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# MISCELLANEOUS RESULTS ON PENDANT DOMINATION OF GRAPHS

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## Abstract

A pendant dominating set is defined by Nayaka et al., so that a dominating set that has a pendant vertex is a pendant dominating set. In this paper, we mainly concentrate on the pendant domination number $\gamma_{pe}$ and the upper pendant domination number $\Gamma_{pe}(G)$ for some special graphs and for operation of graphs.

## 1. Introduction

A graph $G = (V, E)$ is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of $G$, called edges. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively. If $e = \{u, v\}$ is an edge, we write $e = uv$, we say that $e$ joins the vertices and; and are adjacent vertices; $u$ and $v$ are incident with $e$. If two vertices are not joined then we say that they are non-adjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other. We denote the number of vertices and edges in $G$ by $|V|$ (or $n$)

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and $|E|$ (or $m$); these two basic parameters are called the order and size of $G$.

In a graph $G$ the degree of a vertex $v$ is defined to be the number of edges incident with is called the degree of a vertex and is denoted by $\deg(v)$, and degree of self-loops is counted twice. The minimum degree $\delta(G)$ of a graph $G$ is $\delta(G) = \min\{\deg(v) : v \in V(\delta)\}$. The maximum degree of a graph $G$ is $\Delta(G) = \max\{\deg(v) : v \in V(\delta)\}$. A vertex having no incident edge is called an isolated vertex. In other words, isolated vertices are those with zero degree. A vertex of degree one is called a pendant vertex or an end vertex. An edge incident on pendant vertex is called pendant edge.

A graph which has neither loops nor multiple edges i.e., each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a simple graph. A wheel $W_{1,n}$ on $(n+1)$-vertices is a graph having $V(W_{1,n}) = \{v_0, v_1, \ldots, v_{n-1}, v_n\}$ as vertex set, then $v_i v_j$ is an edge if and only if $|i-j| = 1$ or $n-1$. A graph in which all vertices are of equal degree is called a regular graph. If the degree of each vertex is $r$, then the graph is called a regular graph of degree $r$ (or) $r$-regular graph. A 3-regular graph is called as cubic graph. A simple graph $G$ is said to be complete if every vertex in $G$ is connected with every other vertex i.e., if $G$ contains exactly one edge between each pair of distinct vertices. The complement $\overline{G}$ of a graph $G$ also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

A graph $G$ is said to be bipartite if the vertex set $V$ of the graph $G$ can be partitioned into two nonempty subsets $V_1$ and $V_2$ such that no two vertices in the same set are adjacent in $G$. A bipartite graph $G$ with vertex partition $V_1$ and $V_2$ is said to be complete bipartite graph, if every vertex in $V_1$ is adjacent to every vertex in $V_2$. A complete bipartite graph with vertex partition $V_1$ and $V_2$ having $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. The $K_{1,n}$ graph is called a star. Clearly $K_{m,n}$ has $mn$ edges. A subgraph of $G$ is a graph having all of its vertices and edges in $G$. If $G_1$ is a subgraph of

$G$, then $G$ is a super graph of $G_1$. In other words, if $G$ and $H$ are two graphs with vertex sets $V(H)$, $V(G)$ and edge sets $E(H)$ and $E(G)$ respectively, such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call $H$ as a subgraph of $G$ or $G$ as a supergraph of $H$. A spanning subgraph is a subgraph containing all the vertices of $G$. In other words, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$ then $H$ is a proper subgraph of $G$ and if $V(H) = V(G)$, then we say that $H$ is a spanning subgraph of $G$. A spanning subgraph need not contain all the edges in $G$. For any set $S$ of vertices of $G$, the vertex induced subgraph $\langle S \rangle$ or simply an induced subgraph is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $\langle S \rangle$ if and only if they are adjacent in $G$. A graph $G$ is connected if every pair of vertices on $G$ are joined by a path. A maximal connected subgraph of $G$ is called a connected component or simply a component of $G$. Otherwise, $G$ is called a disconnected graph. All definitions and terminologies are available in [5].

## 2. Determination of $\gamma_{pe}(G)$ and $\Gamma_{pe}(G)$

By a graph $G$ we mean a finite, simple, non-trivial and connected. The set $D$ of vertices of a graph $G$ is said to be a dominating set if every vertex in $V - D$ is adjacent to a vertex in $D$. A dominating set $D$ is said to be a minimal dominating set if no proper subset of $D$ is a dominating set. The minimum cardinality of a minimal dominating set of a graph $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. The upper domination number $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of $G$ [6]. A set $S$ of vertices of a graph $G$ such that $\langle S \rangle$ has a pendant vertex is called a pendant set of $G$. A pendant set that is also a dominating set is a pendant dominating set [7].

A pendant dominating set was firstly defined by Nayaka et. al., their papers published on pendant domination number $\gamma_{pe}(G)$ [7] and upper pendant domination number $\Gamma_{pe}(G)$ [9] has inspired me to develop further results. Hence in this section we determine the values of pendant domination number $\gamma_{pe}(G)$ and upper pendant domination number $\Gamma_{pe}(G)$ for some special graphs and for some operation of graphs.

A dominating set $S$ of a graph $G$ is said to be a pendant dominating set $\langle PD - set \rangle$ of $G$ if $\langle S \rangle$ has at least one pendant vertex. A pendant dominating set $S$ is said to be a minimal pendant dominating set $\langle MPD - set \rangle$ if no proper subset of $S$ is a pendant dominating set. The pendant domination number $\gamma_{pe}(G)$ and the upper pendant domination number $\gamma_{pe}(G)$ are defined to be

$$\gamma_{pe}(G) = \min \{| S | : S \text{ is a } MPD - \text{set of } G\}$$

$$\Gamma_{pe}(G) = \max \{| S | : S \text{ is a } MPD - \text{set of } G\}.$$

**Definition 2.1** [12]**.** A Bull graph is a simple graph with 5 vertices and 5 edges, in the form of a triangle with two disjoint pendant edges.

The following Figure 1 is a Bull graph.

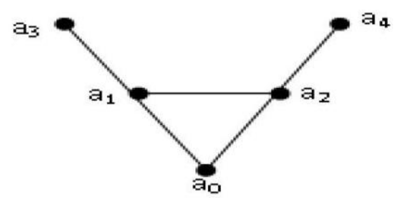


**Figure 1.** Bull graph.

**Theorem 2.2.** *For a bull graph* $G$, $\gamma_{pe}(G) = 2$ *and* $\Gamma_{pe}(G) = 3$.

**Proof.** Let $G$ be a bull graph having vertex set $V = \{a_0, a_1, a_2, a_3, a_4\}$, let us choose a vertex set such that the vertices in the set must form a dominating set, which must also consist pendant vertices such that it results to a pendant dominating set. Keeping in mind the above conditions we choose a vertex set $S = \{a_1, a_2\}$ such that $V - S = \{a_0, a_3, a_4\}$ are adjacent to vertices in $S$. Hence $S$ is a minimal pendant dominating set with minimum cardinality.

Therefore, $\gamma_{pe}(G) = 2$.

Now let us consider another set $S$, such that it consists two pendant vertices and a vertex that is adjacent to any one of the chosen pendant vertex. Hence we obtain a dominating set $S = \{a_3, a_4, a_1\}$ which satisfies the required condition. Where $V - S = \{a_0, a_2\}$ are adjacent to vertices in $S$. Hence $S$ is a minimal pendant dominating set with maximum cardinality, i.e., $\Gamma_{pe}(G) = 3$.

**Definition 2.3** [3]**.** A friendship graph $F_n$ is a simple graph which is constructed by joining $n$-copies of cycle graph $C_3$ with a common vertex.

The following Figure 2 is a friendship graph.

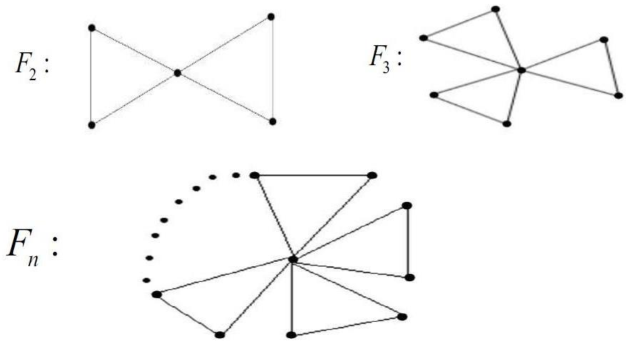


**Figure 2.** Friendship graph.

**Theorem 2.4.** *Let G be a friendship graph. Then* $\gamma_{pe}(F_n) = 2$ *and* $\Gamma_{pe}(F_n) = n + 1$.

**Proof.** Let $F_n$ be a friendship graph consisting of n-copies of cycle graphs $C_3$ having a common vertex, which consists of $(2n + 1)$- vertices and $3n$- edges. Since the common vertex is adjacent to all the other vertices in the graph it forms a dominating set. Now by considering common vertex and a vertex among any one of the $n$-copies of cycles $C_3$ results to a minimal pendant dominating set with minimum cardinality. Therefore, $\gamma_{pe}(F_n) = 2$.

To check the maximum cardinality we exclude common vertex and consider any two vertices among $n$-copies of cycle graph $C_3$ and then a vertex from each of the remaining $(n - 1)$- copies of cycle graph $C_3$ which altogether results to a minimal pendant dominating set with maximum cardinality.

Hence $\Gamma_{pe}(F_n) = 2 + (n - 0) = n + 1$.

**Definition 2.5** [12]**.** A butterfly graph is same as friendship graph $F_2$.

**Corollary 2.6.** *Let G be a butterfly graph. Then* $\gamma_{pe}(G) = 2$ *and* $\Gamma_{pe}(G) = n + 1$.

**Definition 2.7** [1]**.** A diamond graph is a planar undirected graph with 4 vertices and 5 edges. The following Figure 3 is a diamond graph.

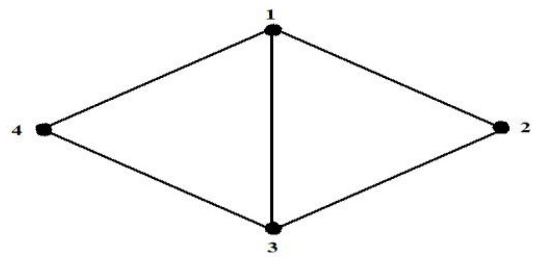


**Figure 3.** Diamond graph.

**Theorem 2.8.** *Let G be a diamond graph. Then* $\gamma_{pe}(G) = 2 = \Gamma_{pe}(G)$.

**Proof.** Let $G$ be a diamond graph whose order is 4 and size 5. By considering any two adjacent vertices a minimal pendant dominating set is obtained. Hence we notice that the minimal pendant dominating set with minimum and maximum cardinality is equal to two. Thus

$$\gamma_{pe}(G) = 2 = \Gamma_{pe}(G).$$

**Definition 2.9** [5]**.** A Petersen graph is a connected bridgeless graph.

The following Figure 4 is a Petersen graph.

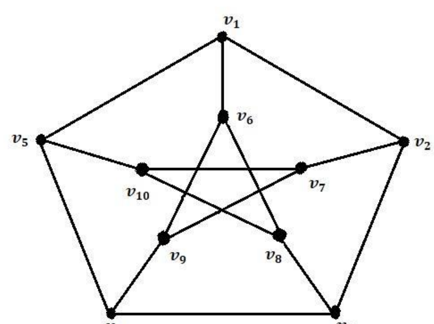


**Figure 4.** Petersen graph.

**Theorem 2.10.** *Let G be a Petersen graph. Then* $\gamma_{pe}(G) = 4 = \Gamma_{pe}(G)$.

**Proof.** Let $G$ be a Petersen graph consisting of $\{v_1, \ldots, v_9, v_{10}\}$ vertices. Consider a set $S$ consisting any two adjacent vertices, then choose vertices such that they are non-adjacent to the chosen vertices and also they are non-adjacent with each other. Hence we obtain a set $S = \{v_1, v_2, v_8, v_9\}$ as a dominating set consisting pendant vertices, where the vertices of $V - S$ are adjacent to the vertices in $S$. Hence $S$ is a minimal pendant dominating set with minimum cardinality. Therefore, $\gamma_{pe}(G) = 4$. The proof for upper pendant domination number follows similarly as that of the pendant domination number. Therefore $\Gamma_{pe}(G) = 4$.

**Definition 2.11** [1]**.** A fish graph is a graph on 6 vertices, whose name derives from its resemblance to a schematic illustration of a fish.

The following Figure 5 is a Fish graph.

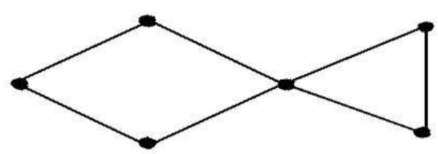

**Figure 5.** Fish graph.

**Theorem 2.12.** *Let G be a fish graph. Then* $\gamma_{pe}(G) = 2$ *and* $\Gamma_{pe}(G) = 4$.

**Proof.** A fish graph is a graph whose order is 6 and size 7. The graph consists two cycles $C_3$ and $C_4$ having a vertex in common. To obtain a dominating set the common vertex and a vertex adjacent to it from $C_4$ can be considered which results to a minimal pendant dominating set with minimum cardinality. i.e., $\gamma_{pe}(G) = 2$. To obtain the upper pendant domination, we exclude the common vertex and consider other two adjacent vertices from $C_3$ and two non-adjacent vertices from $C_4$ which results to a pendant dominating set with maximum cardinality i.e., $\Gamma_{pe}(G) = 4$.

**Definition 2.13** [12]**.** The helm graph $H_n$ is the graph obtained from a $n$-wheel graph by adjoining a pendant edge at each node of the cycle. The helm graph has $(2n + 1)$ vertices and $3n$ edges.

The following Figure 6 is a Helm graph.

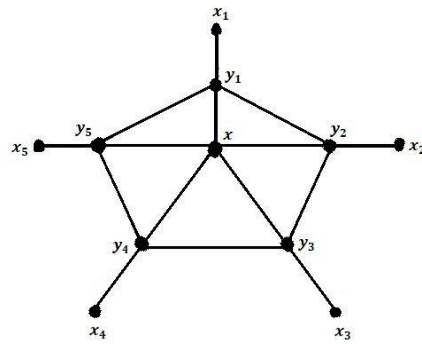


**Figure 6.** Helm graph.

**Theorem 2.14.** *Let* $H_n$ *be a helm graph, then* $\Gamma_{pe}(H_n) = n + 1$.

**Proof.** Let $H_n$ be a helm graph having bipartition *XY*, with $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\} \cup \{x\}$. Let us consider all the vertices of a cycle $C_n$ which results a dominating set $S = \{y_1, y_2, \ldots, y_n\}$ and by considering any one leaf among $x_i$ where, $1 \leq i \leq n$ results to a minimal pendant dominating set with maximum cardinality i.e., $S' = | S | \cup = \{x_i\} = n + 1$.

Hence, $\Gamma_{pe}(H_n) = n + 1$.

**Note.** [8] For any helm graph $H_n$, then $\gamma_{pe}(H_n) = n$.

**Definition 2.15** [2]**.** A cubic symmetric graph is a symmetric cubic i.e., regular of order three having even number of vertices.

The following Figure 7 is a cubic symmetric graph.

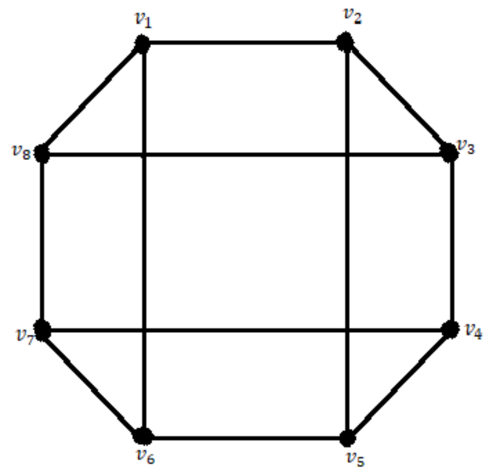


**Figure 7.** Cubic symmetric graph.

**Theorem 2.16.** *Let $G$ be a cubic symmetric graph, then* $\gamma_{pe}(G) = \frac{(n-2)}{2}$ *and* $\Gamma_{pe}(G) = \frac{n}{2}$.

**Proof.** Let $G$ be a cubic symmetric graph of order 8 and size 12. Let $\{v_1, v_2, \ldots, v_8\}$ be the vertex set of $G$. We now choose the dominating set $S = \{v_2, v_5, v_8\}$, which consists at least a pendant vertex. Such that all the vertices of $V - S = \{v_1, v_3, v_4, v_6, v_7\}$ are adjacent to vertices of $S$. Therefore $S$ is a minimal pendant dominating set with minimum cardinality. Hence, $\gamma_{pe}(G) = \frac{(n-2)}{2}$.

To obtain the upper pendant domination we shall choose two pairs of adjacent vertices such that these pairs are non-adjacent with each other. Hence we consider a dominating set consisting pendant vertices, $S = \{v_1, v_2, v_4, v_7\}$. The vertices of $V - S = \{v_3, v_5, v_6, v_8\}$ are adjacent with the vertices of $S$. Therefore, $\Gamma_{pe}(G) = \frac{n}{2}$.

**Definition 2.17** [12]**.** The crown graph $S_n$ for $n \geq 3$ is the graph with vertex set $V = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and an edge from $V$ i.e., $\{x_i y_j : 1 \leq i, j \leq n; i \neq j\}$. Therefore $S_n$ coincides with the complete bipartite graph $S_n$ with horizontal edges removed. The Crown graph $S_4$ is shown in Figure 8.

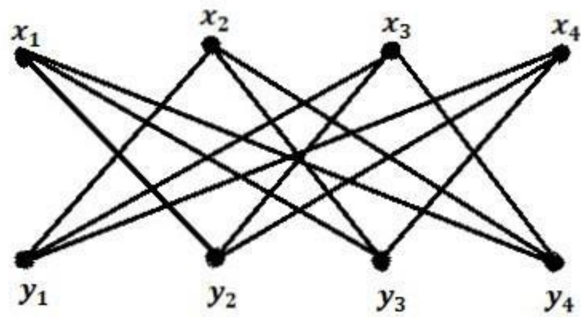


**Figure 8.** Crown graph $S_n$.

**Theorem 2.18.** *Let $S_n$ be a crown graph, then* $\Gamma_{pe}(S_n) = \Gamma(S_n) + 1 = 3.$

**Proof.** Let $S_n$ be a crown graph with the vertex set $V = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. It is clear that $S = \{x_1, y_1\}$ is a minimal dominating set. Now by choosing either vertex $x_i$ or $y_i$ for $i > 1$, forms a set $S' = \{x_1, y_1\} \cup \{x_i\}$ or $\{y_i\}$ that results to a upper pendant dominating set of $S_n$. Therefore, $\Gamma_{pe}(S_n) = \Gamma(S_n) + 1 = 3.$

**Note.** [8] Let $G$ be a crown graph with $2n$ vertices. Then $\gamma_{pe}(G) = \gamma(G) + 1$.

**Observation 2.19.**

1. For a complete bipartite trees $K_{1, n}$ we have $\gamma_{pe}(K_{1,n}) = \Gamma_{pe}(K_{1,n}) = 2$.

2. For a claw graph $K_{1,3}$, we obtain $\gamma_{pe}(K_{1,3}) = \Gamma_{pe}(K_{1,3}) = 2$.

**Theorem 2.20.** *Let G be a graph of order n and size m. If $v_i \in V(G)$ such that $d(v_i) = n - 1$ then,* $\gamma_{pe}(G) = 2$.

**Proof.** Let $G$ be a graph of order and size $m$. Assume that $v_i \in V(G)$ such that $d(v_i) = n - 1$. Then the set is a dominating set of $G$. But $\{v_i\}$ is not pendant dominating set. Thus, $\{v_i, v_j \in V(G); v_i v_j \in E(G)\}$, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$ is a pendant dominating set and the result follows. We can apply Theorem 2.19 on the identity graph of a multigroup. For details about the identity graph of a multigroup, see [10, 11].

## 3. Results on $\gamma_{pe}$ and $\Gamma_{pe}$ for Operation of Graphs

**Observation 3.1.**

1. The following three operations union, intersection and ring sum of graphs are commutative. Hence we have the following results :

i. $\gamma_{pe}(G_1 \cup G_2) = \gamma_{pe}(G_2 \cup G_1)$; $\Gamma_{pe}(G_1 \cup G_2) = \Gamma_{pe}(G_2 \cup G_1)$.

ii. $\gamma_{pe}(G_1 \cap G_2) = \gamma_{pe}(G_2 \cap G_1)$; $\Gamma_{pe}(G_1 \cap G_2) = \Gamma_{pe}(G_2 \cap G_1)$.

iii. $\gamma_{pe}(G_1 \oplus G_2) = \gamma_{pe}(G_2 \oplus G_1)$; $\Gamma_{pe}(G_1 \oplus G_2) = \Gamma_{pe}(G_2 \oplus G_1)$.

**Theorem 3.2.** *Let $G_1$ and $G_2$ be any two graphs, $(G_1 + G_2)$ is the join of the graphs. Then $\gamma_{pe}(G_1 + G_2) = 2 = \Gamma_{pe}(G_1 + G_2)$.*

**Proof.** Let $G_1$ and $G_2$ be any two graphs having vertex sets $V_1 = (x_1, x_2, \ldots, x_n)$ and $V_2 = (y_1, y_2, \ldots, y_n)$ respectively. Then by definition for join of graphs $(G_1 + G_2)$ we have union of the graphs such that each vertex of $V_1$ is adjacent to all the other vertices of $V_2$. Hence it is obvious that the set $S$ consists exactly two vertices i.e., a vertex chosen from vertex set $V_1$ and the other from $V_2$ results to a pendant dominating set. Say $S = \{(x_i, y_i) \backslash i = 1, 2, \ldots, n\}$, Where the vertices of $V - S$ are adjacent to the vertices of $S$. Hence minimum and maximum cardinality of a minimal pendant dominating set remains identical. Therefore, $\gamma_{pe}(G_1 + G_2) = 2 = \Gamma_{pe}(G_1 + G_2)$.

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