Advances and Applications in Mathematical Sciences
Volume 20, Issue 10, August 2021, Pages 2279-2290

THE INFLUENCE OF AN ADDITIVE OUTLIER ON THE VARIANCE OF AN $MA(1)$ MODEL

R. SURESH and J. V. JANHAVI

Department of Statistics
Bangalore University
Bengaluru-560056, India
E-mail: sureshramaiah@gmail.com
sureshr@bub.ernet.in
jvjanhavi@gmail.com

Abstract

In this work, the effect of an Additive Outlier (AO) on the variance the residuals and series of a Moving Average model of order 1, $(MA(1))$, is algebraically derived. The expressions obtained are split (partitioned) in such a manner that one can see the contribution of an AO to the variance of the residuals and to the series. Through a simulation study, the contamination due to the AO in the variance of the series has been obtained for the model. This analysis reveals that the magnitude of the AO (δ), the sample size (n) and the sign of the parameter (θ) of the underlying model influences the variance of the residuals and the series. This fact is vital while drawing inferences from a time series data contaminated by an AO.

1. Introduction

Outliers can result for many external or internal reasons. Measurement (recording or typing) errors, classification mistakes in sampling or some non repetitive exogenous interventions can have effects in the form of outliers (isolated or patchy). Economic and business time series are sometimes subject to the influence of strikes, outbreaks of wars, sudden change in the market structure of some group of commodities, technical change or new equipment in a communication system, or simply unexpected pronounced changes in weather etc.

2010 Mathematics Subject Classification: Primary 62M10.
Keywords: Variance; Residuals; Moving average; Additive Outlier; Simulation Study.
Received March 1, 2021; Accepted March 9, 2021

There is abandunt literature which discusses the effects of outliers in Time series analysis. Its presence has deleterious effects on end results, like model misspecification, biased parameter estimation, and poor forecasts (Hillmer [4] and Ledolter [6] studied the effect of additive outliers on forecasts). Other works of same nature are especially on Sample Autocorrelation Function. Some of them are due to: Chan [2] who study on the Sample Autocorrelation Function (SACF) of the series. Further, Maronna et al. [7] have discussed the effect of a Patch of Additive Outliers on the first order SACF of the series. The same has been extended by Suresh [8] to higher orders.

Importance of the variance of the residuals and the variance of the time series are well known in Time Series Analysis. Some of the main instances where the variance used are in residual analysis (construction of band), testing of significance of parameters, Sample Autocorrelation Functions (SACFs) and Partial Sample Autocorrelation Functions (PACFs), model selection criteria like Akaike Information Criterion (AIC), Schwarz-Bayesian Information Criterion (BIC) and in forecasting (point, interval and density). Having seen the importance of variance in time series analysis, this makes it relevant to study the effect of outliers on the variance of the residuals and that of the series along the similar lines as that discussed on the SACF.

2. Variance of the $MA(1)$ model in the presence of an Additive Outlier

Having known the importance of the variances of the residuals and the series, the expressions for them in the presence of an AO are obtained in this section.

Let $\{X_t\}$ be a Moving Average process of order q, $MA(q)$, defined as

$$X_t = \theta(B) a_t, \tag{1}$$

where $\{a_t\}$ is a sequence of independent and identically distributed Gaussian variables with mean zero and variance σ_a^2 and the polynomial $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$ is a polynomial in B, which is the backshift operator such that $BX_t = X_{t-1}$ and $\theta(B) = 0$ have all roots outside the unit circle. The

$MA\ (q)$ model is then invertible. Further $\theta(B)$ is assumed to have all distinct roots. Outliers can take several forms in time series. The formal definitions and a classification of outliers in a time series context were first proposed by Fox [3]. One can refer to Tsay [9] to know more about other types of Outliers in Time Series. When outliers or structural changes occur, $\{X_t\}$ is disturbed and unobservable. We assume that the series $\{X_t\}$ as defined in (1) follows Box-Jenkins univariate time series model (Box et al. [1]). In this case, it is assumed that the observed series $\{Y_t\}$ follows the model

$$Y_t = f(t) + X_t, \tag{2}$$

where $f(t)$ a parametric is function representing the exogenous disturbances of X_t. In practice, $f(t)$ is specified by data analysts based on the substantive information of the disturbances and the process $\{Y_t\}$. It is assumed that $f(t)$ is of the form

$$f(t) = \delta_0 \frac{\omega(B)}{\delta(B)} I_t^T, \tag{3}$$

where

$$I_t^T = \begin{cases} 1 & if \quad t = T, \\ 0 & if \quad t \neq T, \end{cases} \tag{4}$$

is an indicator variable signifying the occurrence of a disturbance at the time point T, $\omega(B) = 1 - \omega_1 B - \omega_2 B^2 - \ldots - \omega_s B^s \ldots$ and $\delta(B) = 1 - \delta_1 B - \delta_2 B^2 - \ldots - \delta_r B^r$ are polynomials in B of degrees s and r, respectively, and δ_0 is a constant denoting the initial impact of the disturbance. The model in (2) turns out to be an Additive Outlier (AO) model in the special case of $f(t)$ obtained by taking $\delta_0 = \delta$ and $\frac{\omega(B)}{\delta(B)} = 1$ (3).

It is this type of outliers that affect a single observation X_T. After this disturbance, the series returns to its normal path as if nothing has happened. The effect caused by AO at time $t = T$, with the magnitude of the effect denoted by δ is given by

$$Y_t = X_t + \delta I_t^T. \tag{5}$$

We confine our discussion to the variance of the residuals and series of MA(1) model in the presence of an AO. Therefore, using (1) equation (5) can be written as

$$Y_t = \theta(B) a_t + \delta I_t^{(T)}, \tag{6}$$

where $\theta(B) = 1 - \theta B$.

Explicit expressions for the estimate of the variance of the residuals $(\hat{\sigma}_{\epsilon}^2)$ and variance of the series $(\hat{\sigma}_y^2)$ have been derived for the MA (1) model in the presence of an AO by using the results obtained by Janhavi and Suresh [5]. We know that the variance of the time series $\{X_t\}$ generated by a MA (1) model is given by

$$\sigma_x^2 = (1 + \theta^2)\sigma_a^2, \tag{7}$$

where σ_a^2 is the variance of the errors $\{a_t\}$.

The estimate of the variance of the $\{X_t\}$ series which follows an MA (1) model is therefore

$$\hat{\sigma}_x^2 = (1 + \hat{\theta}^2)\hat{\sigma}_a^2. \tag{8}$$

In the above expression, $\hat{\sigma}_a^2$ is calculated using

$$\hat{\sigma}_a^2 = \frac{\sum_{t=1}^{n} (\hat{a}_t - \bar{\hat{a}})^2}{n} \text{ and } \bar{\hat{a}} = \frac{\sum_{t=1}^{n} \hat{a}_t}{n}. \tag{9}$$

Similarly, let us denote the variance of the contaminated series $\{Y_t\}$ as

$$\sigma_y^2 = (1 + \theta^2)\sigma_{\epsilon}^2, \tag{10}$$

where σ_{ϵ}^2 is of the variance of the contaminated errors, $\{\epsilon_t\}$. The estimate of the variance of the $\{Y_t\}$ series is therefore

$$\hat{\sigma}_y^2 = (1 + \hat{\theta}^2)\hat{\sigma}_{\epsilon}^2. \tag{11}$$

As done earlier, $\hat{\sigma}_{\epsilon}^2$ is calculated using the contaminated residuals $\{e_t\}$.

$$\hat{\sigma}_{\epsilon}^2 = \frac{\sum_{t=1}^{n} (e_t - \bar{e})^2}{n} \text{ and } \bar{e} = \frac{\sum_{t=1}^{n} e_t}{n}. \tag{12}$$

Note that the errors which are contaminated with an AO are defined as,

$$\epsilon_t = \frac{Y_t}{\theta(B)},$$

where $\theta(B) = 1 - \theta B$.

It has been shown by Janhavi and Suresh [5] that for a time series of size n, the contaminated residuals $\{e_t\}$ can be obtained by using the uncontaminated residuals $\{\hat{a}_t\}$ through the relationship (13). Residuals at time $t < T$ (i.e., before the occurrence of AO) and $t \geq T$ (i.e., from the time of occurrence of AO) are as given below.

$$e_t = \begin{cases} \hat{a}_t & t < T, \ (i.e, \ t = 1(1)T - 1) \\ \delta + \hat{a}_t & t = T, \\ \delta \hat{\theta}^j + \hat{a}_t & t = T + j, \ (j = 1(1)n - T). \end{cases} \tag{13}$$

From the above equation it is clear that in an MA (1) model, the AO affects all the subsequent residuals from the time of its occurrence. In (13), without loss of generality, we shall assume that the magnitude of the AO δ is known.

Using (13), the mean of the contaminated residuals is

$$\bar{e} = \bar{\hat{a}} + \frac{\delta\left(\sum_{j=0}^{n-T} \hat{\theta}^j\right)}{n}. \tag{14}$$

The amount of contamination in the mean of the residuals of a $MA(1)$ model which contains an AO is

$$\frac{\delta\left(\sum_{j=0}^{n-T} \hat{\theta}^j\right)}{n}.$$

Consider

$$\hat{\sigma}_\epsilon^2 = \frac{\sum_{t=1}^{n} e_t^2}{n} - \bar{e}^2. \tag{15}$$

From (13), it turns out that the first part of the RHS of (15) is

$$\frac{\sum_{t=1}^{n} e_t^2}{n} = (\hat{a}_1^2 + \hat{a}_2^2 + \ldots + \hat{a}_{T-1}^2 + (\delta + \hat{a}_T)^2 + (\delta\hat{\theta} + \hat{a}_{T+1})^2$$

$$+ (\delta\hat{\theta}^2 + \hat{a}_{T+2})^2 + \ldots + (\delta\hat{\theta}^{n-T} + \hat{a}_n)^2)/n$$

$$= \frac{\sum_{t=1}^{n} \hat{a}_t^2}{n} + (\delta^2 + 2\delta\hat{a}_T + \delta^2\hat{\theta}^2 + 2\delta\hat{\theta}\hat{a}_{T+1} + \delta^2\hat{\theta}^4$$

$$+ 2\delta\hat{\theta}^2\hat{a}_{T+2} + \ldots + \delta^2\hat{\theta}^{2n-2T} + 2\delta\hat{\theta}^{n-T}\hat{a}_n)/n.$$

Plugging (14) and the above in (15), we get

$$\hat{\sigma}_\epsilon^2 = \frac{\sum_{t=1}^{n-T} \hat{a}_t^2}{n} + \frac{\delta^2}{n}\sum_{j=0}^{n-T}(\hat{\theta}^2)^j + \frac{2\delta}{n}\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+j} - \left(\bar{\hat{a}} + \frac{\delta(\sum_{j=0}^{n-T}\hat{\theta}^j)}{n}\right)^2. \tag{16}$$

Arranging the terms and rewriting, we get

$$\hat{\sigma}_\epsilon^2 = \left(\frac{\sum_{t=1}^{n-T} \hat{a}_t^2}{n} - \bar{\hat{a}}^2\right) + \frac{\delta^2}{n}\sum_{j=0}^{n-T}(\hat{\theta}^2)^j + \frac{2\delta}{n}\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+j}$$

$$- \frac{\delta^2}{n^2}\left(\sum_{j=0}^{n-T}\hat{\theta}^j\right)^2 - 2\bar{\hat{a}}\frac{\delta}{n}\sum_{j=0}^{n-T}\hat{\theta}^j. \tag{17}$$

Using (9) in (17) and further simplifying leads to variance of the contaminated residuals of $MA(1)$ model as

$$\hat{\sigma}_{\epsilon}^2 = \hat{\sigma}_a^2 + \frac{\delta}{n}\left\{\delta\sum_{j=0}^{n-T}(\hat{\theta}^2)^j + 2\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+j} - \frac{\delta}{n}\left(\sum_{j=0}^{n-T}\hat{\theta}^j\right)^2 - 2\bar{\hat{a}}\sum_{j=0}^{n-T}\hat{\theta}^j\right\}. \tag{18}$$

The expression (18) gives the variance of the residuals contaminated by a single AO in a $MA(1)$ model. Hence the second part of the LHS of (18) namely,

$$\frac{\delta}{n}\left\{\delta\sum_{j=0}^{n-T}(\hat{\theta}^2)^j + 2\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+j} - \frac{\delta}{n}\left(\sum_{j=0}^{n-T}\hat{\theta}^j\right)^2 - 2\bar{\hat{a}}\sum_{j=0}^{n-T}\hat{\theta}^j\right\}, \tag{19}$$

is the amount of contamination in the variance of the residuals of a $MA(1)$ model due to the presence of an Additive Outlier.

We know that,

$$\hat{\sigma}_y^2 = (1+\hat{\theta}^2)\hat{\sigma}_{\epsilon}^2. \tag{20}$$

Subtituting (18) in (20), we get

$$\hat{\sigma}_y^2 = (1+\hat{\theta}^2)\left[\hat{\sigma}_a^2 + \frac{\delta}{n}\left\{\delta\sum_{j=0}^{n-T}(\hat{\theta}^2)^j + 2\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+j} - \frac{\delta}{n}\left(\sum_{j=0}^{n-T}\hat{\theta}^j\right)^2 - 2\bar{\hat{a}}\sum_{j=0}^{n-T}\hat{\theta}^j\right\}\right]$$

$$= (1+\hat{\theta}^2)\hat{\sigma}_a^2 + (1+\hat{\theta}^2)\frac{\delta}{n}\left\{\delta\sum_{j=0}^{n-T}(\hat{\theta}^2)^j + 2\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+j} - \frac{\delta}{n}\left(\sum_{j=0}^{n-T}\hat{\theta}^j\right)^2 - 2\bar{\hat{a}}\sum_{j=0}^{n-T}\hat{\theta}^j\right\}$$

$$\hat{\sigma}_y^2 = \hat{\sigma}_x^2 + (1+\hat{\theta}^2)\frac{\delta}{n}\left\{\delta\sum_{j=0}^{n-T}(\hat{\theta}^2)^j + 2\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+j} - \frac{\delta}{n}\left(\sum_{j=0}^{n-T}\hat{\theta}^j\right)^2 - 2\bar{\hat{a}}\sum_{j=0}^{n=T}\hat{\theta}^j\right\}.$$

Futher simplification gives us variance of the $MA(1)$ series contaminated by a single AO as

$$\hat{\sigma}_y^2 = \hat{\sigma}_x^2 + (1 + \hat{\theta}^2)\frac{\delta}{n}\left\{\delta\left(\frac{1-(\hat{\theta}^2)^{n-T+1}}{1-\hat{\theta}^2}\right) + 2\sum_{j=0}^{n-T}\hat{\theta}^j\hat{a}_{T+1} - \frac{\delta}{n}\left(\frac{1-\hat{\theta}^{n-T+1}}{1-\hat{\theta}}\right)^2 - 2\bar{\hat{a}}\left(\frac{1-\hat{\theta}^{n-T+1}}{1-\hat{\theta}}\right)\right\}. \tag{21}$$

Let

$$\hat{\sigma}_y^2 = \hat{\sigma}_x^2 + \hat{\sigma}_\delta^2, \tag{22}$$

where, $\hat{\sigma}_\delta^2$ (the second term on RHS of (21)) is the estimate of the amount of contamination in the estimate of the variance of contaminated $MA(1)$ series $\{Y_t\}$.

3. Simulation Study

The effect of an AO on the variance of an $MA(1)$ model, which was algebraically derived in the previous section depends on the length (n) of the time series data, parameter of the model (θ) and the magnitude of the AO (δ). In order to evaluate these variances for specific values of $n\ (100, 200, 300)$, $\theta(|\theta| < 1)$ and $\delta\ (5, 10, 15)$, simulation study was carried out. The results obtained are presented in this section. The contaminated variances and the uncontaminated variances across the values n, θ and δ are presented in the form of graphs in figure: 1. Further, table: 1 has been presented in a format that would reveal clearly the contaminated variances and the amount of contamination in it across different sets of values n, θ and δ.

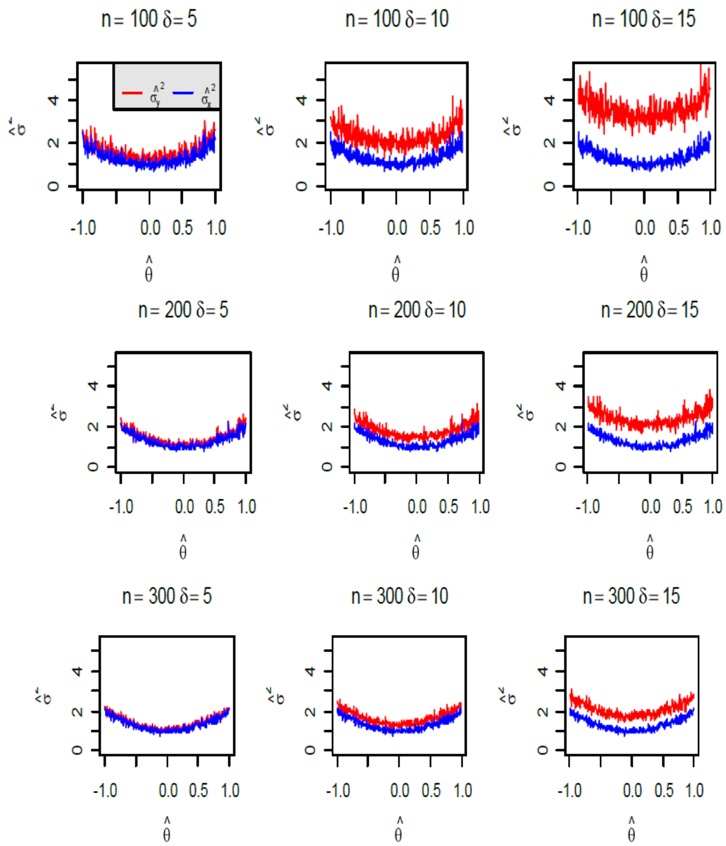

Figure 1. Estimate of the variance of $MA(1)$ model in the presence of an AO.

	$\hat{\sigma}_y^2$ and Amount of Contamination ($\hat{\sigma}_\delta^2$)								
			θ						
n	δ	$\hat{\sigma}_y^2$ and $\hat{\sigma}_\delta^2$	**-0.98**	**-0.65**	**-0.30**	**0.05**	**0.30**	**0.65**	**0.98**
100	5	$\hat{\sigma}_y^2$	2.53	1.74	1.31	1.08	1.53	1.42	2.09
		$\hat{\sigma}_\delta^2$	0.03	0.29	0.24	0.10	0.31	0.19	0.16
	10	$\hat{\sigma}_y^2$	3.08	2.57	2.06	1.68	2.32	2.10	2.64
		$\hat{\sigma}_\delta^2$	0.58	1.12	0.99	0.70	1.10	0.87	0.71
	15	$\hat{\sigma}_y^2$	4.13	3.87	3.31	2.79	3.60	3.31	3.70
		$\hat{\sigma}_\delta^2$	1.63	2.42	2.24	1.81	2.38	2.08	1.77
200	5	$\hat{\sigma}_y^2$	2.47	1.69	1.15	1.12	1.02	1.47	1.98
		$\hat{\sigma}_\delta^2$	0.26	0.25	0.11	0.15	0.05	0.12	0.16
	10	$\hat{\sigma}_y^2$	2.87	2.12	1.51	1.52	1.32	1.86	2.37
		$\hat{\sigma}_\delta^2$	0.66	0.68	0.47	0.55	0.35	0.51	0.55
	15	$\hat{\sigma}_y^2$	3.51	2.81	2.13	2.17	1.87	2.50	3.01
		$\hat{\sigma}_\delta^2$	1.30	1.37	1.09	1.20	0.90	1.15	1.19
300	5	$\hat{\sigma}_y^2$	2.20	1.57	1.09	1.15	1.09	1.54	1.93
		$\hat{\sigma}_\delta^2$	0.09	0.14	0.09	0.13	0.07	0.05	0.12
	10	$\hat{\sigma}_y^2$	2.44	1.80	1.37	1.44	1.33	1.79	2.23
		$\hat{\sigma}_\delta^2$	0.33	0.37	0.37	0.42	0.31	0.30	0.42
	15	$\hat{\sigma}_y^2$	2.82	2.21	1.81	1.89	1.74	2.20	2.68
		$\hat{\sigma}_\delta^2$	0.71	0.78	0.81	0.87	0.72	0.71	0.87

Table 1. Estimates of the variance and amount of contamination in the $MA\ (1)$ model.

4. Conclusions

As can be expected, for a fixed δ, the contamination variance decreases as n increases and for a fixed n, the contamination variance increases as δ increases. A similar feature is also observed with regard to the contribution to the contamination variance. The contamination variance as well as the contribution to the contamination variance increase as $|\theta|$ increases when we fix δ and n, and both are minimum when θ is nearer to zero. The table-1 clearly exposes the influence of the model parameter on both the contaminated variance and the magnitude of contamination over fixed values of δ and n. Further, the relative contribution in the presence of an outlier decreases as the contamination variance decreases as n increases and for a fixed n, the contamination variance increases as δ increases. Interestingly when δ is closer to –1, both the magnitude of the contamination variance and the amount of contamination is larger than when θ is closer to 1.

5. Discussion

Several points of interest emerge from the conclusions arrived for $MA(1)$ model presented above.

- The point to be noted is the effect of the model parameter on these two quantities apart from the values of n and δ. The relevant table is formatted in such a way that the role of parameter gets highlighted for easy readability.

- The accompanying graphs for $MA(1)$ model support the information in the table.

- A striking feature of these graphs is the parabolic figure they exhibit in all cases, with the parabola for the contaminated variance being above the uncontaminated one even when the parameter is close to 0.

- This information takes its importance while assessing the reliability of the estimates of the parameter(s), testing of hypothesis, goodness of fit of the model and forecasting techniques when outliers are present.

Acknowledgements

It is a pleasure to thank Prof. K. Sureshchandra, (Retd.), for his valuable suggestions, which improved the presentation of this work.

References

[1] G. E. P. Box, G. M. Jenkins and G. C. Reinsel, Time Series Analysis-Forecasting and Control, Pearson Education, (1994).

[2] W. S. Chan, Understanding the Effect of Time Series Outliers on Sample Auto-correlations, Test 04(01) (1995), 179-186.

[3] A. J. Fox, Outliers in Time Series, Journal of Royal Statistical Society, Series B 34(03) (1972), 350-363.

[4] S. C. Hillmer, Monitoring and adjusting forecasts in the presence of additive outliers, International Journal of Forecasting 3 (1989), 205-215.

[5] J. V. Janhavi and R. Suresh, Role of Structural Coefficients in the Study of the Influence of an Additive Outlier on the Residuals in ARMA (1, 1), AR (1), and MA(1) Models, Journal of Statistics 18 (2011), 67-83.

[6] J. Ledolter, The effect of additive outliers on the forecasts from ARIMA models, International Journal of Forecasting 5 (1989), 231-240.

[7] R. A. Maronna, R. D. Martin and V. J. Yohai, Robust statistics, Wiley, (2006).

[8] R. Suresh, Contributions to Statistical Inference and Forecasting for Time Series in the Presence of Additive Outliers (Unpublished doctoral thesis), Bangalore University, Bengaluru, India (2013).

[9] R. S. Tsay, Outliers, Level Shifts and Variance Changes in Time Series, Journal of Forecasting 7 (1988), 1-20.