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EXISTENCE SOLUTION ON HYERS-ULAM STABILITY OF ATANGANA- BALEANU FRACTIONAL DIFFERENTIAL EQUATIONS WITH DEPENDENCE ON THE LIPSCHITZ FIRST DERIVATIVES

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Abstract

In this article, we study the Hyers-Ulam stability of Atangana-Baleanu fractional differential equations with Mittag-Lefflerkernal is thoughtful of using the lipschitz first derivative conditions. Here we extend the Gronwall inequality in the structure of Hyers-Ulam fractional integral equations. To illustrate the main results we give an example.

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1. Introduction

The Generalized form of an integer order Differential equation is the fractional order differential equation. The fractional differential equations has accomplished the substantial adoration and decisive, for that many research articles have been circulated in this field. Fractional differential equations have been developed many new definitions and it is utilized to develope many mathematical modeling. The different fractional operators can be found in [15] and [21] and it has been showed that the differential equation with fractional order is more accurately than the differential equations with integer order.

The theory of fractional calculus itself and its applications can be found in various area such as rational differential equations arise in rheology, cosmology, fusion low light, Medical area such as HIV/AIDS with treatment compartment [11], and also in recently applied the qualitative analysis in the time of Covid-19 [5] many other branches of science. So that the fractional differential equation is got much more attention in various fields [13, 16, 18, 23].

One of the well known derivatives with fractional orders are Riemann-Liouville, Caputo, Weyl Hadamard and Grunwald-Letnikov, etc., Fractional operators are played a vital role for the mathematical modelling and real world applications. After that Atangana Baleanu delivered few of the new fractional derivatives related to the Caputo and the Riemann Liouville definitions of the fractional order derivatives. In few days many researchers gave much more attention on ABC derivatives along some circumstances in various fields.

The Atangana Baleanu fractional derivatives is usual to the non singularity and the non local conditions of the kernal which is the rationalized Mittag-Leffler function. In recent studies on ABC derivatives Jarad et al. Here they investigated the ordinary differential equation in the form of AB derivative [12]. In [3] give some of the properties of the ABC fractional derivative and it is utilized them to solve the heat transfer model. The ABC fractional derivative is used in the circuit modeling [2]. The fractional diffusion process of circuit board heat dissipation which is simulate ABC fractional derivative [1].

In this article we investigate the Hyers-Ulam stability of

$$({}^{ABC}{}_{a}D^{\alpha}u)(t) - \mu u(t) = f(t, u(t), u'(t, u(t)))\ 1 < \alpha \leq 2,\ t \in C[a, b] \tag{1}$$

$$u(0) = u_0 \tag{2}$$

where ${}^{ABC}{}_{a}D^{\alpha}$ is the left caputo AB fractional derivative. If $f \in C[a, b]$, $f(t, u(t), u'(t, u(t))) = 0$. Consider $\mathfrak{D}u(t) = u'(t, u(t))$. Then (1) becomes

$$({}^{ABC}{}_{a}D^{\alpha}u)(t) - \mu u(t) = f(t, u(t), \mathfrak{D}u(t))\ 1 < \alpha \leq 2. \tag{3}$$

$$u(0) = u_0 \tag{4}$$

The establishment of this article as follows: In section 2 we discuss some needful fractional calculus definitions, properties, definitions, propositions and lemmas. In section 3 we investigate and proved the existence and uniqueness of solutions for AB-fractional derivative results. In final section 4 illustrate an example numerically solved.

2. Preliminaries

In this section we extend some definitions, lemmas and propositions of fractional calculus, which will be used throughout this paper.

Definition 2.1 [14, 17]**.** For $\alpha > 0$, the left Riemann-Liouville fractional integral of order α is given by

$$I_a^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha - 1}u(s)ds. \tag{5}$$

Definition 2.2 [14, 17]**.** For $0 < \alpha < 1$, the left Riemann-Liouville fractional derivative of order α is given by

$$({}_{a}D^{\alpha}u)(t) = \left(\frac{d}{dt}\right)\left(\frac{1}{\Gamma(1 - \alpha)}\int_0^t (t - s)^{-\alpha}u(s)ds\right). \tag{6}$$

Definition 2.3. For $0 \leq \alpha \leq 1$, the Caputo fractional derivative of order $\alpha is given as [14, 17]

$$({}_{a}^{C}D^{\alpha}u)(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-s)^{-\alpha}u'(s)ds \tag{7}$$

Definition 2.4 [3]. Let $u \in H^1(a, b)$, $a < b$, and α in $[0, 1]$. The Caputo Atangana-Baleanu fractional derivative of u of order is defined by

$$({}^{ABC}{}_{a}D^{\alpha}u)(t) = \frac{B(\alpha)}{(1-\alpha)}\int_{0}^{t}u'(s)\,E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{(1-\alpha)}\right]ds. \tag{8}$$

Where E_{α} is the Mittag-Leffler function defined by $E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}$ and $B(\alpha) > 0$ is a normalizing function satisfying $B(0) = B(1) = 1$. The Riemann Atangana-Baleanu fractional derivative of u of order α is defined by

$$({}^{ABC}{}_{a}D^{\alpha}u)(t) = \frac{B(\alpha)}{(1-\alpha)}\frac{d}{dt}\int_{0}^{t}u(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{(1-\alpha)}\right]ds. \tag{9}$$

The associative fractional integral is defined by

$$({}^{AB}{}_{a}D^{\alpha}u)(t) = \frac{1-\alpha}{B(\alpha)}u(t) + \frac{\alpha}{B(\alpha)}(I_{a}^{\alpha}u(t)) \tag{10}$$

where ${}_{a}I^{\alpha}$ a is the left Riemann-Liouville fractional integral given in (5).

Lemma 2.5 [3]. *Let* $u \in H^1(a, b)$, *and* $[0, 1]$. *Then the following relation holds*

$$({}^{ABC}{}_{a}D^{\alpha}u)(t) = ({}^{ABC}{}_{a}D^{\alpha}u)(t) - \frac{B(\alpha)}{(1-\alpha)}u(a)u(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{(1-\alpha)}\right]. \tag{11}$$

Lemma 2.6 [12]. *Suppose that and* $a > 0$, $c(t)\left(1-\frac{1-\alpha}{B(\alpha)}\right)^{-1}$ *is a non-negative, non-degreasing and locally integrable function on* $[a, b]$, $\frac{\alpha dt}{B(\alpha)}\left(1-\frac{1-\alpha}{B(\alpha)}dt\right)^{-1}$ *is non-negative and bounded on* $[a, b]$ *and* $u(t)$ *is non-negative and locally integrable* $[a, b)$ *with*

$$u(t) \leq c(t) + d(t)\left({}^{AB}{}_{a}I^{\alpha}u\right)(t), \tag{12}$$

then

$$u(t) \leq \frac{c(t)\,B(\alpha)}{B(\alpha) - (1-\alpha)dt}\, E_{\alpha}\left(\frac{\alpha\, dt\,(t-a)}{B(\alpha) - (1-\alpha)dt}\right). \tag{13}$$

Theorem 2.7 (Ascoli-Arzela Theorem) [7]. *Let S be a compact metric spaces. Then $M \subset c(\Omega)$ is relatively compact iff M is uniformly bounded and uniformly equicontinuous.*

Theorem 2.8 [7]. (Krasnoselskii Fixed Point Theorem) *Let S be a closed, bounded and convex subset of a real Banach space X and let and be operators on S satisfying the following conditions.*

- $T_1(S) + T_2(S) \subset S$
- T_1 *is a strict contraction on S, (i.e.) there exists* $\alpha k \in [a, b)$ *such that*

$$\| T_1(u) - T_2(u) \| \leq k \| u - v \| \; \forall\, u, v \in S$$

- T_2 *is continuous on S and* $T_2(S)$ *is relatively compact subset of X.*

Then there exists a \$u \in S\$ such that $T_1(u) + T_2(u) = u$.

Preposition 2.9 [20]. *For* $0 \leq \alpha \leq 1$

$$\left({}^{AB}{}_{a}I^{\alpha}\left({}^{ABC}{}_{a}D^{\alpha}u\right)\right)(t) = u(t) - u(a) - u(a)E_{\alpha}(\lambda t^{\alpha}) - \frac{\alpha}{1-\alpha}\, u(\alpha)E_{\alpha,\,\alpha+1}(\lambda t^{\alpha})$$

$$= u(t) - u(a).$$

Preposition 2.10 [8, 10]. $f'(u) \in D$ *satisfy the Lipschitz condition, (i.e.,) There exist a constant* $k > 0$ *such that*

$$\| f'(u) - f'(v) \| \leq k(\| u - v \|),\; u, v \in D, \tag{14}$$

Definition 2.11. A continuous function $u : [a, b] \to \mathbb{R}$ is called a mild solution of the Atangana-Baleanu fractional derivative equation in the sense of Caputo

$$\begin{cases} \left({}^{ABC}{}_{a}D^{\alpha}u\right)(t) - \mu u(t) = f(t)\; 1 < \alpha \leq 2 \\ u(a) = u_0 \end{cases}$$

for all $t \in C[a, b]$, $u(t)$ satisfies the following integral equation.

$$u(t) = u_0 - \mu u(a) + \mu u(t) + {}^{AB}{}_{a}I^{\alpha} f(t).$$

3. Existence and Uniqueness

Let us give some assumptions to solve the existence and uniqueness solutions of the problem (3) and (4) by using Banach contraction principle.

A1 Let us define a continuous function $f \in (C[a, b] \times \Re \times \Re, \Re)$ and $u \in C[a, b]$ and there exists a positive constants $\mathfrak{M}_1, \mathfrak{M}_2$ and such that

$$\| f(t, u_1, v_1) - f(t, u_2, v_2) \| \leq \mathfrak{M}_1(\| u_1 - u_2 \| + \| v_1 + v_2 \|)$$

for each u_1, u_2, v_1, v_2 in $Y, \mathfrak{M}_2 = \max_{t \in R} \| f(t, 0, 0) \|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$. Let $Y = C[\Re, X]$ be the set of continuous functions on $\Re$ with in the Banach space X values.

A2 Let $u' \in C[a, b]$ satisfy the Lipschitz condition. i.e., There exists a positive constants $\mathfrak{N}_1, \mathfrak{N}_2$ and N such that

$$\| \mathfrak{D}(t, u) - \mathfrak{D}(t, v) \| \leq \mathfrak{N}_1(\| u - v \|),$$

for all u, v in $Y \cdot \mathfrak{N}_2 = \max_{t \in D} \| \mathfrak{D}(t, 0) \|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

A3 For each $\lambda > 0$, let $B_{\lambda} \in \{u \in Y : \| u \| \leq \lambda\} \subset Y$ where $\lambda = (1 - \mu - \rho)^{-1}(\| u_0 \|)$ and take ρ is $(\mathfrak{M} + \mathfrak{N}t)$.

A4 For each $\lambda_0 > 0$ let $B_{\lambda_0} \in \{u \in Y : \| u \| \leq \lambda_0\} \subset Y$ then B_{λ_0} is clearly bounded, closed and convex subset in $C([a, b], \Re)$.

Lemma 3.1. *If* (A1) *and* (A2) *are satisfied, then the estimate* $\| \mathfrak{D}u(t) \| \leq t(\mathfrak{N}_1 \| u \| + \mathfrak{N}_2), \| \mathfrak{D}u(t) - \mathfrak{D}v(t) \| \leq \mathfrak{N}t \| u - v \|$ *are satisfied for any* $t \in R$ *and* $u, v \in Y$.

Theorem 3.2. *Let* $u \in C[a, b]$ *such that* $({}^{ABC}{}_{a}D^{\alpha} u)(t) \in C[a, b]$. *Suppose that* $f \in (C[a, b] \times \Re \times \Re, \Re)$ *satisfies* (A1) *and* (A3). *Then iff* $(a, u(a), \mathfrak{D}u(a)) = 0$ *and*

$$\left(\frac{(1-\alpha)}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right) \leq 1$$

then the problem (3) *and* (4) *has a unique solution*.

Proof. Initially we have to prove that $u(t)$ satisfies the (3) and (4) if and only if $u(t)$ satisfies the integral equation.

$$u(t) = u_0 - \mu u(a) + \mu u(t) + {}^{AB}{}_{a}I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \tag{15}$$

Take $u(t)$ satisfy equation (3). Now both sides of (3) we apply the Atangana-Baleanu fractional integral, we get

$$({}^{AB}{}_{a}I^{\alpha}({}^{ABC}{}_{a}D^{\alpha}u))(t) - \mu {}^{AB}{}_{a}I^{\alpha}u(t) = {}^{AB}{}_{a}I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \tag{16}$$

By using (2.9) we formulate the equation we get

$$u(t) - u(a) - \mu {}^{AB}{}_{a}I^{\alpha}u(t) = {}^{AB}{}_{a}I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \tag{17}$$

After all $u(a) = u_0$ from (4) and $f(t, u(t), \mathfrak{D}u(t)) = 0$ then the equation (15) satisfied. If $u(t)$ satisfies (15), it is obvious that $u(a) = u_0$ because by using that $f(t, u(t), \mathfrak{D}u(t)) = 0$.

In equation (15) apply the Atangana-Baleanu fractional derivative both sides and exploit that $({}^{AB}{}_{a}D^{\alpha}({}^{AB}{}_{a}I^{\alpha}u))(t) = u(t)$ we get

$$\begin{aligned}({}^{ABR}{}_{a}D^{\alpha}u)(t) &= u_0({}^{ABR}{}_{a}D^{\alpha}u)(t) - \mu u(a)({}^{ABR}{}_{a}D^{\alpha}1)(t) + \mu({}^{ABR}{}_{a}D^{\alpha}u)(t) \\ &\quad + ({}^{ABR}{}_{a}D^{\alpha}({}^{ABR}{}_{a}I^{\alpha}))(t) f(t, u(t), \mathfrak{D}u(t)).\end{aligned} \tag{18}$$

As follows we have

$$({}^{ABR}{}_{a}D^{\alpha}(u(t) - \mu u(t))) = (u_0 - \mu u(a)) E_{\alpha}\left(-\alpha \frac{(t-a)^{\alpha}}{(1-\alpha)}\right) + f(t, u(t), \mathfrak{D}u(t)) \tag{19}$$

Later, the result is attained by promote from theorem (3.2) in [3]. Here we define the operator

$$Tu(t) = u_0 - \mu u(a) + \mu u(t) + {}^{AB}{}_{a}I^{\alpha} f(t, u(t), \mathfrak{D}u(t))$$

By assumption (A3). $\| u \| \leq \lambda$ we get

$$\| Tu(u) \| \leq \| u_0 \| + \mu \| u \| + \left\| \frac{1-\alpha}{B(\alpha)} f(t, u(t), \mathfrak{D}u(t)) + \frac{\alpha}{B(\alpha)} ({}_a I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \right\|$$

$$\leq \| u_0 \| + \mu\lambda + \frac{1-\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \| + \frac{\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \| ({}_a I^{\alpha})$$

$$\leq \| u_0 \| + \mu\lambda + \frac{1-\alpha}{B(\alpha)} (\mathfrak{M} \| u \| + \mathfrak{N}t \| u \|) + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} (\mathfrak{M} \| u \| + \mathfrak{N}t \| u \|)$$

$$\leq \lambda(1 - \mu - \rho) + \mu\lambda + \left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \right) (\mathfrak{M} \| u \| + \mathfrak{N}t \| u \|)$$

$$\leq \lambda$$

(i.e.,) $\| Tu(t) \| \leq \lambda$. Next we have to prove the uniqueness

$$\| T(u) - T(v) \| = \| u_0 - \mu u(a) + \mu u(t) + {}^{AB}{}_a I^{\alpha} f(t, u(t), \mathfrak{D}u(t))$$

$$- u_0 + \mu u(a) - \mu v(t) - {}^{AB}{}_a I^{\alpha} f(t, v(t), \mathfrak{D}v(t)) \|$$

$$\leq \mu \| u - v \| + \left\| \frac{1-\alpha}{B(\alpha)} f(t, u(t), \mathfrak{D}u(t)) + \frac{\alpha}{B(\alpha)} ({}_a I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \right\|$$

$$- \left\| \frac{1-\alpha}{B(\alpha)} f(t, v(t), \mathfrak{D}v(t)) + \frac{\alpha}{B(\alpha)} ({}_a I^{\alpha} f(t, v(t), \mathfrak{D}v(t)) \right\|$$

$$\leq \mu \| u - v \| + \frac{1-\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) + f(t, v(t), \mathfrak{D}v(t)) \|$$

$$+ \frac{\alpha}{B(\alpha)} ({}_a I^{\alpha}) \| f(t, u(t), \mathfrak{D}u(t)) + f(t, v(t), \mathfrak{D}v(t)) \|$$

$$\leq \mu \| u - v \| + \frac{1-\alpha}{B(\alpha)} (\mathfrak{M} \| u - v \| + \mathfrak{N}t \| u - v \|)$$

$$+ \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(a)} (\mathfrak{M} \| u - v \| + \mathfrak{N}t \| u - v \|)$$

$$\leq \mu \| u - v \| + \left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(a)} \right) (\mathfrak{M} + \mathfrak{N}t) \| u - v \|$$

$$\leq (\mu + \rho) \| u - v \|$$

$$\leq \| u - v \|$$

As $(\mu + \rho) \leq 1$ we have $\| T(u) - T(v) \| \leq \| u - v \|$. Hence, the operator $Tu(t),\ t \in B_\lambda$, showed that the existence and uniqueness conditions and has a fixed point by Banach contraction principle in Banach spaces X. □

Consequently, we investigate the problem (3) and (4) has a fixed point by using another fixed point technique. Krasnoselskii's fixed point theorem.

Theorem 3.3. *If* (A1) *and* (A4) *are satisfied and*

$$q(t_2 - t_1) = [\mathfrak{M}(\| u(t_2) - u(t_1) \| + \mathfrak{N}t \| u(t_2) - u(t_1) \|)]$$

then the problem (3) *and* (4) *has a solution.*

Proof. Here we define the two operator T_1 and T_2 for any constants $\lambda_0 > 0$ \$\l and $u \in B_{\lambda_0}$, as follows

$$(T_1 u)(t) = u_0 - \mu u(a) + \mu u(t) \tag{20}$$

$$(T_2 u)(t) = {}^{AB}{}_{a}I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \tag{21}$$

Clearly, u is a solution (3) and (4) if and only if the operator $T_1 u + T_2 u = u$ has a solution $u \in B_{\lambda_0}$ the proof will divide into three steps.

Step 1. $\| T_1 u + T_2 u \| \leq \lambda_0$, whenever $u \in B_{\lambda_0}$.

For all $u \in B_{\lambda_0}$

$$\| (T_1 u)(t) + (T_2 u)(t) \| \leq \| u_0 \| + \mu \| u \| + \| {}^{AB}{}_{a}I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \|$$

$$\leq \| u_0 \| + \mu \| u \| + \left\| \frac{1 - \alpha}{B(\alpha)} f(t, u(t), \mathfrak{D}u(t)) + \frac{\alpha}{B(\alpha)} ({}_{a}I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \right\|$$

$$\leq \| u_0 \| + \mu\lambda_0 + \frac{1 - \alpha}{B(\alpha)} (\mathfrak{M} \| u \| + \mathfrak{N}t \| u \|) + \frac{(b - a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} (\mathfrak{M} \| u \| + \mathfrak{N}t \| u \|)$$

$$\leq \lambda(1 - \mu - \rho) + \mu\lambda + \left(\frac{1 - \alpha}{B(\alpha)} + \frac{(b - a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \right) (\mathfrak{M} + \mathfrak{N}t) \| u \|$$

$$\leq \lambda_0$$

Therefore

$$\| (T_1 u)(t) + (T_2 u)(t) \| \leq \lambda_0 \text{ for all } u \in B_{\lambda_0}.$$

Step 2. T_1 is a contraction on B_{λ_0} for all B_{λ_0}. Based on the assumption (A4) and (20) we have

$$\| (T_1 u)(t) + (T_2 u)(t) \| = \| u_0 - v_0 \| [(1 + \mu) \| u - v \|] \leq R \| u_0 - v_0 \|$$

Since $R = 1$ where $(1 + \mu) \| u - v \|$ this shows that is a contraction.

Step 3. T_2 is completely continuous operator.

Initially, we have to prove that T_2 is continuous on B_{λ_0}. For all $u_n, u \subset B_{\lambda_0}, n = 1, 2, 3, \ldots$ with $\lim_{n \to \infty} \| u_n - u \| = 0,$ then we get $\lim_{n \to \infty} u_n(t) = u(t),$ for $t \in [a, b]$

Thus by (A1), we have

$$\lim_{n \to \infty} f(t, u_n(t), \mathfrak{D} u_n(t)) = f(t, u(t), \mathfrak{D} u(t))$$

for $t \in [a, b]$ we conclude that

$$\sup_{s \in [a, b]} \| f(t, u_n(t), \mathfrak{D} u_n(t)) - f(t, u(t), \mathfrak{D} u(t)) \| \to 0 \text{ as } n \to \infty$$

On the other hand, for $t \in [a, b]$

$$\| (T_1 u_n)(t) + (T_2 u)(t) \| \leq \| {}^{AB}{}_a I^\alpha f(t, u_n(t), \mathfrak{D} u_n(t)) - {}^{AB}{}_a I^\alpha f(t, u(t), \mathfrak{D} u(t)) \|$$

$$\leq \frac{1 - \alpha}{B(\alpha)} (\| f(t, u_n(t), \mathfrak{D} u_n(t)) - f(t, u(t), \mathfrak{D} u(t)) \|)$$

$$+ \frac{\alpha}{B(\alpha)} (\| f(t, u_n(t), \mathfrak{D} u_n(t)) - f(t, u(t), \mathfrak{D} u(t)) \|) ({}_a I^\alpha (t))$$

$$\leq \frac{1 - \alpha}{B(\alpha)} \sup_{s \in [a, b]} \| f(t, u_n(t), \mathfrak{D} u_n(t)) - f(t, u(t), \mathfrak{D} u(t)) \|$$

$$\leq \frac{(b - a)^\alpha}{B(\alpha) \Gamma(\alpha)} \sup_{s \in [a, b]} \| f(t, u_n(t), \mathfrak{D} u_n(t)) - f(t, u(t), \mathfrak{D} u(t)) \|$$

$$\leq \left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right) \sup_{s\in[a,\,b]} \| f(t, u_n(t), \mathfrak{D}u_n(t)) - f(t, u(t), \mathfrak{D}u(t)) \|$$

Therefore, $\| (T_1u_n)(t) + (T_2u)(t) \| \to 0$ as $n \to \infty$. Hence T_2 is continuous on B_{λ_0}.

Let $T_2u \cdot u \in B_{\lambda_0}$ we have to prove that it is relatively compact which is agreeable to prove that the function $T_2u \cdot u \in B_{\lambda_0}$ uniformly bounded and equicontinuous, and for each $t \in [a, b]$.

$\| T_2u \| \leq \lambda_0$, for all $u \in B_{\lambda_0}$ Hence $(T_2u)(t), u \in B_{\lambda_0}$ is bounded uniformly. Instantly we prove that $(T_2u)(t), u \in B_{\lambda_0}$ is equicontinuous, For all $u \in B_{\lambda_0}$ and $a \leq t_1 \leq t_2 \leq t$, we get

$$\| (T_2u_n)(t_2) + (T_2u)(t_1) \| \leq \| {}^{AB}{}_{a}I^{\alpha} f(t_2, u_n(t_2), \mathfrak{D}u_n(t_2)) - {}^{AB}{}_{a}I^{\alpha} f(t_1, u(t_1), \mathfrak{D}u(t_1)) \|$$

$$\leq \frac{1-\alpha}{B(\alpha)} (\| f(t_2, u_n(t_2), \mathfrak{D}u_n(t_2)) - {}^{AB}{}_{a}I^{\alpha} f(t_1, u(t_1), \mathfrak{D}u(t_1)) \|)$$

$$+ \frac{\alpha}{B(\alpha)} ({}_{a}I^{\alpha})(\| f(t_2, u_n(t_2), \mathfrak{D}u_n(t_2)) - f(t_1, u(t_1), \mathfrak{D}u(t_1)) \|)$$

$$\leq \frac{1-\alpha}{B(\alpha)} (\mathfrak{M}\| u(t_2) - u(t_1) \| + \mathfrak{N}t\| u(t_2) - u(t_1) \|)$$

$$+ \frac{\alpha}{B(\alpha)} (\| f(t_2, u_n(t_2), \mathfrak{D}u_n(t_2)) - f(t_1, u(t_1), \mathfrak{D}u(t_1)) \|)({}_{a}I^{\alpha})(t_2 - t_1)$$

$$\leq \frac{1-\alpha}{B(\alpha)} \sup_{s\in[a,\,b]} \| f(t, u_n(t), \mathfrak{D}u_n(t)) - f(t, u(t), \mathfrak{D}u(t)) \|$$

$$\leq \frac{1-\alpha}{B(\alpha)} q(t_2 - t_1) + \frac{\alpha}{B(\alpha)} q(t_2 - t_1) \frac{(t_2 - t_1)^{\alpha}}{\alpha\Gamma(\alpha)}$$

$$\leq \left(\frac{1-\alpha}{B(\alpha)} + \frac{(t_2 - t_1)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right) q(t_2 - t_1)$$

$\| (T_2u_n)(t_2) + (T_2u)(t_1) \| \to 0$ as $t_2 \to t_1$. Hence the operator T_2 is a equicontinuous on B_{λ_0} Thus T_2 is relatively compact on B_{λ_0}.

Hence T_2 is relatively compact subset of X by theorem (2.7) and by theorem (2.8) we conclude that T_2 has at least one fixed point. Therefore the operator T has a fixed point u which is the solution of (3) and (4). □

4. Example

Consider the given problem

$$\left({}^{ABC}{}_{a}D^{\frac{3}{2}}u\right)(t) - \mu u(t) = \frac{2t}{1+3e^{t}} \frac{u(t)+u'(t)}{1+u(t)}$$

Here consider

$$\alpha = \frac{3}{2},\ \mu = 1,\ b = 1,\ f(t, u(t), \mathfrak{D}u(t)) = \frac{2t}{1+3e^{t}} \frac{u(t)+u'(t)}{1+u(t)},$$

$$B(\alpha) = 1 \tag{22}$$

$$u(0) = 1 \tag{23}$$

we have that

$$f(t, u(0), \mathfrak{D}u(0)) = 0$$

and

$$u'(t) \in C[0, 1]$$

Now the given problem satisfy the Lipschitzconditions. Let

$$f(t, u, v) = \frac{2t}{1+3e^{t}} \frac{u+v}{1+u'},\ t \in [0, 1]$$

Now

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le \frac{2t}{1+3e^{t}} (|u_1 - v_1| + |u_2 - v_2|)$$

for all $t \in [0, 1]$, $u_1, u_2, v_1, v_2 \in \Re$

$$\le \frac{1}{2} |u - v|$$

Hence $\mu + \rho = \frac{1}{2}$, Then

$$\mu + \rho\left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-\alpha)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right) = \frac{1}{2}\left(1 - \alpha + \frac{1}{\Gamma(\alpha)}\right)$$

From the theorem (16), and equation (22) and (23) has a unique solution. It can be written as

$$u(t) = \lim_{n\to\infty} u_n(t)$$

Where

$$u_n(t) = 1 + \frac{1}{2}u_{n-1}(t) + \frac{1}{2}\left({}^{AB}{}_{0}I^{\alpha}\right)(tu_{n-1}(t)),\ n = 0, 1, 2, \ldots$$

Otherwise

$$= 1 + \frac{1}{2}u_{n-1}(t) + \frac{1-\alpha}{2}tu_{n-1}(t) + \frac{\alpha}{2\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_{n-1}(s)ds,\ n = 1, 2, 3, \ldots$$

we solve the equation (22) and (23) we apply the method proposed by Mekkaoui and Atangana in [22], utilizing from the two-step Lagrange polynomial interpolation.

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