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TOTAL COLORING OF LINE GRAPH AND SQUARE GRAPH FOR CERTAIN GRAPHS

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Abstract

A total coloring of a graph is a coloring to the elements (vertices and edges) of the graph G, for which any adjacent vertices or edges and incident elements are colored differently. The total chromatic number of G is the minimum number of colors that needed in a total coloring. In this paper, we have determined the total chromatic number for $L(P_n^+)$, $L(T_n)$, $(P_n^+)^2$ and T_n^2.

1. Introduction and Preliminaries

Let us assume all graphs are finite, simple and undirected graph G with the vertex set $V(G)$ and the edge set $E(G)$ respectively. Let $f : V(G) \cup E(G) \to C$ be a total coloring of G, where C is set of colors and satisfies the given conditions

(a) $f(a) \neq f(b), \forall\ a, b \in V(G)$ are any two adjacent vertices

(b) $f(e_1) \neq f(e_2), \forall\ e_1, e_2 \in E(G)$ are two any adjacent edges and

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(c) $f(a) \neq f(e), \forall\ e \in E(G)$ is incident with any vertex $a \in V(G)$

The minimum number of colors needed in a total coloring of G is called the total chromatic number of G, and it is denoted by $\chi''(G)$.

In graph theory the famous conjecture i.e. the total coloring conjecture was independently introduced by Behzad [1] and Vizing [5] declared that for any simple graph G can be total colored with $\leq \Delta(G)+2$. Rosenfeld [3] and Vijayaditya [4] verified the total coloring conjecture, for any graph G with $\Delta \leq 3$. Muthuramakrishnan et al. [2] proved that the total chromatic number of line graph of star graph and square graph of bistar graph. In a simple connected graph G, the square [2] of G is the graph G^2 obtained from joining the edges to G, if $d(u, v) = 2$ in G, where $d(u, v)$ is distance between any two vertices. The line graph [2] of G, denoted by $L(G)$, is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent, where the corresponding edges are adjacent in G. The Twig graph T_n is obtained from a path by adding two pendant edges exactly to each internal vertex of the path. The Comb graph is obtained by joining one pendent edge to each vertex of a path and it is denoted by P_n^+. In this work, we investigate the total chromatic number for $L(P_n^+)$, $L(T_n)$, $(P_n^+)^2$ and T_n^2 respectively.

2. Main Results

Theorem 2.1. *For any* $n \geq 4$, $\chi''(L(P_n^+)) = 5$.

Proof. Let $V(P_n^+) = \{u_k, v_k : 1 \leq k \leq n\}$ and

$$E(P_n^+) = \{s_k, v_k : 1 \leq k \leq n\} \cup \{w_k = v_k v_{k+1} : 1 \leq k \leq n-1\}$$

By the construction of line graph, the vertex and edge sets of $L(P_n^+)$ are given by,

Let $V(L(P_n^+)) = \{u_k, v_k : 1 \leq k \leq n\}$ and

$$E(L(P_n^+)) = \{s_k, w_k, w_k s_{k+1} : 1 \leq k \leq n-1\} \cup \{w_k w_{k+1} : 1 \leq k \leq n-2\}$$

Construct a total coloring $f : V(L(P_n^+)) \cup E(L(P_n^+)) \to \{1, 2, 3, 4, 5\}$ as follows:

For $1 \le k \le n$,

$$f(S_k) = 4, \tag{2.1}$$

For $1 \le k \le n-1$,

$$f(w_k) = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 2, & \text{if } k \text{ is even} \end{cases} \tag{2.2}$$

$$f(s_k w_k) = \begin{cases} 2, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even} \end{cases} \tag{2.3}$$

$$f(w_k s_{k+1}) = 5 \tag{2.5}$$

We observe the equation (2.1) to (2.5), by the construction, $\chi''(L(P_n^+)) \le 5$.

Since $\Delta = 4$ and $\chi''(L(P_n^+)) \ge \Delta + 1 \ge 4 + 1 \ge 5$. Therefore $\chi''(L(P_n^+)) = 5$.

Thus *f* is total colored with 5 colors.

Theorem 2.2. For any $n \ge 4$, $\chi''(L(T_n)) = 7$.

Proof. Let $V(T_n) = \{u'_k, v'_k, w'_k : 1 \le k \le n-2\}$ and

$E(T_n) = \{u'_k, v'_{k+1}, w'_k v'_{k+1} : 1 \le k \le n-2\} \cup \{v'_k, v'_{k+1} : 1 \le k \le n-1\}$

By construction of line graph, the vertex set and edge set of $L(T_n)$ as given below:

Let $V(L(T_n)) = \{u_k, w_k : 1 \le k \le n-2\} \cup \{v_k : 1 \le k \le n-1\}$ and

$E(L(T_n)) = \{u_k v_k, u_k v_{k+1} u_k w_k, v_k w_k, w_k v_{k+1}, v_k v_{k+1} : 1 \le k \le n-2\}$

Define a total coloring $f : V(L(T_n)) \cup E(L(T_n)) \to \{1, 2, 3, 4, 5, 6, 7\}$ as follows:

For $1 \le k \le n-1$

$$f(v_k) = \begin{cases} 2, & \text{if } k \text{ is even} \\ 1, & \text{if } k \text{ is odd} \end{cases} \tag{2.6}$$

For $1 \leq k \leq n-2$,

$$f(u_k) = 3,\ f(w_k) = 6, \tag{2.7}$$

$$f(u_k v_{k+1}) = 5,\ f(v_k w_k) = 7, \tag{2.8}$$

$$f(u_k w_k) = 4,\ f(v_k v_k) = 6, \tag{2.9}$$

$$f(v_k v_{k+1}) = \begin{cases} 3, & \text{if } k \text{ is odd} \\ 4, & \text{if } k \text{ is even} \end{cases} \tag{2.10}$$

$$f(w_k v_{k+1}) = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 2, & \text{if } k \text{ is even} \end{cases} \tag{2.11}$$

From equation (2.6) to (2.11), by the construction, $\chi''(L(T_n)) \leq 7$. Since $\Delta = 6$ and $\chi''(L(T_n)) \geq \Delta + 1 \geq 6 + 1 \geq 7$. Therefore $\chi''(L(T_n)) = 7$. Thus f is total colored with 7 colors.

Theorem 2.3. *Let $(P_n^+)^2$ be the square graph of comb graph, then* $\chi''(P_n^+)^2 = 8$.

Proof. Let $V(P_n^+)^2 = \{u_k, v_k : 1 \leq k \leq n\}$ and

$$E(P_n^+) = \{u_k v_k, : 1 \leq k \leq n\} \cup \{v_k v_{k+1}, : 1 \leq k \leq n-2\}$$

$$\cup \{u_k v_{k+1}, v_k v_{k+1}, v_k u_{k+1} : 1 \leq k \leq n-1\}$$

Construct a total coloring $f : V(P_n^+)^2 \cup E(P_n^+)^2 \to \{1, 2, \ldots, 8\}$ as follows:

For $1 \leq k \leq n$,

$$f(v_k) = \begin{cases} 1, & \text{if } k = 4, 7, 10, 13, \ldots \\ 2, & \text{if } k = 5, 8, 11, 14, \ldots \\ 3, & \text{if } k = 6, 9, 12, 15, \ldots \end{cases} \tag{2.12}$$

$$f(u_k) = 4,\ f(u_k v_k) = 8, \tag{2.13}$$

For $1 \leq k \leq n-1$,

$$f(v_{k+1} u_k) = 6,\ f(v_k u_{k+1}) = 7 \tag{2.14}$$

$$f(v_k v_{k+1}) = \begin{cases} 4, & \text{if } k \text{ is odd} \\ 5, & \text{if } k \text{ is even} \end{cases} \tag{2.15}$$

For $1 \leq k \leq n-2$,

$$f(v_k v_{k+2}) = \begin{cases} 2, & \text{if } k = 4, 7, 10, 13, \ldots \\ 3, & \text{if } k = 5, 8, 11, 14, \ldots \\ 1, & \text{if } k = 6, 9, 12, 15, \ldots \end{cases} \tag{2.16}$$

From equation (2.12) to (2.16), by the construction, $\chi''(L(P_n^+)^2) \leq 8$. Since $\Delta = 7$ and $\chi''(P_n^+)^2 \geq \Delta + 1 \geq 7 + 1 = 8$. Therefore $\chi''(L(P_n^+)^2) = 8$. Thus f is total colored with 8 colors.

Theorem 2.4. *Let T_n^2 be the square graph of twig graph. Then $\chi''(T_n^2) = 11$.*

Proof. Let $V(T_n^2) = \{u_k, w_k : 1 \leq k \leq n-2\} \cup \{v_k : 1 \leq k \leq n\}$ and

$$E(T_n^2) = \begin{cases} \{u_k, v_{k+1}, w_k v_{k+1}, u_k v_k, u_k v_{k+2}, u_k w_k, v_k w_k, u_k v_k, v_k v_{k+2}, \\ w_k v_{k+1} : 1 \leq k \leq n-2\} \cup \{v_k v_{k+1} : 1 \leq k \leq n-1\} \end{cases}$$

Construct a total coloring $f : V(T_n^2) \cup E(T_n^2) \to \{1, 2, \ldots, 11\}$ as follows:

For $1 \leq k \leq n$,

$$f(v_k) = \begin{cases} 1, & \text{if } k = 4, 7, 10, 13, \ldots \\ 2, & \text{if } k = 5, 8, 11, 14, \ldots \\ 3, & \text{if } k = 6, 9, 12, 15, \ldots \end{cases} \tag{2.17}$$

For $1 \leq k \leq n-1$,

$$f(v_k v_{k+1}) = \begin{cases} 3, & \text{if } k = 4, 7, 10, 13, \ldots \\ 1, & \text{if } k = 5, 8, 11, 14, \ldots \\ 2, & \text{if } k = 6, 9, 12, 15, \ldots \end{cases} \tag{2.18}$$

For $1 \leq k \leq n-2$,

$$f(u_k) = 11,\ f(w_k) = 8, \tag{2.19}$$

$$f(u_k v_k) = 8,\ f(w_k v_{k+1}) = 9, \tag{2.20}$$

$$f(u_k w_k) = 1,\ f(v_k w_k) = 11, \tag{2.21}$$

$$f(w_k v_{k+1}) = 10,\ f(u_k v_{k+1}) = 7 \tag{2.22}$$

$$f(v_k v_{k+2}) = \begin{cases} 5, & \text{if } k = 4, 7, 10, 13, \ldots \\ 6, & \text{if } k = 5, 8, 11, 14, \ldots \\ 4, & \text{if } k = 6, 9, 12, 15, \ldots \end{cases} \tag{2.23}$$

$$f(v_k v_{k+2}) = \begin{cases} 4, & \text{if } k = 4, 7, 10, 13, \ldots \\ 5, & \text{if } k = 5, 8, 11, 14, \ldots \\ 6, & \text{if } k = 6, 9, 12, 15, \ldots \end{cases} \tag{2.24}$$

From equation (2.17) to (2.24), by construction, $\chi''(T_n^2)^2 \leq 11$. Since $\Delta = 10$ and $\chi''(T_n^+) \geq \Delta + 1 \geq 10 + 1 = 11$. Therefore $\chi''(L(T_n^2)) = 11$. Thus f is total colored with 11 colors.

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