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SOME FIXED POINT THEOREMS FOR CONTRACTIVE CONDITION OF INTEGRAL TYPE IN INTUITIONISTIC GENERALIZED FUZZY METRIC SPACES

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Abstract

In this paper, we shall establish some fixed point theorems for mappings with the contractive condition of integrable type of complete intuitionistic generalized fuzzy metric spaces $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$. We also use Lebesgue-integrable mapping to obtain new results. In this paper by using the main idea of the work, we introduce the concept of A-fuzzy contractive mappings.

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1. Introduction

Since the concept of fuzzy sets was introduced by Zadeh [16], there are many study results in this area. Some of them are dedicated to generalize the definition of fuzzy set. Atanassov [4] introduce the concept of intuitionistic fuzzy sets, Park [12] defined and studied a notion of intuitionistic fuzzy metric spaces, as a natural generalization of fuzzy metric spaces due to Kromosil, Michalek [9], George, Veeramani [7], Amari and Moutawakli [1] and Liu et. al., [10] respectively, define the property (E. A.) and common property (E.A.) and utilize the same to prove common fixed point theorems in metric spaces. Branciari [5] gave a fixed point result for a single mapping satisfying Banach's contraction principle for an integral type inequality. The authors [3, 5, 6, 14] proved fixed point theorems using generalized contractive conditions of integral type. In this paper, we shall establish some fixed point theorems for mappings with the contractive condition of integrable type of complete intuitionistic generalized fuzzy metric spaces $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$. We also use Lebesgue-integrable mapping to obtain new results.

2. Preliminaries

Definition 2.1. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm, if it satisfies the following conditions:

(i) $*$ is associative and commutative,

(ii) $*$ is continuous,

(iii) $a^* 1 = a$ for all $a \in [0, 1]$,

(iv) $a^* b \leq c^* d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in ([0, 1])$.

Examples of continuous t-norm are $a^* b = ab$ and $a^* b = \min\{a, b\}$, $a^* b = \min\{a + b - 1, 0\}$ and $a^* b = \dfrac{ab}{\min\{a, b, \lambda\}}$, for $0 < \lambda < 1$ are continuous t-conforms.

Definition 2.2. A binary operation $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-conform, if $\Diamond$ satisfies the following conditions:

(i) $\Diamond$ is commutative and associative,

(ii) $\Diamond$ is continuous,

(iii) $a \Diamond 0 = a$ for all $a \in [0, 1]$,

(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of continuous t-norm are $a \Diamond b = \min\{a + b, 1\}$, $a \Diamond b = \max\{a, b\}$ are continuous t-conforms.

Definition 2.3. A 5-tuple $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ is called a intuitionistic generalized fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, $\Diamond$ a continuous t-conorm and $\mathcal{M}$ and $\mathcal{N}$ are fuzzy sets on $X^3 \times (0, \infty)$, satisfying the following conditions:

For each $x, y, z, a \in X$ and $t, s > 0$.

(a) $\mathcal{M}(x, y, z, t)^* \mathcal{N}(x, y, z, t) \leq 1$,

(b) $\mathcal{M}(x, y, z, t) > 0$,

(c) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(d) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,

(e) $\mathcal{M}(x, y, a, t)^* \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(f) $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ continuous,

(g) $\mathcal{N}(x, y, z, t) > 0$,

(h) $\mathcal{N}(x, y, z, t) = 0$, if and only if $x = y = z$,

(i) $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}t)$, where p is a permutation function,

(j) $\mathcal{N}(x, y, z, a, t) \Diamond \mathcal{N}(a, z, z, s) \geq \mathcal{N}(x, y, z, t + s)$,

(k) $\mathcal{N}(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Then $(\mathcal{M}, \mathcal{N})$ is called a intuitionistic generalized fuzzy metric on X.

Example 2.4. Let (X, d) be a metric space. Denote $a^*b = ab$ and $a \lozenge b = \min\{a+b, 1\}$ for all $a, b \in [0, 1]$ and let $\mathcal{M}_d$ and $\mathcal{N}_d$ be fuzzy sets on $X^3 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_d(x, y, z, t) = \frac{ht^n}{ht^n + mD^*(x, y, z)}, \ \mathcal{N}_d(x, y, z, t) = \frac{D^*(x, y, z)}{kt^n + mD^*(x, y, z)},$$

for all $h, k, m, n \in \mathbb{R}^+$.

If $h = k = m = n = 1$, we get

$$\mathcal{M}_d(x, y, z, y) = \frac{t}{t + D^*(x, y, z)}, \ \mathcal{N}_d(x, y, z, t) = \frac{D^*(x, y, z)}{t + D^*(x, y, z)}.$$

We call this intuitionistic generalized fuzzy metric induced by a metric D^*. The standard intuitionistic generalized fuzzy metric and $(X, M_d, N_d, *, \lozenge)$ is an intuitionistic generalized fuzzy metric space.

Definition 2.5. Let $(X, \mathcal{M}, \mathcal{N}, *, \lozenge)$ be an intuitionistic generalized fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ are defined by

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r, \mathcal{N}(x, y, y, t) < r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \leq A$.

Definition 2.6.

(i) A sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \to 1$ and $\mathcal{N}(x, x, x_n, t) \to 0$, as $n \to \infty$ for each $t > 0$.

(ii) A sequence $\{X_n\}$ is called a Cauchy sequence if for each $0 < \in < 1$ and $t > 0$, there exist $n_0 \in \mathcal{N}$ such that $\mathcal{M}(X_n, X_n, X_m, t) > 1 - \in$ and $\mathcal{N}(X_n, X_n, X_m, t) < \in$ for each $n, m \geq n_\circ$.

(iii) A intuitionistic generalized fuzzy metric $(X, \mathcal{M}, \mathcal{N}, *, \lozenge)$ is said to be complete if every Cauchy sequence is convergent.

Definition 2.7. A sequence $\{x_n\}$ in an intuitionistic generalized fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ is called intuitionistic fuzzy contractive sequence if there exists $0 < k < 1$ such that $\left[\frac{1}{\mathcal{M}(x_{n+1}, x_{n+2}, t)} - 1\right] \leq$ $\lambda\left[\frac{1}{\mathcal{M}(x_n, x_n, x_{n+1}, t)} - 1\right]$ and $\mathcal{N}(x_{n+1}, x_{n+1}, x_{n+2}, t) \leq k\, \mathcal{N}(x_n, x_n, x_{n+1}, t)$, for all n and $t > 0$.

Definition 2.8. Let $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ be an intuitionistic generalized fuzzy metric space. A Self map f on X is said to be intuitionistic generalized fuzzy contractive whenever there exists

$$k \in (0, 1). \left[\frac{1}{\mathcal{M}(f(x), f(y), f(z), t)} - 1\right] \leq k\left[\frac{1}{\mathcal{M}(x, y, z, t)} - 1\right] \text{ and}$$

$$\mathcal{N}(f(x), f(y), f(z), t) \leq k\, \mathcal{N}(x, y, z, t), \text{ for all } x, y, z \in X \text{ and } t > 0.$$

Definition 2.9 Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and A be the set of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying.

(A1) α is continuous on the set $\mathbb{R}_+^3$ (With respect to the Euclidean metric on $\mathbb{R}^3$),

(A2) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all a, b.

3. Main Results

Theorem 3.1. *Let $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ be a complete intuitionistic generalized fuzzy metric space, $c \in (0, 1)$ and let $f : X \to X$ be a mapping such that for each $x, y, z \in X, t > 0$,*

$$\int_0^{\frac{1}{\mathcal{M}(fx, fy, fz, t)}-1} \varphi(s)ds \leq c \int_o^{\frac{1}{\mathcal{M}(x, y, z, t)}-1} \varphi(s)ds \tag{3.1.1}$$

$$\int_0^{\mathcal{N}(fx, fy, fz, t)} \varphi(s)ds \leq c\int_0^{\mathcal{N}(x, y, z, t)} \varphi(s)ds. \tag{3.1.2}$$

Where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, +\infty)$, nonnegative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(s)ds > 0$. Then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \to +\infty} f^n x = a$.

Proof. Step 1. We have,

$$\int_0^{\frac{1}{\mathcal{M}(f^n x,\, f^{n+1}x,\, f^{n+1}x,\, t)}-1} \varphi(s)ds \leq c^n \int_0^{\frac{1}{\mathcal{M}(x,\, fx,\, fx,\, t)}-1} \varphi(s)ds.$$

This follows immediately by iterating (3.1.1) n times:

$$\int_0^{\frac{1}{\mathcal{M}(f^n x,\, f^{n+1}x,\, f^{n+1}x,\, t)}-1} \varphi(s)ds \leq c \int_0^{\frac{1}{\mathcal{M}(f^{n-1} x,\, f^{n}x,\, t)}-1} \varphi(s)ds$$

$$\leq \ldots \leq c^n \int_0^{\frac{1}{\mathcal{M}(x,\, fx,\, fx,\, t)}-1} \varphi(s)ds.$$

As a consequence, since $c \in (0, 1)$, we get

$$\int_0^{\frac{1}{\mathcal{M}(f^n x,\, f^{n+1}x,\, f^{n+1}x,\, t)}-1} \varphi(s)ds \leq c^n \int_0^{\frac{1}{\mathcal{M}(x,\, fx,\, fx,\, t)}-1} \varphi(s)ds \to 0^+.$$

Step 2. We have, $\frac{1}{\mathcal{M}(f^n x,\, f^{n+1}x,\, f^{n+1}x,\, t)} - 1 \to 0$ as $n \to \infty$.

Suppose that $\lim_{n \to +\infty} \operatorname{Sup}\left(\frac{1}{\mathcal{M}(f^n x,\, f^{n+1}x,\, f^{n+1}x,\, t)} - 1\right) = \frac{\varepsilon}{t} > 0$. Then there exists $m_\varepsilon \in \mathbb{N}$ and a sequence $\{f^{n_m}\}_{m \geq m_\varepsilon}$, such that $\left(\frac{1}{\mathcal{M}(f^n x, f^{n+1}x, f^{n+1}x, t)} - 1\right) \to \frac{\varepsilon}{t} > 0$ as $m \to +\infty$ and $\left(\frac{1}{\mathcal{M}(f^{n_m} x,\, f^{n_m+1}x,\, f^{n_m+1}x,\, t)} - 1\right) \geq \frac{\varepsilon}{2t}$, for $m \geq m_\varepsilon$. Thus, by Step 1 and the sign of φ, we have the following contradiction:

$$0 = \lim_{m \to +\infty} \int_0^{\frac{1}{\mathcal{M}(f^{n_m+1} x,\, f^{n_m+1}x,\, f^{n_m+1}x,\, t)}-1} \varphi(s)ds \geq \int_0^{\frac{\varepsilon}{2t}} \varphi(s)ds > 0.$$

Step 3. For each $x \in X, \{f^{n_x}\}_{n \in \mathbb{N}}$ is a Cauchy sequence that is for all $\varepsilon > 0$

there exists $m_\varepsilon \in \mathbb{N}$, for all $m, n \in N$, $m > n > m_\varepsilon$: $\frac{1}{\mathcal{M}(f^m x, f^n x, f^n x, t}$ $-1 < \frac{\varepsilon}{t}$.

Suppose that there exist a $\varepsilon > 0$ such that $l \in \mathbb{N}$ there are $m_l, n_l \in N$ with $m_l > n_l > l$ such that $\frac{1}{\mathcal{M}(f^{m_l}x, f^{n_l}x, f^{n_l}x, t)} - 1 \geq \frac{\varepsilon}{t}$. Then we choose the sequence $\{m_l\}_{l \in N}, \{n_l\}_{l \in N}$ such that for each $l \in \mathbb{N}, m_l$ is minimal in $\frac{1}{\mathcal{M}(f^{m_l}x, f^{n_l}x, f^{n_l}x, t)} - 1 \geq \frac{\varepsilon}{t}$. But $\frac{1}{\mathcal{M}(f^{n}x, f^{n_l}x, f^{n_l}x, t)} - 1 < \frac{\varepsilon}{t}$, for each $h \in \{n_l + 1, \ldots m_l - 1\}$. Now, we analyze the properties of $\frac{1}{\mathcal{M}(f^{m_l}x, f^{n_l}x, f^{n_l}x, t)} - 1$ and $\frac{1}{\mathcal{M}(f^{m_l+1}x, f^{n_l+1}x, f^{n_l+1}x, t)} - 1$. In the first, we have $\frac{1}{\mathcal{M}(f^{m_l}x, f^{n_l}x, f^{n_l}x, t)} - 1 \to \frac{\varepsilon^+}{t}$, as $l \to +\infty$.

Now, by the triangular inequality and step 2,

$$\frac{\varepsilon}{t} \leq \frac{1}{\mathcal{M}(f^{m_l}x, f^{n_l}x, f^{n_l}x, t)} - 1 \leq \frac{1}{\mathcal{M}(f^{m_l}x, f^{m_l}x, f^{m_l-1}x, t)} - 1$$

$$+ \frac{1}{\mathcal{M}(f^{m_l-1}x, f^{n_l}x, f^{n_l}x, t)} - 1$$

$$< \frac{1}{\mathcal{M}(f^{m_l}x, f^{m_l}x, f^{m_l-1}x, t)} - 1$$

$$+ \frac{\varepsilon}{t} \to \frac{\varepsilon}{t}, \text{ as } l \to \infty.$$

Further, there exists $\mu \in \mathbb{N}$ such that for each natural number $\upsilon > \mu$,

$$\frac{1}{\mathcal{M}(f^{m_\upsilon+1}x, f^{n_\upsilon+1}x, f^{n_\upsilon+1}x, t)} - 1 < \frac{\varepsilon}{t}.$$

In fact, if there exists a sub sequence $\{\nu_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that, $\frac{1}{\mathcal{M}(f^{m_{\upsilon_p}+1}x, f^{n_{\upsilon_p}+1}x, f^{k_{\upsilon_p}+1}x, t)} - 1 < \frac{\varepsilon}{t}$, then

$$\frac{\varepsilon}{t} \leq \frac{1}{\mathcal{M}(f^{m_{\nu_p}+1}x, f^{n_{\nu_p}+1}x, f^{n_{\nu_p}+1}x, t)} - 1$$

$$\leq \frac{1}{\mathcal{M}(f^{m_\nu p+1}x, f^{m_\nu p}x, f^{m_\nu p}x, t)} - 1 + \frac{1}{\mathcal{M}(f^{m_\nu p}x, f^{n_\nu p}x, f^{n_\nu p}x, t)} - 1$$

$$+ \frac{1}{\mathcal{M}(f^{n_\nu p}x, f^{n_\nu p+1}x, f^{n_\nu p+1}x, t)} - 1 \to \frac{\varepsilon}{t}, \text{ as } k \to +\infty \text{ and from (3.1.1),}$$

$$\int_0^{\frac{1}{\mathcal{M}(f^{m_\nu p+1}x, f^{n_\nu p+1}x, f^{n_\nu p+1}x, t)}-1} \varphi(s)ds \leq c \int_0^{\frac{1}{\mathcal{M}(f^{m_\nu p}x, f^{n_\nu p}x, f^{n_\nu p}x, t)}-1} \varphi(s)ds. \quad (3.1.3)$$

Letting now $k \to +\infty$ in both sides of (3.1.3), we have $\int_0^{\varepsilon} \varphi(s)ds \leq c\int_0^{\varepsilon} \varphi(s)ds.$

Which is a contradiction being $c \in (0, 1)$ and the integral being positive.

Therefore, for certain $\mu \in \mathbb{N}, \frac{1}{\mathcal{M}(f^{m_\nu+1}x, f^{n_\nu+1}x, f^{n_\nu+1}x, t)} - 1 < \frac{\varepsilon}{t},$ for all $\nu > \mu$. Finally, we prove the stronger property that there exist a $\sigma_\varepsilon \in (0, \varepsilon)$ and a $v_\varepsilon \in \mathbb{N}$ such that for each $v > v_\varepsilon (v \in \mathbb{N})$, we have $\frac{1}{\mathcal{M}(f^{m_\nu+1}x, f^{n_\nu+1}x, f^{n_\nu+1}x, t)} - 1 < \frac{\varepsilon - \sigma_\varepsilon}{t}$.

Suppose the existence of the subsequence $\{v_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that,

$$\frac{1}{\mathcal{M}(f^{m_\nu p+1}x, f^{n_\nu p+1}x, f^{n_\nu p+1}x, t)} - 1 \to \frac{\varepsilon}{t}, \text{ as } k \to +\infty.$$

Also, we have,

$$\int_0^{\frac{1}{\mathcal{M}(f^{m_\nu p+1}x, f^{n_\nu p+1}x, f^{n_\nu p+1}x, t)}-1} \varphi(s)ds \leq c \int_0^{\frac{1}{\mathcal{M}(f^{m_\nu p}x, f^{n_\nu p}x, f^{n_\nu p}x, t)}-1} \varphi(s)ds.$$

Letting $k \to +\infty$, we have again the contradiction that, $\int_0^{\frac{\varepsilon}{t}} \varphi(s)ds \leq c\int_0^{\frac{\varepsilon}{t}} \varphi(s)ds.$

In conclusion of this step, we can prove the Cauchy character of $\{f^{n_x}\}_{n \in \mathbb{N}} (x \in X)$.

In fact, for each natural number $\nu > \nu_\varepsilon (\nu_\varepsilon$ as above), we have

$$\frac{\varepsilon}{t} \leq \frac{1}{\mathcal{M}(f^{m_\nu}x, f^{n_\nu}x, f^{n_\nu}x, t)} - 1$$

$$\leq \frac{1}{\mathcal{M}(f^{m_\nu}x,\, f^{m_\nu+1}x,\, t)} - 1 + \frac{1}{\mathcal{M}(f^{m_\nu+1}x,\, f^{n_\nu+1}x,\, f^{n_\nu+1}x,\, t)} - 1$$

$$+ \frac{1}{\mathcal{M}(f^{n_\nu+1}x,\, f^{n_\nu}x,\, f^{n_\nu}x,\, t)} - 1$$

$$< \frac{1}{\mathcal{M}(f^{m_\nu}x,\, f^{m_\nu+1}x,\, f^{m_\nu+1}x,\, t)} - 1 + (\varepsilon - \sigma_\varepsilon)$$

$$+ \frac{1}{\mathcal{M}(f^{n_\nu}x,\, f^{n_\nu+1}x,\, f^{n_\nu}x,\, t)} - 1.$$

When $\nu \to +\infty$, we have $\varepsilon < \varepsilon - \sigma_\varepsilon$, which is a contradiction. This proves Step 3.

Step 4. Existence of a fixed point. Since $(X, \mathcal{M}, \mathcal{N}, *, \lozenge)$ is a complete intuitionistic generalized fuzzy metric space, there exists a point $a \in X$ such that $a = \lim_{n\to\infty} f^n x$.

Further a is a fixed point. In fact, suppose that $\frac{1}{\mathcal{M}(a,\, fa,\, fa,\, t)} - 1 > 0$.

Then

$$0 < \frac{1}{\mathcal{M}(a, fa, fa, t)} - 1 \leq \frac{1}{\mathcal{M}(a, f^{n+1}x, f^{n+1}x, t)} - 1 + \frac{1}{\mathcal{M}(f^{n+1}x, fa, fa, t)} - 1 \to 0,$$

$$\text{as } n \to +\infty. \tag{3.1.4}$$

Because $\mathcal{M}(a,\, a,\, f^{n+1}x,\, t)$ and $\mathcal{M}(f^{n+1}x,\, fa,\, fa,\, t)$ converges to 1 as $n \to +\infty$. For the first one it is obvious, while the second one we have

$$\int_0^{\frac{1}{\mathcal{M}(f^{n+1}x,\, fa,\, fa,\, t)} - 1} \varphi(s)ds \leq c\int_0^{\frac{1}{\mathcal{M}(f^{n}x,\, a,\, a,\, t)} - 1} \varphi(s)ds \to 0,\, as\ n \to +\infty.$$

Now if $\mathcal{M}(f^{n+1}x, fa, fa, t)$ does not converge to 1 as $n \to \infty$, then there exists a subsequence $\{f^{n_\nu+1}x\}_{n\in\mathbb{N}} \subseteq \{f^{n+1}x\}_{n\in\mathbb{N}}$, such that $\frac{1}{\mathcal{M}(f^{n_\nu+1}x,\, a,\, a,\, t)} \geq \frac{\varepsilon}{t}$, for a Certain $\varepsilon < 0$. Thus, we have following contradictions:

$$0 < \int_0^{\frac{\varepsilon}{t}} \varphi(s)ds \leq \int_0^{\frac{1}{\mathcal{M}(f^{n_\nu+1}x,\, fa,\, fa,\, t)}-1} \varphi(s)ds, \text{ as } \nu \to +\infty.$$

Step 5: Uniqueness of the fixed point. Suppose that there are two distinct points $a, b, c \in X$ such that $fa = a$, $fb = b$ and $fc = c$. Then by (3.1.1), we have the contradiction,

$$0 < \int_0^{\frac{1}{\mathcal{M}(a,\, b,\, c,\, t)}-1} \varphi(s)ds = \int_0^{\frac{1}{\mathcal{M}(fa,\, fb,\, fc,\, t)}-1} \varphi(s)ds$$

$$\leq c\int_0^{\frac{1}{\mathcal{M}(a,\, b,\, c,\, t)}-1} \varphi(s)ds < \int_0^{\frac{1}{\mathcal{M}(a,\, b,\, c,\, t)}-1} \varphi(s)ds.$$

The final step also proves that for each $x \in X$, $\lim_{n \to +\infty} f^n x = a = fa$.

The proof is completed.

Now we give remark and examples concerning these contractive mappings of integral type, which clarify the connection between our result and the classical ones.

Remark 3.2. Theorem (3.1) is a generalization of the Banach principle, letting $\varphi(s) = 1$ for each $s \geq 0$ in (3.1.1), we have,

$$\int_0^{\frac{1}{\mathcal{M}(fx,\, fy,\, fz,\, t)}-1} \varphi(s)ds = \frac{1}{\mathcal{M}(fx,\, fy,\, fz,\, t)} - 1 \leq c\left(\frac{1}{\mathcal{M}(fx,\, fy,\, fz,\, t)} - 1\right)$$

$$= c\int_0^{\frac{1}{\mathcal{M}(x,\, y,\, z,\, t)}-1} \varphi(s)ds.$$

Thus, a Banach fuzzy contraction also satisfies (3.1.1). The converse is not true as we will see in follow examples.

Example 3.3. Let $f : R^+ \to R^+$ be defined by $fx = x + 2$, $\varphi \equiv -2$, $\mathcal{M}(x,\, y,\, z,\, t) = \frac{t}{t + D(x,\, y,\, z)}$ and $\mathcal{N}(x,\, y,\, z,\, t) = \frac{D(x,\, y,\, z)}{t + D(x,\, y,\, z)}$ and d be the Euclidean distance function. Then for an arbitrary $c \in (0, 1)$, we have

$$\int_0^{\frac{1}{\mathcal{M}(fx,\, fy,\, fz,\, t)}-1} \varphi(s)ds = -2\left(\frac{1}{\mathcal{M}(fx,\, fy,\, fz,\, t)} - 1\right) = -2\left(\frac{1}{\mathcal{M}(x,\, y,\, z,\, t)} - 1\right)$$

$$\leq -2c\left(\frac{1}{\mathcal{M}(fx,\, fy,\, fz,\, t)} - 1\right) = c\int_0^{\frac{1}{\mathcal{M}(x,\, y,\, z,\, t)}-1} \varphi(s)ds.$$

Thus (3.1.1) is satisfied with $\varphi \equiv -2$ and for all $c \in (0, 1)$, but f, being a translation on R^+, has no fixed points.

Definition 3.4 A self map f on intuitionistic generalized fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ is said to be an A-fuzzy contraction if it satisfies the condition

$$\frac{1}{\mathcal{M}(fx, fy, fz, t)} - 1 \leq \alpha\Big(\frac{1}{\mathcal{M}(x, y, z, t)} - 1, \frac{1}{\mathcal{M}(x, fx, fx, t)} - 1, \frac{1}{\mathcal{M}(y, fy, fy, t)} - 1,$$

$$\frac{1}{\mathcal{M}(z, fz, fz, t)} - 1\Big),$$

for all $x, y, z \in X$ and some $\alpha \in A$.

Lemma 3.5. *Let a self-map f on intuitionistic generalized fuzzy metric spaces $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ for all $x, y, z \in X, t > 0$ and some $\beta \in \left[0, \frac{1}{2}\right)$, satisfying*

$$\frac{1}{\mathcal{M}(fx,\, fy,\, fz,\, t)} - 1$$

$$\leq \beta \max \left\{ \begin{array}{l} \frac{1}{\mathcal{M}(fx,\, x,\, x,\, t)} + \frac{1}{\mathcal{M}(fy,\, y,\, y,\, t)} + \frac{1}{\mathcal{M}(fz,\, z,\, z,\, t)} - 3, \\ \frac{1}{\mathcal{M}(fz,\, z,\, z,\, t)} + \frac{1}{\mathcal{M}(x,\, y,\, z,\, t)} - 2\, \frac{1}{\mathcal{M}(fy,\, y,\, y,\, t)} + \frac{1}{\mathcal{M}(x,\, y,\, z,\, t)} - 2, \\ \frac{1}{\mathcal{M}(fx,\, x,\, x,\, t)} + \frac{1}{\mathcal{M}(fx,\, y,\, z,\, t)} - 2 \end{array} \right\}$$

is an A-fuzzy contraction.

Proof. Define the map $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ as $\alpha(u, v, w) = \beta \max\{u + v, + w, u + w\}$

for all $u, v, w \in \mathbb{R}_+$, where β is any fixed number in $\left[0, \frac{1}{2}\right)$. Then $\alpha \in A$. First, note that α is continuous, second for $u \le \alpha(u, v, v) = \beta \max\{t + u + v, v + v, u + v, t + v\}$ we consider the following cases.

Case I: $\max\{u + v, u + v, v + v\} = u + v$. In this case, $u \le \frac{\beta}{1-\beta} v$, with $k = \frac{\beta}{1-\beta} \in [0, 1)$.

Case II: $\max\{u + v, v + u, v + v\} = 2v$. In this case $u \le kv$ with $k = 2\beta \in [0, 1)$.

Similarly, for $u \le \alpha(v, u, v)$ or $u \le \alpha(v, v, u)$ we have $u \le kv$ for some $k \in [0, 1)$.

Hence,

$$\frac{1}{\mathcal{M}(fx, fy, fz, t)} - 1 \le \beta \max \begin{Bmatrix} \frac{1}{\mathcal{M}(fx, x, x, t)} + \frac{1}{\mathcal{M}(fy, y, y, t)} + \frac{1}{\mathcal{M}(fz, z, z, t)} - 3 \\ \frac{1}{\mathcal{M}(fz, z, z, t)} + \frac{1}{\mathcal{M}(x, y, z, t)} - 2, \\ \frac{1}{\mathcal{M}(fy, y, y, t)} + \frac{1}{\mathcal{M}(x, y, z, t)} - 2, \\ \frac{1}{\mathcal{M}(fx, x, x, t)} + \frac{1}{\mathcal{M}(x, y, z, t)} - 2 \end{Bmatrix}$$

$$= \alpha\left(\frac{1}{\mathcal{M}(x, y, z, t)} - 1, \frac{1}{\mathcal{M}(x, x, x, t)} - 1 \frac{1}{\mathcal{M}(fy, y, y, t)} - 1, \frac{1}{\mathcal{M}(fz, z, z, t)} - 1\right),$$

by the construction of α. Thus, f is an A-contraction.

Example 3.6. Let $X = \{0, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$, $\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$ and $\mathcal{N}(x, y, z, t) = \frac{D^*(x, y, z)}{t + D^*(x, y, z)}$, with usual metric relative to real line, f be a self-map on X, given by $fx = \begin{cases} 12 & x = 0 \\ 11 & otherwise. \end{cases}$ One can easily verify that f satisfies

$$\frac{1}{\mathcal{M}(fx,\ fy,\ fz,\ t)} - 1 \le \beta \max \begin{Bmatrix} \frac{1}{\mathcal{M}(fx,\ x,\ x,\ t)} + \frac{1}{\mathcal{M}(fy,\ y,\ y,\ t)} + \frac{1}{\mathcal{M}(fz,\ z,\ z,\ t)} - 3, \\ \frac{1}{\mathcal{M}(fz,\ z,\ z,\ t)} + \frac{1}{\mathcal{M}(x,\ y,\ z,\ t)} - 2, \\ \frac{1}{\mathcal{M}(fy,\ y,\ y,\ t)} + \frac{1}{\mathcal{M}(x,\ y,\ z,\ t)} - 2, \\ \frac{1}{\mathcal{M}(fx,\ x,\ x,\ t)} + \frac{1}{\mathcal{M}(x,\ y,\ z,\ t)} - 2 \end{Bmatrix},$$

for all $x,\ y,\ z \in X$ and some $\beta \in \left[0,\ \frac{1}{2}\right)$.

Hence, by lemma (3.5), f is an A-fuzzy contraction.

Theorem 3.7. *Let f be a self-map of a complete intuitionistic generalized fuzzy metric space*, $(X,\ \mathcal{M},\ \mathcal{N},\ *,\ \Diamond)$ *satisfying the following condition*:

$$\int_0^{\frac{1}{\mathcal{M}(fx,\ fy,\ fz,\ t)}-1} \varphi(s)ds \le \alpha\left(\int_0^{\frac{1}{\mathcal{M}(x,\ y,\ z,\ t)}-1} \varphi(s)ds,\ \int_0^{\frac{1}{\mathcal{M}(x,\ fx,\ fx,\ t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}(y,\ fy,\ fy,\ t)}-1} \varphi(s)ds,\ \int^{\frac{1}{\mathcal{M}(z,\ fz,\ fz,\ t)}-1} \varphi(s)ds\right) \quad (3.7.1)$$

for each $x,\ y,\ z \in X$ and $t > 0$ with some $\alpha \in A$, where $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. With finite integral) on each compact subset of $[0,\ +\infty)$, nonnegative and such that for each

$$\varepsilon > 0,\ \int_0^{\varepsilon} \varphi(s)ds > 0. \quad (3.7.2)$$

Then *f* has a unique fixed point $p \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n x = p$.

Proof. Let $x_0 \in X$ is an arbitrary and define $x_n + 1 = fx_n$. From (3.7.1), for each integer $n \ge 1$, we get,

$$\int_0^{\frac{1}{\mathcal{M}(x_n x_{n+1},\ x_{n+2},\ t}-1} \varphi(s)ds = \int_0^{\frac{1}{\mathcal{M}(fx_{n-1},\ fx_n,\ fx_{n+1},\ t}-1} \varphi(s)ds$$

$$\leq \alpha\Bigg(\int_0^{\frac{1}{\mathcal{M}(x_{n-1}, x_n, x_{n+1}, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(x_{n-1}, fx_{n-1}, fx_{n-1}, t)}-1} \varphi(s)ds,$$

$$\int_0^{\frac{1}{\mathcal{M}(x_n, fx_n, fx_n, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(x_{n+1}, fx_{n+1}, fx_{n+1}, t}-1} \varphi(s)ds\Bigg)$$

$$= \alpha\Bigg(\int_0^{\frac{1}{\mathcal{M}(x_{n-1}, x_n, x_{n+1}, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(x_{n-1}, x_n, x_n, t)}-1} \varphi(s)ds,$$

$$\int_0^{\frac{1}{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t)}-1} \varphi(s)ds\Bigg).$$

Then by axiom (A2) of function α,

$$\int_0^{\frac{1}{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds \leq k\int_0^{\frac{1}{\mathcal{M}(x_{n-1}, x_n, x_{n+1}, t)}-1} \varphi(s)ds \qquad (3.7.3)$$

for some $k \in [0, 1)$ as $\alpha \in A$. In this fashion, one can obtain

$$\int_0^{\frac{1}{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds \leq k\int_0^{\frac{1}{\mathcal{M}(x_{n-1}, x_n, x_{n+1}, t)}-1} \varphi(s)ds$$

$$\leq k^2\int_0^{\frac{1}{\mathcal{M}(x_{n-1}, x_n, x_{n+1}, t)}-1} \varphi(s)ds$$

$$\leq k^n\int_0^{\frac{1}{\mathcal{M}(x_0, x_1, x_2, t)}-1} \varphi(s)ds.$$

Taking limit as $n \to +\infty$, we get $\lim_n \int_0^{\frac{1}{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds = 0$ as $k \in [0, 1)$. Which, from (3.7.2) implies that,

$$\lim_n \frac{1}{\mathcal{M}(x_n, x_{n+1}x_{n+2}, t)} - 1 = 0. \qquad (3.7.4)$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not a

Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{m_i\}$ and $\{n_i\}$ such that

$$m_i < n_i < m_{i+1} \text{ with } \frac{1}{\mathcal{M}(x_{m_i}, x_{n_i}, t)} - 1 \geq \frac{\varepsilon}{t}, \frac{1}{\mathcal{M}(x_{m_i}, x_{n_{i-1}} x_{n_{i-1}}, t)} - 1 < \frac{\varepsilon}{t}. \quad (3.7.5)$$

Now, we have

$$\frac{1}{\mathcal{M}(x_{m_{i-1}}, x_{n_i}, x_{n_i}, t)} - 1 \leq \frac{1}{\mathcal{M}(x_{m_{i-1}}, x_{m_i}, x_{m_i}, t)} - 1 + \frac{1}{\mathcal{M}(x_{m_i}, x_{n_{i-1}}, x_{n_{i-1}}, t)} - 1$$

$$< \frac{1}{\mathcal{M}(x_{m_{i-1}}, x_{m_i}, x_{m_i}, t)} - 1 + \frac{\varepsilon}{t} \quad (3.7.6)$$

So, by (3.7.4) and (3.7.6).

We get

$$\lim_i \int_0^{\frac{1}{\mathcal{M}(x_{m_{i-1}}, x_{n_{i-1}} x_{n_{i-1}}, t)}} \varphi(s) ds \leq \int_0^{\varepsilon} \varphi(s) ds. \quad (3.7.7)$$

Using (3.7.3), (3.7.5) and (3.7.7), we have,

$$\int_0^{\varepsilon} \varphi(s) ds \leq \int_0^{\frac{1}{\mathcal{M}(x_{m_i}, x_{n_i} x_{n_i}, t)} - 1} \varphi(s) ds$$

$$\leq k \int_0^{\frac{1}{\mathcal{M}(x_{m_{i-1}}, x_{n_i - 1} x_{n_{i-1}}, t)} - 1} \varphi(s) sd \leq \int_0^{\varepsilon} \varphi(s) ds.$$

Which is a contradiction (since $k \in [0, 1)$).

Thus $\{x_n\}$ is Cauchy and hence, is convergent.

Call the limit p. From (3.7.1), we get

$$\int_0^{\frac{1}{\mathcal{M}(fp, x_{n+1}, x_{n+2}, t)} - 1} \varphi(s) ds \leq \int_0^{\frac{1}{\mathcal{M}(fp, fx_n, fx_{n+1}, t)} - 1} \varphi(s) ds$$

$$\leq \alpha \left(\int_0^{\frac{1}{\mathcal{M}(p, x_n, x_{n+1}, t)} - 1} \varphi(s) ds, \int_0^{\frac{1}{\mathcal{M}(p, fp, fp, t)} - 1} \varphi(s) ds, \right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}(p, fx_n, fx_n, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(x_{n+1}, fx_{n+1}, fx_{n+1}, t)}-1} \varphi(s)ds\right).$$

Taking limit as $n \to \infty$, we get,

$$\int_0^{\frac{1}{\mathcal{M}(fp, p, p, t)}-1} \varphi(s)ds \leq \alpha\left(0, \int_0^{\frac{1}{\mathcal{M}(p, fp, fp, t)}-1} \varphi(s)ds, 0, 0\right).$$

So, by the axiom (A2) of function $\int_0^{\frac{1}{\mathcal{M}(fp, p, p, t)}-1} \varphi(s)ds \leq k.0 = 0.$

Which implies that $\frac{1}{\mathcal{M}(fp, p, p, t)} = 1$ or $fp = p$ (by 3.7.2).

Next, suppose that $w \neq p$ be another fixed point of f. From (3.7.1), we have

$$\int_0^{\frac{1}{\mathcal{M}(z, w, w, t)}-1} \varphi(s)ds = \int_0^{\frac{1}{\mathcal{M}(fz, fw, fw, t)}-1} \varphi(s)ds$$

$$\leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}(z, w, w, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(z, fz, fz, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}(w, fw, fw, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(w, fw, fw, t)}-1} \varphi(s)ds\right)$$

$$\leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}(z, w, w, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(z, z, z, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}(w, w, w, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(w, w, w, t)}-1} \varphi(s)ds\right)$$

$$= \alpha\left(\int_0^{\frac{1}{\mathcal{M}(z, w, w, t)}-1} \varphi(s)ds, 0, 0, 0\right).$$

So, by axiom (A2) of function α, $\int_0^{\frac{1}{\mathcal{M}(z, w, w, t)}-1} \varphi(s)ds = 0$.

Which implies that $\frac{1}{\mathcal{M}(z, w, w, t)} = 1$ or $p = w$ (by 3.7.2).

Hence the fixed point is unique. Next theorem describes a common fixed point of two self-maps on X having two related metrics in integral setting.

Theorem 3.8. *Let* $(X, \mathcal{M}_d, \mathcal{N}_d, *, \Diamond)$ *and* $(X, \mathcal{M}_\delta, \mathcal{N}_\delta, *, \Diamond)$ *be intuitionistic generalized fuzzy metric spaces with two fuzzy metric*

$$M_d(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}, N_d(x, y, z, t) = \frac{D^*(x, y, z)}{t + D^*(x, y, z)},$$

$$\mathcal{M}_\delta(x, y, z, t) = \frac{1}{t + \delta(x, y, z)}, \mathcal{N}_\delta(x, y, z, t) = \frac{\delta(x, y, z)}{t + \delta(x, y, z)},$$ *satisfying the following conditions*:

(i) *For all* $x, y, z \in X \int_0^{\frac{1}{\mathcal{M}_d(x, y, z, t)}} \varphi(s)ds \leq \int_0^{\frac{1}{\mathcal{M}_\delta(x, y, z, t)}-1} \varphi(s)ds$ *and*

$$\int_0^{\frac{1}{\mathcal{N}_d(x, y, z, t)}} \varphi(s)ds \leq \int_0^{\frac{1}{\mathcal{N}_\delta(x, y, z, t)}-1} \varphi(s)ds$$

(ii) $(X, \mathcal{M}_d, \mathcal{N}_d, *, \Diamond)$ *is complete,*

(iii) S, T *are self-maps on X such that T is continuous with respect to d and*

$$\int_0^{\frac{1}{\mathcal{M}_\delta(Tx, sy, Tz, t)}-1} \varphi(s)ds \leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}_\delta(x, y, z, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(x, Tx, Tx, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}_\delta(y, sy, sy, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(z, Tz, Tz, t)}-1} \varphi(s)ds\right) \quad (3.8.1)$$

For each $x, y, z \in X$ and $t > 0$, with some $\alpha \in A$ where $\varphi : [0, +\infty) \to [0, 1)$ is a Lebesgue-integrable mapping which is summable (i.e., With finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)ds > 0. \tag{3.8.2}$$

Then T and S have a unique common fixed point $q \in X$.

Proof. For each integer $n \geq 0$, we define $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Then from (3.8.1), we get

$$\int_0^{\frac{1}{\mathcal{M}_\delta(x_1, x_2, x_3, t)}-1} \varphi(s)ds = \int_0^{\frac{1}{\mathcal{M}_\delta(Tx_0, Sx, Tx_2, t)}-1} \varphi(s)ds$$

$$\leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, Tx_0, Tx_0, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}_\delta(x_1, Sx_1, Sx_1, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(x_2, Tx_2, Tx_2, t)}-1} \varphi(s)ds\right)$$

$$\leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_1, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}_\delta(x_1, x_2, x_2, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(x_2, x_3, x_3, t)}-1} \varphi(s)ds\right).$$

Then by the action (A2) function $\int_0^{\frac{1}{\mathcal{M}_\delta(x_1, x_2, x_3, t)}-1} \varphi(s)ds \leq k \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds$, for some $k \in [0, 1)$. Similarly, one can show that, $\int_0^{\frac{1}{\mathcal{M}_\delta(x_2, x_3, x_4, t)}-1} \varphi(s)ds \leq k \int_0^{\frac{1}{\mathcal{M}_\delta(x_1, x_2, x_3, t)}-1} \varphi(s)ds$, for some $k \in [0, 1)$.

In general, for any $r \in N$ odd or even.

$$\int_0^{\frac{1}{\mathcal{M}_\delta(x_r, x_{r+1}, x_{r+2}, t)}-1} \varphi(s)ds \leq k \int_0^{\frac{1}{\mathcal{M}_\delta(x_{r-1}, x_r, x_{r+1}, t)}-1} \varphi(s)ds.$$

Thus, for any $n \in N$ odd or even, one can easily obtain that,

$$\int_0^{\frac{1}{\mathcal{M}_\delta(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds \leq k^n \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds.$$

Then, by the condition (i) of the theorem, we obtain,

$$\int_0^{\frac{1}{\mathcal{M}_d(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds \leq \int_0^{\frac{1}{\mathcal{M}_\delta(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds$$

$$\leq k^n \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds$$

Taking limit as $n \to \infty$, we get, $\lim_n \int_0^{\frac{1}{\mathcal{M}_\delta(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds = 0.$

As $k \in [0, 1)$, which from (3.8.2) implies that $\lim_n \frac{1}{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t)} - 1 = 0$ or $\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) = 1$. We now show that $\{x_n\}$ is a Cauchy sequence with respect to $(X, \mathcal{M}_d, \mathcal{N}_d, *, \lozenge)$. For any integer $p > 0$

$$\int_0^{\frac{1}{\mathcal{M}_\delta(x_n, x_{n+p}, x_{n+p+1}, t)}-1} \varphi(s)ds \leq \int_0^{\frac{1}{\mathcal{M}_\delta(x_n, x_{n+p}, x_{n+p+1}, t)}-1} \varphi(s)ds$$

$$\leq \int_0^{\frac{1}{\mathcal{M}_\delta(x_n, x_{n+1}, x_{n+2}, t)}-1} \varphi(s)ds + \int_0^{\frac{1}{\mathcal{M}_\delta(x_{n+1}, x_{n+2}, x_{n+3}, t)}-1} \varphi(s)ds$$

$$+ \ldots + \int_0^{\frac{1}{\mathcal{M}_\delta(x_{n+p-1}, x_{n+p}, x_{n+p+1}, t)}-1} \varphi(s)ds$$

$$\leq k^n \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds + k^{n+1} \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds$$

$$+ \ldots + k^{n+p-1} \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds$$

$$\leq \frac{k^n}{1-k} \int_0^{\frac{1}{\mathcal{M}_\delta(x_0, x_1, x_2, t)}-1} \varphi(s)ds \to (s)ds \to 0 \; as \; n \to +\infty.$$

Since $k \in [0, 1)$. Therefore, $\{x_n\}$ is Cauchy.

Hence, by the completeness of X, $\{x_n\}$ converges to some $q \in X$.

i.e. $\frac{1}{\mathcal{M}_d(x_n, q, q, t)} - 1 \to 0$ or $\mathcal{M}_d(x_n, q, q, t) = 1 \; as \; n \to +\infty$, for some $q \in X$.

Since, T is continuous with respect to d, we get,

$$0 = \lim_n \int_0^{\frac{1}{\mathcal{M}_d(x_{2n+1}, q, q, t)}-1} \varphi(s)ds = \lim_n \int_0^{\frac{1}{\mathcal{M}_d(Tx_{2n}, q, q, t)}-1} \varphi(s)ds$$

$$= \lim_n \int_0^{\frac{1}{\mathcal{M}_d(Tq, q, q, t)}-1} \varphi(s)ds.$$

So, by (3.8.2), $\frac{1}{\mathcal{M}_d(Tq, q, q, t)} - 1 = 0$ or $\mathcal{M}_d(Tq, q, q, t) = 1$.

i.e., $Tq = q$. Now, by (3.8.1), we have

$$\int_0^{\frac{1}{\mathcal{M}(q, Sq, Sq, t)}-1} \varphi(s)ds = \int_0^{\frac{1}{\mathcal{M}(Tq, Sq, Tq, t)}-1} \varphi(s)ds = \int_0^{\frac{1}{\mathcal{M}_\delta(Tq, Sq, Tq, t)}-1} \varphi(s)ds$$

$$\leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}_\delta(q, q, q, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(q, Tq, Tq, t)}-1} \varphi(s)ds, \right.$$

$$\left. \int_0^{\frac{1}{\mathcal{M}_\delta(q, Sq, Sq, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(q, Tq, Tq, t)}-1} \varphi(s)ds \right)$$

$$\leq \alpha\left(0, 0, \int_0^{\frac{1}{\mathcal{M}_\delta(q, Sq, Sq, t)}-1} \varphi(s)ds, 0\right).$$

Then, by axiom (A2) of function α,

$\int_0^{\frac{1}{\mathcal{M}_\delta(q,\, Sq,\, q,\, t)}-1} \varphi(s)ds \leq k.0 = 0$ and by (3.8.2), $\mathcal{M}_\delta(q,\, Sq,\, q,\, t) = 1$ or $Sq = q$.

Thus q is a common fixed point of S and T.

Let $w \neq q$ be another common fixed point S and T in x. Then by (3.7.1), we have

$$\int_0^{\frac{1}{\mathcal{M}_\delta(q,\, w,\, w,\, t)}-1} \varphi(s)ds = \int_0^{\frac{1}{\mathcal{M}_\delta(Tq,\, Sw,\, Tw,\, t)}-1} \varphi(s)ds$$

$$\leq \left(\int_0^{\frac{1}{\mathcal{M}(q,\, w,\, w,\, t)}-1} \varphi(s)ds,\ \int_0^{\frac{1}{\mathcal{M}(q,\, Tq,\, Tq,\, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}_\delta(w,\, Sw,\, Sw,\, t)}-1} \varphi(s)ds,\ \int_0^{\frac{1}{\mathcal{M}_\delta(w,\, Tw,\, Tw,\, t)}-1} \varphi(s)ds\right)$$

$$\leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}_\delta(q,\, w,\, w,\, t)}-1} \varphi(s)ds,\ 0,\ 0,\ 0\right).$$

$$\leq k.0 = 0 \text{ as } \alpha \in A.$$

Then, by (3.8.1) we have $\frac{1}{\mathcal{M}_\delta(q,\, w,\, w,\, t)} - 1 = 0$ or $\mathcal{M}_\delta(q,\, w,\, w,\, t) = 1$, hence $q = w$.

If $S = T$, then the Theorem (3.8) gives as follows.

Corollary 3.9. *Let* $(X,\, \mathcal{M}_d,\, \mathcal{N}_d,\, *,\, \Diamond)$ *and* $(X, \mathcal{M}_\delta, \mathcal{N}_\delta, *, \Diamond)$ *be intuitionistic generalized fuzzy metric spaces with two fuzzy metric* $M_d(x,\, y,\, z,\, t) = \frac{t}{t + D^*(x,\, y,\, z)}$, $N_d(x,\, y,\, z,\, t) = \frac{D^*(x,\, y,\, z)}{t + D^*(x,\, y,\, z)}$, $\mathcal{M}_\delta(x,\, y,\, z,\, t) = \frac{t}{t + \delta(x,\, y,\, z)}$, $\mathcal{N}_\delta(x,\, y,\, z,\, t) = \frac{\delta(x,\, y,\, z)}{t + \delta(x,\, y,\, z)}$, *satisfying the following conditions*:

(i) *For all* $x, y, z \in X$, $\int_0^{\frac{1}{\mathcal{M}_d(x, y, z, t)}-1} \varphi(s)ds \leq \int_0^{\frac{1}{\mathcal{M}_\delta(x, y, z, t)}-1} \varphi(s)ds$ *and* $\int_0^{\frac{1}{\mathcal{N}_d(x, y, z, t)}-1} \varphi(s)ds \leq \int_0^{\frac{1}{\mathcal{N}_d(x, y, z, t)}-1} \varphi(s)ds$

(ii) $(X, \mathcal{M}_d, \mathcal{N}_d, *, \lozenge)$ *is complete,*

(iii) *T is self-map on X such that T is continuous with respect to d and*

$$\int_0^{\frac{1}{\mathcal{M}_\delta(Tx, Ty, Tz, t)}-1} \varphi(s)ds \leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}_\delta(x, y, z, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(x, Tx, Tx, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}_\delta(y, Ty, Ty, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}_\delta(z, Tz, Tz, t)}-1} \varphi(s)ds\right)$$

for each $x, y, z \in X$ *and* $t > 0$ *with some* $\alpha \in A$ *where* $\varphi : [0, +\infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping which is summable* (*i.e., With finite integral*) *on each compact subset of* $[0, +\infty)$, *nonnegative, and such that for each* $\varepsilon > 0, \int_0^{\varepsilon} \varphi(s)ds > 0.$

Then T has a unique common fixed point $q \in X$.

Example 3.10. Consider X an example (3.6) $\mathcal{M}(x, y, z, t) = \dfrac{t}{t + D^*(x, y, z)}$ and $\mathcal{N}(x, y, z, t) = \dfrac{D^*(x, y, z)}{t + D^*(x, y, z)}$, with usual metric relative to real line.

Define f on X by $f_x = \begin{cases} 12 & x = 0 \\ 11 & \text{otherwise.} \end{cases}$

Let $\varphi : R_+ \to R_+$ be given by $\varphi(S) = \dfrac{s-1}{s}$ for all $S \in \mathbb{R}_+$. Then $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$ nonnegative and such that for each $\varphi > 0, \int_0^{\varepsilon} \varphi(S)ds > 0.$

Now, as we know for example (3.6) a self map of satisfying

$$\frac{1}{\mathcal{M}(fx, fy, fz, t)} - 1 \leq \beta \max\left\{\frac{1}{\mathcal{M}(fx, x, x, t)} + \frac{1}{\mathcal{M}(fy, y, y, t)} + \frac{1}{\mathcal{M}(fz, z, z, t)} - 3,\right.$$

$$\frac{1}{\mathcal{M}(fz, z, z, t)} + \frac{1}{\mathcal{M}(fx, y, z, t)} - 2, \frac{1}{\mathcal{M}(fy, y, y, t)} + \frac{1}{\mathcal{M}(x, y, z, t)} - 2$$

$$\left.\frac{1}{\mathcal{M}(fx, x, x, t)} + \frac{1}{\mathcal{M}(x, y, z, t)} - 2\right\}.$$

For all $x, y, z \in X$ and some $\beta \in \left[0, \frac{1}{2}\right)$ is an A-fuzzy contraction. We have,

$$\int_0^{\frac{1}{\mathcal{M}(fx, fy, fz, t)}-1} \varphi(s)ds \leq \alpha\left(\int_0^{\frac{1}{\mathcal{M}(x, y, z, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(x, fx, fx, t)}-1} \varphi(s)ds,\right.$$

$$\left.\int_0^{\frac{1}{\mathcal{M}(y, fy, fy, t)}-1} \varphi(s)ds, \int_0^{\frac{1}{\mathcal{M}(z, fz, fz, t)}-1} \varphi(s)ds\right)$$

$$= \beta \max\left\{\int_0^{\frac{1}{\mathcal{M}(fx,x,x,t)}+\frac{1}{\mathcal{M}(fy,y,y,t)}-1\frac{1}{\mathcal{M}(fz,z,z,t)}-3} \varphi(s)ds,\right.$$

$$\int_0^{\frac{1}{\mathcal{M}(fx, x, x, t)}-1\frac{1}{\mathcal{M}(x, y, z, t)}-2} \varphi(s)ds,$$

$$\int_0^{\frac{1}{\mathcal{M}(fy, y, y, t)}+\frac{1}{\mathcal{M}(x, y, z, t)}-2} \varphi(s)ds,$$

$$\left.\int_0^{\frac{1}{\mathcal{M}(fz, z, z, t)}+\frac{1}{\mathcal{M}(x, y, z, t)}-2} \varphi(s)ds\right\}.$$

Which is satisfied for all $x, y, z \in X, t > 0$ and some $\beta \in \left[0, \frac{1}{2}\right)$.

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