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ON FACE IRREGULAR LABELING OF TYPE (1, 1, 0) OF CERTAIN FAMILIES OF PLANE GRAPHS

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Abstract

A total k-labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ of a connected plane graph $G = (V, E, F)$ is called face irregular if distinct faces have distinct weights (i.e.) $w_\phi(f) \neq w_\phi(g), f, g \in F$. The weight of the face f is defined as

$$w_\phi(f) = \sum_{v \in f} \phi(v) + \sum_{e \in f} \phi(e),$$

where the sums are taken over all the vertices and the edges surrounding that face f under ϕ. The minimum k for which a plane graph G has a face irregular total k-labeling is called total face irregularity strength of type (1, 1, 0) of G and is denoted by $tfs(G)$.

In this paper, we investigate the total face irregularity strength of type (1, 1, 0) of certain families of plane graphs, namely H_n, A_n, B_n and $C_4 \times P_n$.

1. Introduction

Plane graph G is a graph embedded in the plane such that no pair of lines intersect. The graph divides the plane into a number of regions called faces. The graphs considered here are 2-connectedplane graphs (i.e., every face is bounded by a cycle) that are finite, undirected, without loops and multiple

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edges. The set of vertices, edges and faces of G are denoted by $V(G)$, $E(G)$ and $F(G)$ respectively.

A k-labeling of a graph G is a map from the set of graph elements to the set of positive integers $\{1, 2, \ldots, k\}$. k-labelings are classified according to their corresponding domain set as in the following way.

Domain set	Name of the k- labelings	Type of k-labelings
V(G)	Vertex k-labeling	(1,0,0)
E(G)	Edge k-labeling	(0,1,0)
F(G)	Face k-labeling	(0,0,1)
V(G) ∪ ***E(G)***	Total k-labeling	(1,1,0)
V(G) ∪***F(G)***	Vertex-face k-labeling	(1,0,1)
E(G) ∪ ***F(G)***	Edge-face k-labeling	(0,1,1)
V(G) ∪ ***E(G)*** ∪ ***F(G)***	Entire k-labeling	(1,1,1)

G. Chartrand et al. [6] introduced and studied an edge k-labeling ζ of a graph G such a way that the weight sum (sum of the labels which are incident with a vertex) $w_{\zeta}(x) \neq w_{\zeta}(y)$ for distinct vertices $x, y \in V(G)$, called irregular assignments and the minimum value of k for which G has an irregular assignment is called irregularity strength of the graph G and denoted as $s(G)$.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [7], [8], [16]. Amar and Togni [2] established irregularity strength of trees, Nierhoff [14] conjectured that the edges of every connected graph of order at least 3 can be assigned labels from {1, 2, 3} such that for all pairs of adjacent vertices the sums of the labels of the incident edges are different. Motivated by these papers and by a book of Wallis [18], Baca et al., [5] started to investigate the total edge irregularity strength of a graph G, an invariant analogous to the irregularity strength for total labeling.

The authors of [5] were determined several bounds and exact values of $tvs(G)$ for different types of graphs. In addition, they were derived lower and upper bounds of the total edge-irregularity strength of any graph $G(V, E)$ as follows.

$$\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq tes(G) \leq |E(G)|.$$

Ivanco and Jendrol [10] proved that for any tree T the total edge irregularity strength is equal to its lower bound, i.e.,

$$tes[T] = \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{|\Delta(T)| + 1}{2} \right\rceil\right\}.$$

Recently the authors of [12] determined a new upper bound for the total vertex irregularity strength of graphs. They have proved that

$$tvs(G) \leq 3\lceil n/\delta \rceil + 1$$

for each graph of order n with minimum degree $\delta > 0$.

Hafidhyah Dwi Wahyuna and Diari Indriati [9] investigated the total edge irregularity strength of generalized butterfly graph, BF_n, for $n \geq 2$ and found that $tes(BF_n) = \left\lceil \frac{4n}{3} \right\rceil$. Nurdin Hinding et al., [15] investigated the total edge irregularity strength $tes(G)$ and total vertex irregularity strength $tvs(G)$ of diamond graphs Br_n and proved that $tes(Br_n) = \left\lceil \frac{5n - 3}{3} \right\rceil$, while $tvs(Br_n) = \left\lceil \frac{n + 1}{3} \right\rceil$. Tri Winarshi and Diari Indriati [17] determined that the total edge irregularity strength of the (n, t)- kite graph with $n \geq 3$ and $t \geq 1$ is $\left\lceil \frac{n + t + 2}{3} \right\rceil$. Recently authors of [11] studied the total edge irregularity strength for some families of graphs like shadow graph of cycle and path, total graph of cycle and path, lotus inside a circle and double wheel graph.

Ali Ahmad et al. [1] determined the edge irregularity strength of some chain graphs and the join of two graphs and conjectured that the edge irregularity strength of chain graph with n blocks whose blocks are combination of C_m is $\left\lceil \frac{nm + 1}{2} \right\rceil$, for $n \geq 2$ and $m \geq 5$.

Martin Baca et al. [4] studied the face irregularity strength of type (1, 1, 1) as entire face irregularity strength, denoted by $efs(G)$. "For a 2-connected plane graph $G = (V, E, F)$, they defined an entire k-labeling $\Psi : V \cup E \cup F \to \{1, 2, \ldots, k\}$ such that the associated face weights are distinct for different faces and it is called face irregular entire k-labeling. The weight of a face f under an entire k-labeling Ψ, $w_{\Psi}(f)$ is the sum of labels carried by that face and all the edges and vertices incident with the face. The minimum k for which a plane graph G admits face irregular entire k-labeling is called the entire face irregularity strength of G."

In [3], Martin Baca et al., generalized the face irregular k-labeling as the k-labeling of type (α, β, γ) where $\alpha, \beta, \gamma \in \{0, 1\}$, the associated face weight of a face $f \in F(G)$ is defined as

$$w_{\phi(\alpha,\beta,\gamma)}(f) = \alpha \sum_{v\sim f} \phi(v) + \beta \sum_{e\sim f} \phi(e) + \gamma\phi(f),$$

where the sums are taken over all vertices and all edges adjacent to the face f, respectively.

Note that the trivial case $(\alpha, \beta, \gamma) = (0, 0, 0)$ is not allowed.

A k-labeling α of type (α, β, γ) of the plane graph G is defined to be a face irregular k-labeling of type (α, β, γ) if for every two different faces f and g of G there is

$$w_{\phi(\alpha,\beta,\gamma)}(f) \neq w_{\phi(\alpha,\beta,\gamma)}(g).$$

The face irregularity strength of type (α, β, γ) of a plane graph G, denoted by $fs_{(\alpha,\beta,\gamma)}(G)$, is the smallest integer k such that G admits a face irregular k-labeling of type (α, β, γ).

The characteristic called total face irregularity strength, was introduced by K. Muthugurupackiam in [13] which is of type (1, 1, 0).

"A total k-labeling $\phi : V \cup E \to \{1, 2, \ldots, k\}$ of a connected plane graph $G = (V, E, F)$ is called face irregular if distinct faces have distinct weights (i.e.) $w_{\phi}(f) \neq w_{\phi}(g)$. The weight of the face f is defined as

$$w_{\phi}(f) = \sum_{v \in f} \phi(v) + \sum_{e \in f} \phi(e),$$

where the sums are taken over all the vertices and the edges surrounding that face f under ϕ. The minimum k for which a plane graph G has a face irregular total k-labeling is called total face irregularity strength of type (1, 1, 0) of G and is denoted by $tfs(G)$."

The following result gives the lower bound of total face irregularity strength of a plane graph.

Result 1.1 [13]. Let G be a plane graph contains m_i, r_i-sided faces, where $1 < i < s$ and let $r_i < r_{i+1}$ for all i and $\sum | m_i | = | F |$. Then,

$$tfs(G) \geq \left\lceil \frac{2r_i + | F | - 1}{2r_s} \right\rceil.$$

The main aim of this paper is to obtain the exact values of the total face irregularity strength of type (1, 1, 0) for certain families of plane graphs, namely H_n, A_n, B_n and $C_4 \times P_n$.

2. Face Irregular Total Labeling of Type (1, 1, 0) of Some Families of Plane Graphs

Definition 2.1. A honeycomb H_n, $n \geq 2$ is defined as a plane graph consisting of $2n - 2$ six sided faces and one external infinite face as shown in the following figure.

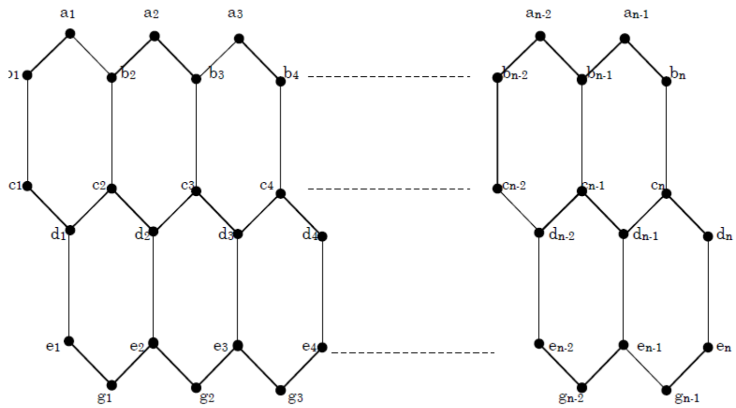

Figure 1. Honeycomb H_n.

Theorem 2.2. *Total face irregularity strength of type* (1, 1, 0) *of honeycomb* H_n *is* $\left\lceil \frac{2n+9}{12} \right\rceil$, $n > 2$.

Proof. Let $V(H_n) = \{a_i, g_i / 1 \le i \le n-1\} \cup \{b_i, c_i, d_i, e_i / 1 \le i \le n\}$ be the vertex set and $E(H_n) = \{a_i b_i, a_i b_{i+1}, e_i g_i, g_i e_{i+1} / 1 \le i \le n-1\} \cup \{b_i c_i, c_i d_i, d_i e_i / 1 \le i \le n\}$ be the edge set of the Honeycomb H_n respectively. Since H_n consisting of $2n-2$ six sided faces, let us denote the faces such that for $1 \le i \le n-1$, f_i be the i^{th} face surrounded by the vertices namely $a_i, b_i, c_i, d_i, c_{i+1}$ and b_{i+1} and f_{n-1+i} be the $(n-1+i)^{th}$ face surrounded by the vertices namely $c_{i+1}, d_i, e_i, g_i, e_{i+1}$ and d_{i+1}.

Define a total k-labeling of type $(1, 1, 0)\, \phi : V \cup E \left\{1, 2, \ldots, \left\lceil \frac{2n+9}{12} \right\rceil \right\}$ as follows.

(i) $\phi(a_i) = \left\lceil \frac{i+3}{6} \right\rceil$ if $1 \le i \le n-1$,

(ii) $\phi(b_i) = \phi(e_i) = \left\lceil \frac{i+3}{6} \right\rceil$ if $1 \le i \le n$,

(iii) $\phi(c_i) = \left\lceil \frac{i+4}{6} \right\rceil$ if $1 \le i \le n$,

(iv) $\phi(d_i) = \left\lceil \frac{i+5}{6} \right\rceil$ if $1 \le i \le n$,

(v) $\phi(g_i) = \left\lceil \frac{i+4}{6} \right\rceil$ if $1 \le i \le n-1$,

(vi) $\phi(a_i b_i) = \phi(e_i g_i) = \left\lceil \frac{i}{6} \right\rceil$ if $1 \le i \le n-1$,

(vii) $\phi(a_i b_{i+1}) = \phi(e_{i+1} g_i) = \left\lceil \frac{i+1}{6} \right\rceil$ if $1 \le i \le n-1$,

(viii) $\phi(b_i c_i) = \left\lceil \frac{i}{6} \right\rceil$ if $1 \le i \le n$,

(ix) $\phi(c_i d_i) = \left\lceil \frac{i+2}{6} \right\rceil$ if $1 \le i \le n$,

(x) $\phi(c_{i+1} d_i) = \left\lceil \frac{i+2}{6} \right\rceil$ if $1 \le i \le n-1$,

(xi) $\phi(d_i e_i) = \left\lceil \frac{i+1}{6} \right\rceil$ if $1 \le i \le n$,

Under the labeling ϕ the face weights of the six-sided faces f_i are $12, 13, 14, \ldots, 2n+9$, all are distinct. Also the weight of the external face is much more than $2n+9$. Thus ϕ is a total face irregular k-labeling of H_n. Thus,

$$tfs(H_n) \le \left\lceil \frac{2n+9}{12} \right\rceil. \tag{1}$$

By Result 1.1

$$tfs(H_n) \ge \left\lceil \frac{2n+9}{12} \right\rceil. \tag{2}$$

From (1) and (2),

$$tfs(H_n) = \left\lceil \frac{2n+9}{12} \right\rceil.$$

Illustration 2.3.

Total face irregularity strength of the Honeycomb H_5 is 2.

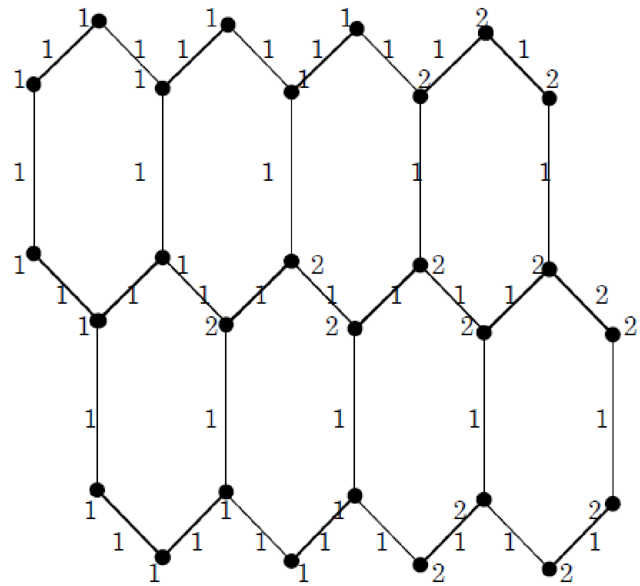

Figure 2. Total face irregular 2-labeling of H_5.

Definition 2.4. For $n \geq 2$, we define a plane graph by A_n consisting of $3n - 3$ five sided faces and one external infinite face as shown in the following figure.

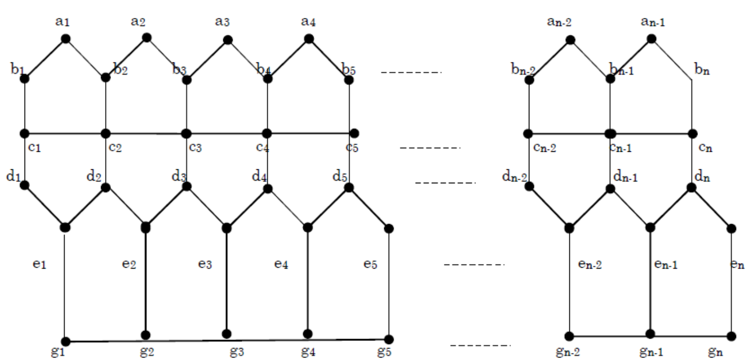

Figure 3. Graph A_n.

Theorem 2.5. *Total face irregularity strength of type* (1, 1, 0) *of* A_n *is* $\left\lceil \frac{3n+6}{10} \right\rceil$, $n \geq 2$.

Proof. Let $V(A_n) = \{a_i / 1 \leq i \leq n-1\} \cup \{b_i, c_i, d_i, e_i, g_i / 1 \leq i \leq n\}$ be the vertex set and $E(A_n) = \{a_i b_i, a_i b_{i+1}, c_i c_{i+1}, e_i d_{i+1}, g_i g_{i+1} / 1 \leq i \leq n-1\}$ $\cup \{b_i c_i, c_i d_i, d_i e_i, e_i g_i / 1 \leq i \leq n\}$ be the edge set of A_n respectively. In A_n, there are $3n-3$ five sided faces and one external infinite face. For $1 \leq i \leq n-1$, let f_{3i-2} be the $(3i-2)^{th}$ five sided face surrounded by the vertices a_i, b_i, c_i, c_{i+1} and b_{i+1}, let f_{3i-1} be the $(3i-1)^{th}$ five sided face surrounded by the vertices c_i, d_i, e_i, d_{i+1} and c_{i+1} and let f_{3i} be the $3i^{th}$ five sided face surrounded by the vertices $e_i, g_i, g_{i+1}, e_{i+1}$ and d_{i+1}.

Define a total k-labeling of type (1, 1, 0) $\phi : V \in E \to \left\{1, 2, \ldots, \left\lceil \frac{3n+6}{10} \right\rceil\right\}$ as follows.

(i) $\phi(a_i) = \left\lceil \frac{i}{3} \right\rceil$, $1 \leq i \leq n-1$.

(ii) $\phi(b_i) = \left\lceil \frac{i+1}{3} \right\rceil$, $1 \leq i \leq n$.

(iii) $\phi(c_i) = \left\lceil \frac{i+2}{4} \right\rceil$, $1 \leq i \leq n$.

(iv) $\phi(d_i) = \left\lceil \frac{i+1}{3} \right\rceil$, $1 \leq i \leq n$.

(v) $\phi(e_i) = \left\lceil \frac{i+4}{4} \right\rceil$, $1 \leq i \leq n$.

(vi) $\phi(g_i) = \left\lceil \frac{i+2}{4} \right\rceil$, $1 \leq i \leq n$.

(vii) $\phi(a_i b_i) = \left\lceil \frac{i}{4} \right\rceil$, $1 \leq i \leq n-1$.

(viii) $\phi(a_i b_{i+1}) = \left\lceil \frac{i}{3} \right\rceil$, $1 \le i \le n-1$.

(ix) $\phi(b_i c_i) = \left\lceil \frac{i+1}{3} \right\rceil$, $1 \le i \le n$.

(x) $\phi(c_i d_i) = \left\lceil \frac{i+1}{3} \right\rceil$, $1 \le i \le n$.

(xi) $\phi(d_i e_i) = \left\lceil \frac{i}{3} \right\rceil$, $1 \le i \le n$.

(xii) $\phi(d_{i+1} e_i) = \left\lceil \frac{i}{3} \right\rceil$, $1 \le i \le n-1$.

(xiii) $\phi(e_i g_i) = \left\lceil \frac{i+1}{3} \right\rceil$, $1 \le i \le n$.

(xiv) $\phi(c_i c_{i+1}) = \left\lceil \frac{i+1}{3} \right\rceil$, $1 \le i \le n-1$.

(xv) $\phi(g_i g_{i+1}) = \left\lceil \frac{i}{3} \right\rceil$, $1 \le i \le n-1$.

Under the labeling ϕ weights of the five-sided faces are $10, 11, 12, \ldots, 3n+6$ and the weight of the external face is much more than $3n+6$ and hence ϕ is a total face irregular k-labeling of the plane graph A_n.

Thus,

$$tfs(A_n) = \left\lceil \frac{3n+6}{10} \right\rceil. \quad (3)$$

By Result 1.1

$$tfs(A_n) = \left\lceil \frac{3n+6}{10} \right\rceil. \quad (4)$$

From (3) and (4),

$$tfs(A_n) = \left\lceil \frac{3n+6}{10} \right\rceil.$$

Illustration 2.6. Total face irregularity strength of type (1, 1, 0) of A_5 is 3.

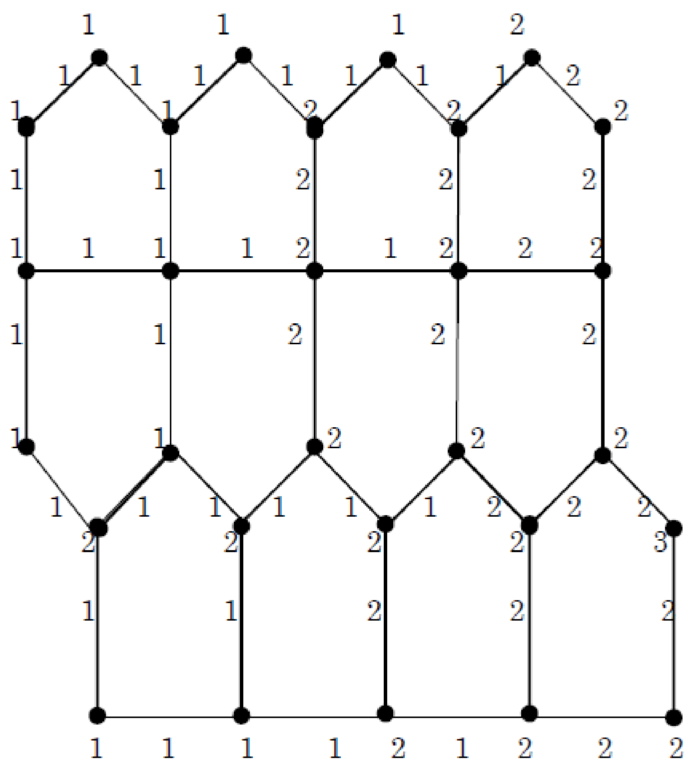

Figure 4. Total face irregular 3-labeling of A_5.

Definition 2.7. A plane graph consists of $4n-4$ three sided faces and one external infinite face as shown in the following figure is defined by B_n, for $n \geq 2$.

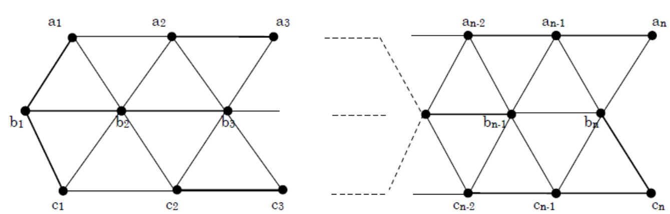

Figure 5. The graph B_n.

Theorem 2.8. *Total face irregularity strength of type* (1, 1, 0) *of* B_n *is* $\left\lceil \frac{4n+1}{6} \right\rceil$, $n \geq 2$.

Proof. Let $V(B_n) = \{a_i b_i, c_i / 1 \leq i \leq n\}$ be the vertex set and $E(B_n) = \{a_i a_{i+1}, a_i b_{i+1}, b_i b_{i+1}, c_i c_{i+1}, c_i b_{i+1} / 1 \leq i \leq n-1\} \cup \{a_i b_i, b_i c_i / 1 \leq i \leq n\}$ be the edge set of B_n respectively. In B_n, there are $4n - 4$, three sided faces. For $1 \leq i \leq n-1$, let f_{4i-3} be then $(4i-3)^{th}$ three sided face surrounded by the vertices are a_i, b_i and b_{i+1}, let f_{4i-2} be the $(4i-2)^{th}$ three sided face surrounded by the vertices b_i, c_i and b_{i+1}, let f_{4i-1} be the $(4i-1)^{th}$ three sided face surrounded by the vertices a_i, b_{i+1} and a_{i+1} and let f_{4i} be the $4i^{th}$ three sided face surrounded by the vertices $b_{i+1} c_i$ and c_{i+1}.

Define a total k-labeling of type (1, 1, 0) $\phi : V \in E \to \left\{1, 2, \ldots, \left\lceil \frac{4n+1}{6} \right\rceil\right\}$ as follows. For $1 \leq i \leq n$,

(i) $$\phi(a_i) = \begin{cases} 2\left\lceil \frac{i}{3} \right\rceil, & 1 \neq 1 (\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil - 1, & i \equiv 1 (\text{mod } 3) \end{cases}$$

(ii) $$\phi(a_i a_{i+1}) = 2\left\lceil \frac{i}{3} \right\rceil, \; 1 \leq i \leq n-1$$

(iii) $$\phi(b_i) = \begin{cases} 2\left\lceil \frac{i}{3} \right\rceil - 1, & i \not\equiv 0 (\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil, & i \equiv 0 (\text{mod } 3) \end{cases}$$

(iv) $$\phi(b_i b_{i+1}) = \begin{cases} 2\left\lceil \frac{i}{3} \right\rceil, & i \not\equiv 1 (\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil - 1, & i \equiv 1 (\text{mod } 3), \; 1 < i < n-1 \end{cases}$$

(v) $$\phi(c_i) = \begin{cases} 2\left\lceil \frac{i}{3} \right\rceil, & i \not\equiv 1 (\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil - 1, & i \equiv 1 (\text{mod } 3) \end{cases}$$

(vi) $\phi(a_ib_{i+1}) = \phi(c_ic_{i+1}) = \phi(b_ic_i) = \begin{cases} 2\left\lceil \frac{i}{3} \right\rceil + 1, \; i \equiv 0 \,(\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil - 1, \; i \equiv 1 \,(\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil, \; i \equiv 2 \,(\text{mod } 3), \; 1 < i < n-1 \end{cases}$

(vii) $\phi(a_ib_i) = \begin{cases} 2\left\lceil \frac{i}{3} \right\rceil - 1, \; i \not\equiv 0 \,(\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil, \; i \equiv 0 \,(\text{mod } 3) \end{cases}$

(viii) For $2 \le i \le n$,

$$\phi(b_ic_{i-1}) = \begin{cases} 2\left\lceil \frac{i}{4} \right\rceil + 1, \; i \equiv 1 \,(\text{mod } 3) \\ 2\left\lceil \frac{i}{3} \right\rceil, \; i \not\equiv 1 \,(\text{mod } 3) \end{cases}$$

Under the labeling ϕ the weights of the three-sided faces are $6, 7, 8, \ldots, 4n+1$ and the weight of the external face is much more than $4n+1$, all are distinct. Thus B_n is a face irregular plane graph and

$$tfs(B_n) \le \left\lceil \frac{4n+1}{6} \right\rceil. \tag{5}$$

By Result 1.1,

$$tfs(B_n) \le \left\lceil \frac{4n+1}{6} \right\rceil. \tag{6}$$

From (5) and (6),

$$tfs(B_n) \le \left\lceil \frac{4n+1}{6} \right\rceil.$$

Illustration 2.9. The total face irregularity strength of type (1, 1, 0) of the graph B_5 is 4.

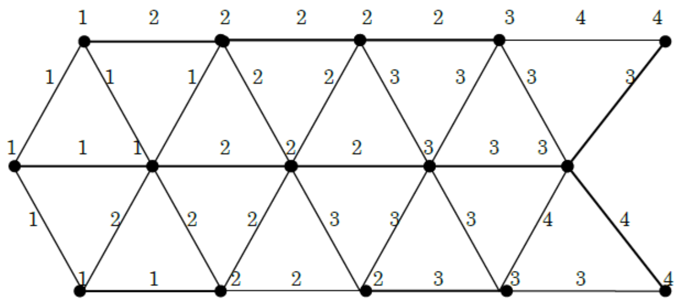

Figure 6. Total face irregular 4-labeling of B_5.

Definition 2.10. For $n \geq 2$, construct the plane graph by the Cartesian product of the cycle C_4 and the path P_n and denoted by $C_4 \times P_n$ consisting of $4n - 3$ bounded four sided faces and one external unbounded four sided face as shown in the following figure.

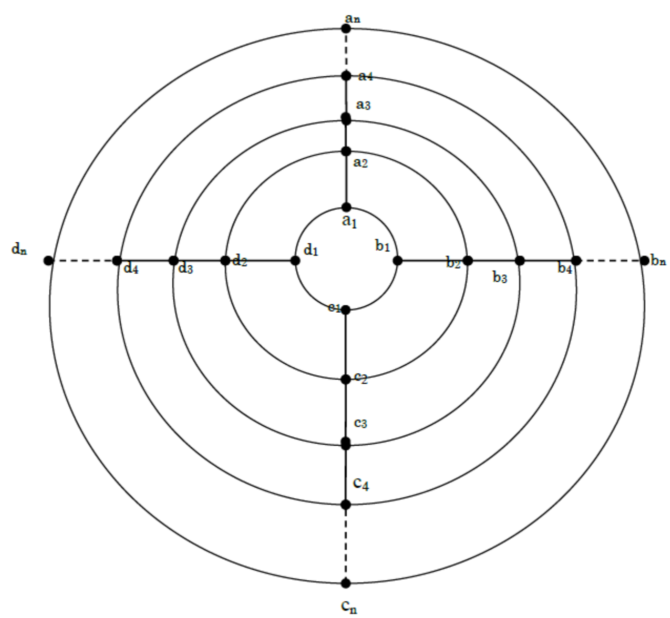

Figure 7. The graph $C_4 \times P_n$.

Theorem 2.11. *Total face irregularity strength of type* (1, 1, 0) *of* C_4, P_n *is* $\left\lceil \frac{4n+5}{8} \right\rceil$, $n \geq 2$.

Proof. Let $V(C_4 \times P_n) = \{a_i, b_i, c_i, d_i / 1 \leq i \leq n\}$ be the vertex set and $E(C_4 \times P_n) = \{a_i b_i, b_i c_i, c_i d_i, d_i a_i / 1 \leq i \leq n\} \cup \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1} / 1 \leq i \leq n-1\}$ be the edge set of $C_4 \times P_n$ respectively. In $C_4 \times P_n$, there are $4n - 3$ four sided faces. Let f_1 be the four sided face surrounded by the vertices a_1, b_1, c_1 and d_1, for $1 \leq i \leq n-1$ let f_{4i-2} be the $(4i-2)^{th}$ four sided face surrounded by the vertices a_{i+1}, b_{i+1}, b_i and a_i, let f_{4i-1} be the $(4i-1)^{th}$ four sided face surrounded by the vertices b_{i+1}, c_{i+1}, c_i and b_i, let f_{4i} be the $4i^{th}$ four sided face surrounded by the vertices c_{i+1}, d_{i+1}, d_i and c_i and let f_{4i+1} be the $(4i+1)^{th}$ four sided face surrounded by the vertices $d_{i+1}, a_{i+1} a_i$ and d_i. Let f_{4n-2} be the unbounded four sided face bounded by a_n, b_n, c_n and d_n.

Define a total k-labeling of type (1, 1, 0) $\phi : V \in E \to \left\{1, 2, \ldots, \left\lceil \frac{4n+5}{8} \right\rceil \right\}$ as follows.

(i) $\phi(a_i) = \left\lceil \frac{i}{2} \right\rceil, 1 \leq i \leq n.$

(ii) $\phi(b_i) = \phi(c_i) = \left\lceil \frac{i+1}{2} \right\rceil, 1 \leq i \leq n.$

(iii) $\phi(d_1) = 1$ and $\phi(d_i) = \left\lceil \frac{i+2}{2} \right\rceil, 2 \leq i \leq n.$

(iv) $\phi(a_i \, a_{i+1}) = \left\lceil \frac{i}{2} \right\rceil, 1 \leq i \leq n-1.$

(v) $\phi(b_1 \, b_2) = 1$ and $\phi(b_i \, b_{i+1}) = \left\lceil \frac{i+1}{2} \right\rceil, 2 \leq i \leq n-1.$

(vi) $\phi(c_i \, c_{i+1}) = \left\lceil \frac{i}{2} \right\rceil, 1 \leq i \leq n-1.$

(vii) $\phi(d_i \, d_{i+1}) = \left\lceil \frac{i+3}{2} \right\rceil, 1 \leq i \leq n-1.$

(viii) $\phi(a_i\ b_i) = \phi(b_i\ c_i) = \left\lceil \frac{i}{2} \right\rceil$, $1 \leq i \leq n$.

(ix) $\phi(c_i\ d_i) = \phi(d_i\ a_i) = \left\lceil \frac{i+1}{2} \right\rceil$, $1 \leq i \leq n$.

Under the labeling ϕ weights of the four sided are $8, 9, 10, \ldots, 4n+4$ and the weight of the external face is $4n+5$ all are distinct. Thus, $C_4 \times P_n$ is a face irregular plane graph and

$$tfs(C_4 \times P_n) \leq \left\lceil \frac{4n+5}{8} \right\rceil. \tag{7}$$

By Result 1.1

$$tfs(C_4 \times P_n) \leq \left\lceil \frac{4n+5}{8} \right\rceil. \tag{8}$$

From (7) and (8),

$$tfs(C_4 \times P_n) \leq \left\lceil \frac{4n+5}{8} \right\rceil.$$

Illustration 2.12. Total face irregularity strength of type (1, 1, 0) of $C_4 \times P_n$ is 3.

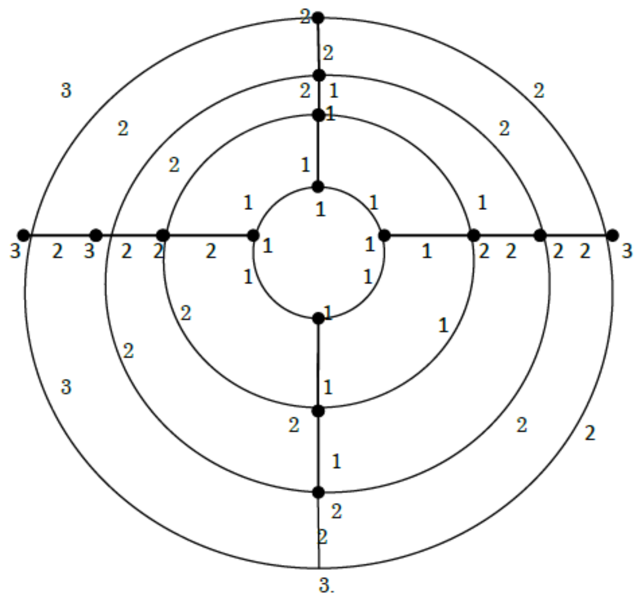

Figure 8. Total face irregular 3-labeling of $C_4 \times P_n$.

3. Conclusion

In this paper, we determined the exact value of the total face irregularity strength of type (1, 1, 0) of H_n, A_n, B_n and $C_4 \times P_n$, for $n \geq 2$.

Open Problem 3.1. Determine the total face irregularity strength of type (1, 1, 0) of $C_m \times P_n, m \geq 5, n \geq 2$.

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