, b : a^n = b^2 = 1, ab = ba^{-1} \rangle$. If G acts on $\Omega$ by conjugation, then $|\Omega| = 3\binom{n}{2}$ represented as follows: There are n elements in the form of $(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq 2n$ and $\frac{n}{2}$ elements are in the form of $(a^i b, a^{\frac{n}{2}+i} b), 0 \leq i \leq 2n$.*

**Theorem 5.** [17] *Let G be a semi-dihedral group,*

$G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$, *where* $n \geq 3$. *If* $G$ *acts on* $\Omega$ *by conjugation, then* $|\Omega| = 3(2^{n-2})$, *represented as follows: There are* $2^{n-1}$ *elements in the form of* $(a^{2^{n-1}}, a^i b), 0 \leq i \leq 2^n$, *where i is even and there are* $2^{n-2}$ *elements in the form of* $(a^i b, a^{i+2^{n-1}} b), 0 \leq i \leq 2^n$ *where i is even.*

**Theorem 6.** [17] *Let* $G$ *be a quasi-dihedral group,* $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$, *where* $n \geq 3$. *If* $G$ *acts on* $\Omega$ *by conjugation, then* $|\Omega| = 3$, *given as follows: There are two elements in the form of* $(a^{2^{n-1}}, a^{2^{n+1}i} b), 0 \leq i \leq 2^n$, *and one element is in the form of* $(a^{2^{n-1}} b, b)$.

## 3. Main Results

This section introduces the main results, where the generalized non-commuting graph is presented, followed by result on the generalized non-commuting graph of dihedral groups, semi-dihedral groups and quasi-dihedral groups.

### 3.1 Generalized Non-Commuting Graph of Finite Groups

In this section, a new graph called the generalized non-commuting graph is introduced. The definition of the generalized non-commuting graph is given in the following.

**Definition 3.** Suppose $G$ be a finite non-abelian group, and $\Omega$ be a non-empty subset of $G \times G$. Then the generalized non-commuting graph denoted by $\Gamma_{\Omega}^{G \cdot N}$ is a graph whose vertices are non-central elements of subsets of $G$ i.e. $|V(\Gamma_{\Omega}^{G \cdot N})| = |\Omega| - |A|$, where $A = \{\omega \in \Omega : g\omega = \omega g : g \in G\}$. Two vertices $w_1, w_2 \in \Omega$ are adjacent if $w_1 w_2 \neq w_2 w_1$ for all $w_1, w_2 \in \Omega$

The following lemma illustrates the case that the generalized non-commuting graph is null.

**Lemma 1.** *Let* $G$ *be an abelian group. Then the generalized non-*

*commuting graph is null.*

**Proof.** Since $G$ is an abelian group, thus $Z(G) = G$. Therefore, $|V(\Gamma_{\Omega}^{G\cdot N})| = |\Omega - A|$. Thus $|V(\Gamma_{\Omega}^{G\cdot N})| = 0$ since $|\Omega| = |A|$. Hence, is $\Gamma_{\Omega}^{G\cdot N}$ null.

In the following, let $\Omega$ be the non-empty subset of $G \times G$ of size two that is in the form of $(a, b)$, where $a, b$ commute and $|a| = |b| = 2$.

The following theorem proves the connectivity of the non-commuting graph.

**Theorem 7.** *The generalized non-commuting graph is a connected graph.*

**Proof.** From [15], $\Gamma_{\Omega}^{G\cdot C}$ consists of $\frac{n}{2}$ complete components of $K_3$ graph, where two vertices $w_1, w_2$ belong to the same component if $w_1w_2 = w_2w_1$. Now, let $w_1w_2 \in \Gamma_{\Omega}^{G\cdot N}$ such that $w_1w_2 \neq w_2w_1$, i.e. $w_1$ and $w_2$ are connected by an edge. From the definition of $\Gamma_{\Omega}^{G\cdot C}$, it is very clear that $w_1, w_2$ belong to two different components of $\Gamma_{\Omega}^{G\cdot C}$. Suppose $w_3 \in \Gamma_{\Omega}^{G\cdot N}$. If $w_3 \in w_1$ component of $\Gamma_{\Omega}^{G\cdot C}$, then $w_3w_2 \neq w_2w_3$. If $w_3 \in w_1$ component of $\Gamma_{\Omega}^{G\cdot C}$, then $w_3w_1 \neq w_1w_3$. Also, if $w_3 \notin w_1w_2$ components of $\Gamma_{\Omega}^{G\cdot C}$, then $w_3w_2 \neq w_2w_3$, $w_3w_1 \neq w_1w_3$. In either case $w_3$ is connected by an edge, hence $\Gamma_{\Omega}^{G\cdot C}$ is a connected graph.

### 3.2 The Generalized Non-Commuting Graph of Dihedral Groups, Semi-dihedral Groups and Quasi-dihedral Groups

In this section, we compute the generalized non-commuting graph of dihedral groups of order $2n$ where $n$ is even, the generalized non-commuting graph of semi-dihedral groups and quasi-dihedral groups. We begin with the generalized commuting graph of dihedral groups.

**Theorem 8.** *Let $G$ be a dihedral group of order $2n$ where $n$ is even. Let $\Omega$ be a non-empty subset of $G \times G$ Then the generalized non-commuting graph*

$\Gamma_{\Omega}^{G \cdot N}$ *is a K-regular graph, where* $K = |V(\Gamma_{\Omega}^{G \cdot N})| - 3$ *and* $E(\Gamma_{\Omega}^{G \cdot N}) = \frac{|V(\Gamma_{\Omega}^{G \cdot N})|(|V(\Gamma_{\Omega}^{G \cdot N})| - 1)}{2} |V(\Gamma_{\Omega}^{G \cdot N})|.$

**Proof.** According to Theorem 4, the number of elements of $\Omega$ is equal to $\frac{3n}{2}$. Based on Definition 3, $|V(\Gamma_{\Omega}^{G \cdot N})| = \frac{3n}{2}$. Since two vertices $w_1, w_2 \in \Omega$ are adjacent if $w_1 w_2 \neq w_2 w_1$, thus the vertices which are in the form of $(a^i b, a^{i+\frac{n}{2}} b)$ and the vertices $(a^j b, a^{j+\frac{n}{2}} b)$ $0 \leq i, j \in 2n$ are adjacent. The vertices that are in the form of $(a^i b, a^{i+\frac{n}{2}} b)$ are adjacent to the vertices $(a^j b, a^{\frac{n}{2}}), 0 \leq i, j \in 2n,$ where $i \neq j$. The vertices which are in the form of $(a^i b, a^{i+\frac{n}{2}} b)$ and the vertices $(a^j b, a^{\frac{n}{2}} b), 0 \leq i, j \in 2n, i \neq j$ are joined by edges. The vertices that are in the form of $(a^j b, a^{\frac{n}{2}})$ are adjacent to the vertices $(a^j b, a^{j+\frac{n}{2}} b), 0 \leq i, j \nleq 2n$ where $i \neq j$. The vertices which are in the form of $(a^i b, a^{\frac{n}{2}})$ and the vertices $(a^{j+\frac{n}{2}} b, a^{\frac{n}{2}}), 0 \leq i, j \leq 2n, i \neq j$ are connected. The vertices that are in the form of $(a^i b, a^{\frac{n}{2}}), (a^j b, a^{\frac{n}{2}}),$ $0 \leq i, j \nleq 2n, i \neq j$ are adjacent to each other. The vertices which are in the form of $(a^{i+\frac{n}{2}} b, a^{\frac{n}{2}})$ are adjacent to the vertices $(a^j b, a^{j+\frac{n}{2}} b), 0 \leq i, j \leq 2n,$ where $i \neq j$. The vertices that are in the form of $(a^{i+\frac{n}{2}} b, a^{\frac{n}{2}})$ and the vertices $(a^{j+\frac{n}{2}} b, a^{\frac{n}{2}}), 0 \leq i, j \leq 2n, i \neq j$ are adjacent, and the vertices which are in the form of $(a^{i+\frac{n}{2}} b, a^{\frac{n}{2}})$ are connected to the vertices $(a^j b, a^{j+\frac{n}{2}}),$ $0 \leq i, j \leq 2n,$ where $i \neq j$. If $\Gamma_{\Omega}^{G \cdot N}$ is a complete graph, then degree of each vertex in $\Gamma_{\Omega}^{G \cdot N}$ is $\deg(\Gamma_{\Omega}^{G \cdot N}(w)) = |V(\Gamma_{\Omega}^{G \cdot N}(w))| - 1,$ where $w \in \Omega$. Since the generalized commuting graph of Theorem 1 consists of $\frac{n}{2}$ complete components of $K_3$ thus it is clear that each vertex in $\Gamma_{\Omega}^{G \cdot C}$ has degree two, it follows that $\deg(\Gamma_{\Omega}^{G \cdot N}(w)) = |V(\Gamma_{\Omega}^{G \cdot N}(w))| - 1 - 2 = |V(\Gamma_{\Omega}^{G \cdot N}(w))| - 3$ Thus $\Gamma_{\Omega}^{G \cdot N}$ is regular since all vertices have the same degree i.e. $|V(\Gamma_{\Omega}^{G \cdot N}(w))| - 3$. Since $\Gamma_{\Omega}^{G \cdot C}$ consists of $\frac{n}{2}$ complete component of $K_3$, hence $E(\Gamma_{\Omega}^{G \cdot C}) = \frac{3n}{2} = V(\Gamma_{\Omega}^{G \cdot N})$. Thus $E(\Gamma_{\Omega}^{G \cdot N}) \frac{|V(\Gamma_{\Omega}^{G \cdot N})||(V(\Gamma_{\Omega}^{G \cdot N})| - 1)}{2}$ $- |V(\Gamma_{\Omega}^{G \cdot N})|$, as claimed.

**Remark 1.** If $G$ is a dihedral group of order $\frac{n}{2}$ where $n$ is odd, then the generalized non-commuting graph, $\Gamma_{\Omega}^{G \cdot N}$ cannot be computed since there is not any subset of a set $\Omega$ in which $|a| = |b| = 2$ and $ab = ba$, as the center of $G$ in this case is trivial.

Now, the generalized non-commuting graph of semi-dihedral groups and the generalized commuting graph of quasi-dihedral groups.

**Theorem 9.** *Let $G$ be a semi-dihedral group, $G \cong \langle a, b : a^{2^n} = b^2 = 1, ab = ba^{2^{n-1}-1} \rangle$, where $n \geq 3$. Let $\Omega$ be a non-empty subset of $G \times G$. Then the generalized non-commuting graph $\Gamma_{\Omega}^{G \cdot N}$ is a $K$-regular graph, where $K = |V(\Gamma_{\Omega}^{G \cdot N})| - 3$ and* $\frac{|V(\Gamma_{\Omega}^{G \cdot N})|(|V(\Gamma_{\Omega}^{G \cdot N})| - 1)}{2}$ $|V(\Gamma_{\Omega}^{G \cdot N})|$.

**Proof.** According to Theorem $5, |\Omega| = 3(2^{n-2})$. Thus, the number of vertices in is $3(2^{n-2})$. Using the vertices adjacency of $\Gamma_{\Omega}^{G \cdot N}$, the vertices which are in the form of $(a^j b, a^{j+2^{n-1}} b)$, $0 \leq j \leq 2^{n-1}$ are adjacent to the vertices $(a^j b, a^{j+2^{n-1}} b)$, $0 \leq j \leq 2^{n-1}$ where $i \neq j$. The vertices that are in the form of $(a^i b, a^{i+2^{n-1}} b)$, $(a^j b, a^{2^{n-1}})$, $0 \leq i, < 2^{n-1}$, $i \neq j$ are adjacent to each other. The vertices which are in the form of $(a^i b, a^{i+2^{n-1}} b)$, $0 \leq i \leq 2^{n-1}$ are connected to the vertices $(a^{j+2^{n-1}} b, a^{2^{n-1}})$, $0 \leq i \leq 2^{n-1}$, where $i = j$. The vertices that are in the form of $(a^i b, a^{2^{n-1}} b)$, $0 \leq i \leq 2^{n-1}$, are adjacent to the vertices $(a^j b, a^{j+2^{n-1}} b)$, $0 \leq i, j \leq 2^{n-1}$, where $i = j$. The vertices which are in the form of $(a^i b, a^{2^{n-1}})$, $(a^i b, a^{2^{n-1}})$, $0 \leq i, j < 2^{n-1}$, $i \neq j$ are adjacent to each other. The vertices that are in the form of $(a^i b, a^{2^{n-1}})$, are adjacent to the vertices $(a^{j+2n-1} b, a^{2^{n-1}})$, where $0 \leq i, j < 2^{n-1}$ and $i = j$. The vertices which are in the form of $(a^{i+2^{n-1}} b, a^{2^{n-1}})$, $(a^j b, a^{j+2^{n-1}} b)$, $0 \leq i, j < 2^{n-1}$, $i \neq j$ are adjacent to each other. The vertices that are in the form of $(a^{i+2n-1} b, a^{2^{n-1}})$ are adjacent to the vertices $(a^j b, a^{2^{n-1}})$, where $0 \leq i, j < 2^{n-1}$, $i \neq j$. The vertices that are in the form of $(a^{i+2n-1} b, a^{2^{n-1}})$, $0 \leq i \leq 2^{n-1}$ are connected to the vertices $(a^{j+2n-1} b, a^{2^{n-1}})$, $0 \leq i, j \leq 2^{n-1}$ where $i = j$. Since the degree of any vertex in any complete graph is $\deg(\Gamma_\Omega^{G \cdot N}(w)) = | V(\Gamma_\Omega^{G \cdot N}(w)) | - 1$ and since $\Gamma_\Omega^{G \cdot C}$ of Theorem 2 consists of three complete components of $K_3$ and each vertex in $\Gamma_\Omega^{G \cdot C}$ has degree two, it follows that $\deg(\Gamma_\Omega^{G \cdot N}) = | V(\Gamma_\Omega^{G \cdot N}) | - 1 - 2$ $= | V(\Gamma_\Omega^{G \cdot N}) | - 3$. Since $\Gamma_\Omega^{G \cdot C}$ consists of $2^{n-2}$ complete component of $K_3$ thus $E(\Gamma_\Omega^{G \cdot N}) = 3(2^{n-2}) = V(\Gamma_\Omega^{G \cdot N})$. Therefore, $E(\Gamma_\Omega^{G \cdot N})$ $= \frac{| V(\Gamma_\Omega^{G \cdot N}) | (| V(\Gamma_\Omega^{G \cdot N}) | - 1}{2} - | V(\Gamma_\Omega^{G \cdot N}) |$, as claimed.

**Theorem 10.** *Let $G$ be a quasi-dihedral group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$. Let $\Omega$ be a non-empty subset of $G \times G$. Then $\Gamma_\Omega^{G \cdot N} = K_e$.*

Proof Based on Theorem 6, the number of elements of $\Omega$ is three.

According to Definition 3, 3, $|V(\Gamma_{\Omega}^{G\cdot N})| = |\Omega - A|$, thus $|V(\Gamma_{\Omega}^{G\cdot N})| = 3$. By the vertices adjacency of the generalized non-commuting graph, the vertices $w_1, w_2$ are connected if $w_1 \cdot w_2 \neq w_2 \cdot w_1$. Hence, $\Gamma_{\Omega}^{G\cdot N}$ consists of three isolated vertices represented as $(a^{2n-1}, a^{2n-1}ib), 0 \leq i \leq 2^n$ and $(a^{2n-1}b, b)$. Hence, $\Gamma_{\Omega}^{G\cdot N}$ is empty.

## 4. Conclusion

In this paper, the non-commuting graph is successfully extended by defining the generalized non-commuting graph. Some graph properties related to the generalized non-commuting graph are also provided. Besides, the generalized non-commuting graph of dihedral groups, semi-dihedral groups and quasi-dihedral groups are determined.

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