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REAL LIFE APPLICATIONS OF CATENARY AND MATHEMATICAL MODELLING OF A ROPE CATENARY CURVE

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Abstract

In this paper we have studied about catenary curve, its various structures in real life and how it is different from parabola. The various terminology linked with catenary is also discussed for giving a deep insight in catenary curve. The basic mathematical model of catenary curve and the relation between various parameters of catenary is studied. Further the mathematical modelling of a rope catenary curve with equal and unequal supporting rods is studied. We have considered this, first on equal supporting rods and then on unequal rods to find the minimum distance between the supporting rods so that the rope does not touch ground. We have calculated these results only when height of the supporting rods and length of rope is known. The result obtained can be implement in various other fields like bridges, electric wire etc. Also, we perceive how this mathematical equation of catenary astonishingly connects various applications of catenary in real life like bridge, igloo, electric wire, households etc.

1. Introduction

Mathematics is all around us like it is within the petals of a flower, honeycomb of a beehive, etc. thus it is all around us within the world, however, let us simply come back little additional near to us like various things in our house. What if I take my shoestring and if I hold it between my hands, it forms a curve? This curve is thought as catenary shape curve, that is completely different from parabolic conic section. It is a curve of own kind

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that has its mathematical illustration. We can additionally build that curve using household things like the thread or rope etc. The "Catenary" word is acquired from the Latin word referred to as "chain" because of the form that curved shape forms is simply like a chain, or we can say a rope hanging freely below its own weight. The straightforward and delightful piece of arithmetic may be seen everyplace from the strands of a spider's internet to the wires in associate overhead power cable. But why we tend to see this curve showing once more and once more within the word? Answer to the current question will simply justify with the assistance of bubbles. As we all know bubbles form good sphere to attenuate their physical phenomenon of surface tension. However, what if we tend to wish to form a bubble between two rings, it cannot make a sphere rather than that it makes a curve form. To attenuate its physical phenomenon during this case, it makes a curved shape. Thus, like bubbles, a series desires to attenuate its tension so then it forms a curved shape that is not a conic section. Parabola is $y = ax^2 + bx + c$ and on the opposite hand catenary shape is $y = a \cos h(x/a)$ or $y = a(e^{x/2} + e^{x/2})/2$.

The catenary shape curve modifies once the length of chain changes or whenever distance between the two ends changes. This modification may be categorical with the formula $y = a \cos h(x/a)$ however cannot be expressed by the formula of conic section $y = ax^2 + bx + c$.

Griva and Vanderbei [4] thought of the form of a dangling chain, which, in equilibrium, minimizes the P. E. of the chain. Additionally, to the tutorial aspects of this paper, they additionally stressed on the importance of certain modeling problems like complex vs. nonconvex formulations of given drawback. They have given many models of the problem and presented variations within the range of iterations and solution time. Kim et al. [8] recommended a geometrical model and constant conductor model by using mean radius of five conductors within the curve system. Also, they calculate demanded parameter values within the model. By using those, mean radius of five conductors were analyzed by applying the equivalent technique referred as condensed joint matrix.

Yuhung and Chanping [7] planned a completely unique bending moment expression of an oversized sag curve. The expression was derived from the sag identified using bending moment equations, and an answer was found by applying the WKB technique (Wentzel-Kramers-Brillouin method) to beat the complicated downside of boundary layers. Consequently, an easy resolution of assorted mechanical properties during a cable with bending stiffness and huge sag might be obtained. Liu et al. [9] reviewed thoroughly the four aspects on the catenary analysis of high-speed railway, specifically the answer ways for curve equilibrium state, the dynamic modeling ways of curve, non-contact detection ways of curve, and therefore the static and dynamic analysis ways of curve. Additionally, their recent advances are explained. For the low solution accuracy of the initial equilibrium state of curve, the structure finding technique with multi-objective constraint and nonlinear finite element method are introduced to solve the matter. For the catenary's dynamic modeling, considering the influence of environmental wind on the curve, environmental wind simulations and wind tunnel tests are used to acquire the mechanics coefficients and build the wind field on the curve for analysis of its wind vibration properties.

Benet et al. [2] conferred the idea for the mechanical calculation of the curve system during a railway, considering static wire equations. The target was to permit an increment in the speed in railways, by developing a correct mechanical calculation of the electrical wiring referred to as curve. Also, two algorithms for the calculation were developed. In the first one, the precise weight of the wire was considered, whereas in the other, the burden of the wire was estimated as an even horizontal load. Hayes et al. [6] thought about the interaction between a train mounted mechanical device and a railway overhead line, presenting results that might be used to cut back the value of putting in overhead electrification, for instance, by reducing the requirement for costly bridge reconstruction. During this study, the influence of a spread of contact wire gradients on the contact force between the overhead line and mechanical device has been studied by applying a valid finite element model.

Wu et al. [5] addresses the process of sliding electrical contact of pantograph by analyzing key characteristics together with contact resistance, temperature distribution and microstructure. The influence of electrical current on contact resistance was understood. Bai et al. [1] describes the main technologies for the appliance of catenary-powered ship, together with the planning of power-receiving devices and therefore the technology for preventing ship offsets. It additionally makes recommendations supported national policies and therefore the status of catenary-powered ship in the country.

Wang et al. [10] introduced a reliability assessment technique for highspeed railway curve system considering weather circumstances. The atmospheric phenomenon is classified according to IEEE standard, and therefore the failure rate model of curve element is made under three weather circumstances. Then the failure rate and repair rate under totally different weather circumstances are well chosen as random fuzzy variables. Credibility theory is applied to investigate the influence of uncertainties on the reliability assessment of curve system. Finally, fault tree analysis technique is introduced to calculate the reliability indices of the curve system. Case study shows the projected technique achieves reliability assessment for curve of high-speed railroad considering the influence of weather circumstances, and therefore the reliability indices under totally different weather circumstances are obtained. Bruni et al. [3] introduced the foremost representative results from a benchmark exercise addressing the simulation of pantograph-catenary interaction that concerned ten simulation software codes from universities and analysis centers located in nine totally different countries, Especially, the utilization of the finite element technique (FEM) to outline the curve model is predominant and therefore the penalty technique is the most often used approach to represent the pantograph-catenary sliding contact. The modelling of dropper weakening is recognized to have substantial influence on the accuracy of the simulation technique; however, variations exist among the codes in terms of formulation of the nonlinear dropper model.

In this paper we have studied the mathematical modelling of a rope catenary curve with equal and unequal supporting rods. We have considered this, first on equal supporting rods and then on unequal rods to find the minimum distance between the supporting rods so that the rope does not touch ground and the clothes does not get filthy.

2. Terminology linked with catenary

a. Catenary arc: Figure 1 shows an inverted curve arc as a catenary arc. It is sort of a subject arc that has been used since the traditional time to create various things like vaults, buttresses, etc. It is additionally referred to as by the name of a weighted catenary curve.

Figure 1: Catenary Arc**.**

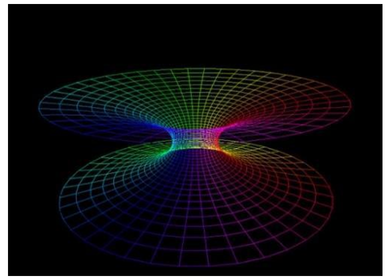

Figure 2. Catenoid.

b. Catenoid: Figure 2 shows the catenary curve as Catenoid. To find the minimum expanse of the curved shape curve, it is revolved regarding associate degree axis and this kind of surface is thought as Catenoid. For instance, once we dip 2 rings in a bubble resolution, they create a catenoid to scale back the physical phenomenon of surface tension. Mathematician Leonhard Euler describes this in year 1744.

c. Tractrix: Figure 3 and 4 shows the catenary curve as Tractrix. Tractrix could be a curled spirally manufactured from catenary curve. In the year 1670 Claude Perrault was the one who discovered it. It has several applications like horn speaker system etc. after we quote tractrix there is another term referred to as Pseudosphere that is outlined as the surface of revolution of tractrix as we will see within the above figure.

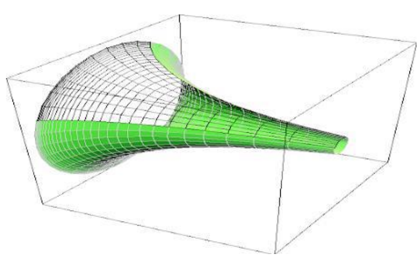

Figure 3. Tractrix.

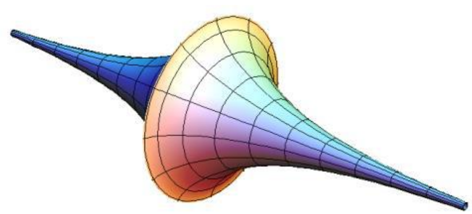

Figure 4. Tractrix as Pseudosphere.

3. Mathematical formation and analysis of the model

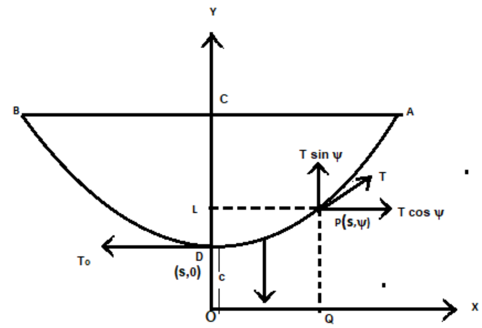

Figure 5. Geometrical representation of catenary with direction of forces.

3.1 Basic mathematical model for Catenary curve ψ

Some terms associated with curved shape in the figure 5 are:-

Span: - The straight length AB is termed as span.

Vertex: - lowest point D is termed as the vertex.

Sag: - vertical displacement CD is termed as the sag.

Parameter of catenary: - the peak of the lowest point from a set point is termed as parameter.

Let a point O vertically below D is taken into an account from the fastened point or the origin of the curve. Let $OD = c,$ then c is the parameter of curve. The curve consists of 2 equal components i.e., Arc BC and Arc AD. Let the cartesian system of the axis outlined at O. Then from the figure 5, the horizontal line is OX and therefore the vertical line is OY. Let the cartesian coordinates be (x, y) Then $OQ = x$ and $PQ = y$ (OX is additionally referred to as directrix of catenary, i.e., the chosen coordinate axis).

Let initial notice the intrinsic equation of the curved shape, so we tend to begin with intrinsic coordinates $(s, \psi),$ where s is the length of the arc and ψ is the angle created by the tangent with x axis. The intrinsic equation of the curved shape is the relation between intrinsic coordinates $(s, \psi),$ of the point P in the figure (5) on the curved shape. This allow us to assume that the vertex D is the point $(s = 0)$ or the pole. Let $DP = s$ and ψ is the angle between the tangents at P makes with the horizontal line.

There from figure (5) the three forces acting of on the arc DP are:

(i) Tension T towards A

(ii) Weight adequate to W_s, vertical downward at the center purpose of the arc DP,

(iii) Strain T_o at the bottom purpose D in the direction away from it.

So, from the equilibrium of forces

$$T \cos \psi = T_o \tag{1}$$

$$T \sin \psi = W_s \tag{2}$$

From (1) and (2) we get

$$\tan \psi = W_s / T_o$$

The horizontal strain at D is $T_o = Wc$ always, Hence, we get $\tan \psi = W_s / Wc \ s = c \tan \psi,$ which is the equation of catenary in intrinsic form or $s = c \dfrac{dy}{dx}$ (3)

Differentiate (3) with respect to x we get: --

$$\frac{ds}{dx} = c \frac{d^2 y}{dx}$$

Now solving the above equation and using the initial condition $x = 0,\ y = 0$ and $x = 0,\ y = c$ we get the cartesian equation of catenary

$$y = c \cos hx / c \tag{4}$$

or

$$y = c(e^{x/c} + e^{-x/c})/2 \tag{5}$$

We can derive the following relation between different parameter of catenary such as:-

- Relation between x and s

$$s = c \cos hx / c \tag{6}$$

- Relation between x and ψ

$$x = c \log(\tan\psi + \sec\psi) \tag{7}$$

▪ Relation between y and ψ

$$y = c \sec\psi \tag{8}$$

▪ Relation between y and s

$$y^2 = c^2 + s^2 \tag{9}$$

3.2 Rope catenary curve with equal and unequal supporting rods

Have you ever noticed once that if we have to suspend cloths outsides to dry them out, we are doing that on what? In fact, on the rope that is tied from each the aspect. Once you place cloths on the rope the rope bend to create the curved shape arc i.e., catenary curve, that rely on the quantity of cloths. A lot of variety of cloths, a lot of bends within the rope and vice versa. So, here we will investigate the concept of rope hanging of our cloths along with the curve on midpoint to find the least distance between the supporting rods, so our cloths do not touch the ground and get filthy. Let us assume that each cloth has the same height and the distance of the supporting rods be x from the midpoint of y axis as we can see in the figure 6. Also, in this paper we have deal with the limiting case in which the curve is tangent to the coordinate axis (the ground) and thus we take off c to reach the equation we are going to use in the problem i.e.

$$y = c \cos h(x/c) - c \tag{10}$$

We will consider this, first on equal rods and then on unequal rods.

3.2.1 For equal supporting rods:

Let the length of the rope be α and two supporting rods having equal height of β (figure 6).

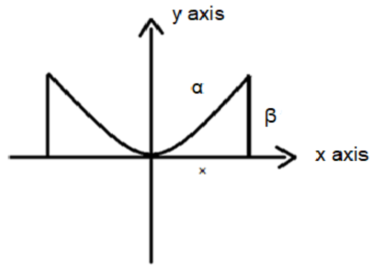

Figure 6. The cloth hanging rope reflecting the catenary curve when heights of supporting rods are equal.

First, we tend to contemplate the equation that determines half the length of the rope using the arc length formula for the curve represented by the operate in Equation

$$\int_0^x \sqrt{1 + (\frac{dy}{dt})^2}\, dt = \alpha/2 \tag{11}$$

Second, we tend to contemplate the equation that characterize the peak of the supporting rods or the height of the rope β at distance x from the center

$$y(x) = \beta \tag{12}$$

Now solving equation (10), (11) and (12), we get

$$c \sin h(x/c) = \alpha/2 \tag{13}$$

$$c \cos h(x/c) = \beta + c \tag{14}$$

Now solving equation (13) and (14) using the hyperbolic identity and taking $\alpha = 100$ units and $\beta = 20$ units, we have

$$((20 + c)/c)^2 - (50/c)^2 = 1 \text{ or } c = 5.25 \tag{15}$$

Substitution the value of c in equation (13), we get $x = 15.48$ units So, the distance between the rods when the rope touches the ground is $2x = 31$ units and hence accordingly the distance between the rods can be adjusted.

3.2.2 For unequal supporting rods:

Let us consider that the length of the rope be α_1 and two supporting rods having heights of β_1 and β_2 respectively. Let the distance of the rods be x_1 and x_2 respectively from y axis as we can see in the figure 7.

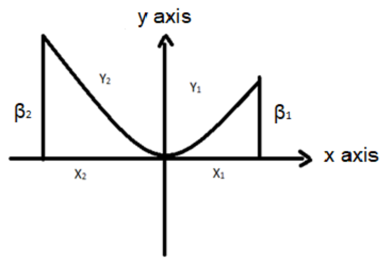

Figure 7. The cloth hanging rope reflecting the catenary curve when heights of supporting rods are unequal.

Now from equation (13) we can have

$$c \sin h(x_1/c) = y_1 \tag{16}$$

$$c \sin h(x_2/c) = y_2 \tag{17}$$

Hence the total length of the rope is $y_1 + y_2 = \alpha_1$ (18)

Height of supporting rod will be:-

$$c \cos h(x_1/c) - c = \beta_1 \tag{19}$$

$$c \cos h(x_2/c) - c = \beta_2 \tag{20}$$

Solving equation (16)-(20), using hyperbolic identity, we have

$$((\beta_1 + c)/c)^2 - (y_1/c)^2 = 1 \tag{21}$$

$$((\beta_2 + c)/c)^2 - (y_2/c)^2 = 1 \tag{22}$$

Solving equation (18), (21) and (22), we get

$$c = (\beta_1 + \beta_2)[\alpha_1^2 - (\beta_1 + \beta_2)^2] \pm 2\alpha_1 \sqrt{}[\beta_1\beta_2\{\alpha_1^2 - (\beta_1 + \beta_2)^2\}]/2(\beta_1 + \beta_2)^2.$$

If we take $\beta_1 = 120$, $\beta_2 = 130$ and $\alpha_1 = 500$, we get $c \approx 187.55$ (23)

now after getting the value of c we can find y_1 and y_2

$$y_1 = \sqrt{}(\beta_1^2 + 2\beta_1 c) = 243.74 \text{ units} \tag{24}$$

$$y_2 = \sqrt{(\beta_2^2 + 2\beta_2 c)} = 256.24 \text{ units} \tag{25}$$

after this we can find the value of x_1 and x_2:-

$$x_1 = c \ln(\beta_1 + y_1 + c)/c = 202.219 \text{ units} \tag{26}$$

$$x_2 = c \ln(\beta_2 + y_2 + c)/c = 209.72 \text{ units} \tag{27}$$

Hence the minimum distance between the supporting rods should be greater than $x_1 + x_2 = 411.93$ units

4. Other Catenary Curves where Mathematical models can be Established

The mathematical results calculated in case of supporting rods distance can be applied in various other real life catenary curves like igloo, electric wire, bridge, anchor rod and spider web etc.

a. Igloo: Figure 8 shows that Igloo is one of the examples of the catenary which is made up of the more complexity of the catenary curve. It holds the position of the catenary curve, due to the uniform thickness of the catenary curve the igloo become short which not only helps in retaining its best equilibrium between the height and the diameter but also avoid the threat of fall in under the mass of compacted snow.

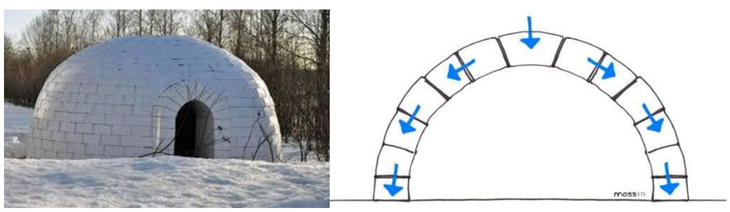

Figure 8. Catenary curve in igloo.

b. Electric wire: Electric wire is another application of catenary curve as we all have seen the electrical wire when hanged in the pole which is not completely linear, or we can say straight but a little low which forms the catenary curve this happens because expansion of the wire (figure 9). This expansion of the electric wire happens because of the gravity and heat produce in the transaction through the medium. Sunlight also plays an important role due to which metal in the electric wire elongates and thus we get our catenary curve.

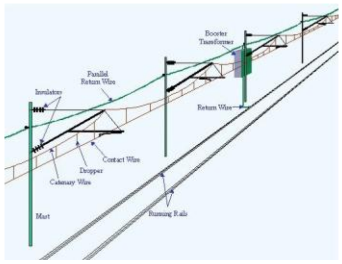

Figure 9. Catenary curve in electric wires.

Figure 10. Catenary curve in bridges.

c. Bridge: Either we talk about hanging bridges or simple bridges (figure 10), the one concept which remains constant is that they both are form with the help of catenary curve. As we seen many times around us that when a chain is tired with poles it never forms linear or straight line but a slight bend in it which forms the curve. The same principle is used in the bridge scenario when the wire is supported with the fixed set and with the presence of the gravity force it forms the catenary curve.

d. Anchor rode: An anchor rode is nothing, but a long line chain connected it can be overcome by catenary made to the gravitational force so all the marine objects such as boat, gliding wind turbines, docks etc. can be used to anchored to the seabed (figure 11). So, whenever the rode gets straight out the chain started to bend and forms the catenary arc so there is no resistance during the process. But to maintaining the catenary shape is not that easy for that there is a support requirement of the other chain too which generally depends on the power of the water flow and the holding capacity of the chains.

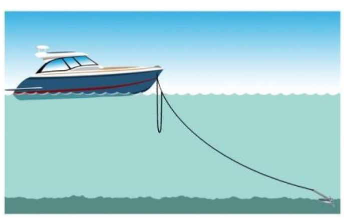

Figure 11. Catenary curve in anchor rod.

Figure 12. catenary curve in spider's web.

e. Spider's web: In the figure 12 you cannot demise the fact that even nature possess spider's web in the shape of the catenary. The web is made up of the silk which is produced by the tip of abdomen of the spider which does not help the spider to lay the eggs safely but also help them to catch their prey. Due to the catenary shape when the prey got stuck to the web it does not broke the web but remain supported in its own weight because of the property of catenary which minimize the tension of the web same happen when eggs got laid.

5. Results and Conclusion

We studied that mathematical equation of catenary plays so important part in our life. We came to know that catenary is Latin word called chain because of the shape it forms under its own weight. People may get confused sometimes and assume that shape of catenary is nothing but a parabola but that is not true as the curve changes when the length of the curve or the distance between the two end changes, which can only be expressed with the formula $y = c \cos h(x/c)$ but cannot be expressed by the formula of parabola $(y = ax^2 + bx = c)$. We have studied some other term which is related to the catenary such as catenary arc, catenoid, tractrix many more. We have studied the mathematical modelling of a rope catenary curve with equal and unequal supporting rods so that hanging cloths does not get spoil due to touching on the ground. In the first case we have considered the height of supporting rods as 20 units and the length of the rope as 100 units. In the second case we have considered the height of supporting rods as 120 units and 130 units and the length of the rope as 500 units. The analytical result has been derived and using it the minimum distance between the supporting rods is calculated in both the cases. The same results can be applied in various application of catenary curve such as igloo, electrical wire, bridge, anchor rode, households, spider's web etc.

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