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A STUDY OF CONVERGENCE BEHAVIOUR OF FIXED POINT ITERATIVE PROCESSES VIA COMPUTER SIMULATION

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Abstract

In this study, two different fixed point iterative algorithms are compared for their rate of convergence and then it is proved that these iterative processes converge equivalently to their fixed points. These iterative processes are also self-compared with each other with respect to their speed of convergence as shown in the comparative tables. To claim this fact, we present some numerical examples illustrating the acceleration of these iterative algorithms to ward their fixed point. Moreover, the convergence behavior of these iterative algorithms is depicted by various graphical representations describing the convergence acceleration of these iterative procedures towards their same fixed point.

1. Introduction

In 1953, the Mann iteration [1] was invented to show the convergence of the sequence to the fixed points of the mappings for which the Banach principle was fail to apply. In 1974, Ishikawa [2] defined a new iteration scheme to derive the convergence of a Lipschitzian pseudo-contractive map whereMann iteration process failed to converge. Noor et al. [3] devised a three-step iteration procedure for $\sigma_1 \in S$ and for all $n \in \mathbb{N}$ as

$$N_n = \begin{cases} \sigma_{n+1} = (1-\omega_n)\sigma_n + \omega_n H\vartheta_n \\ \vartheta_n = (1-\delta_n)\sigma_n + \delta_n H\rho n \\ \rho_n = (1-\varepsilon_n)\sigma_n + \varepsilon_n H\sigma_n \end{cases} \tag{1}$$

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to solve non-linear equations in Banach spaces which is popularly known as the three-step iteration (or the Noor iteration), where the three sequences $\{\omega_n\}$, $\{\delta_n\}$ and $\{\varepsilon_n\}$ are in [0, 1]. Later on, it was shown that the Noor process has the following iterative procedures as special cases:

(i) If we take $\varepsilon_n = 0$, then Noor process reduces to the two-step iterative process

$$I_n = \begin{cases} \sigma_{n+1} = (1-\omega_n)\sigma_n + \omega_n H\vartheta_n \\ \vartheta_n = (1-\delta_n)\sigma_n + \delta_n H\sigma_n \end{cases} \tag{2}$$

which is called the Ishikawa iterative process [2].

(ii) If $\delta_n = \varepsilon_n = 0$, then Noor process becomes

$$M_n \equiv \sigma_{n+1} = (1-\omega_n)\sigma_n + \omega_n H\sigma_n, \tag{3}$$

which is known as the Mann iterative sequence [1].

(iii) If we assume $\omega_n = 1$, then Noor process reduces to the one-step process

$$P_n \equiv \sigma_{n+1} = H\sigma_n \tag{4}$$

which is called the sequence of approximation or the Picard iterative process [5].

Abbas et al. [6] in 2014 and Thakur et al. [7] in 2016 defined the iterative processes as given below and let it be denoted by Itr. P_1 and Itr. P_2 respectively,

$$Itr.P_1 = \begin{cases} \sigma_{n+1} = (1-\omega_n)H\vartheta_n + \omega_n H\rho_n \\ \vartheta_n = (1-\delta_n)H\sigma_n + \delta_n H\rho_n \\ \rho_n = (1-\varepsilon_n)\sigma_n + \varepsilon_n H\sigma_n \end{cases} \tag{5}$$

and

$$Itr.P_2 = \begin{cases} \sigma_{n+1} = (1-\omega_n)H\rho_n + \omega_n H\vartheta_n \\ \vartheta_n = (1-\delta_n)\rho_n + \delta_n H\rho_n \\ \rho_n = (1-\varepsilon_n)\sigma_n + \varepsilon_n H\sigma_n \end{cases} \tag{6}$$

respectively, where the sequences $\{\omega_n\}$, $\{\delta_n\}$ and $\{\varepsilon_n\} \in (0, 1)$.

In this direction, various new and modified iterative procedures [8-11] have been devised and proved weak and strong convergence theorems. Many authors [12-14] have studied and compared the rate of convergence of these iterative sequences to reach their fixed point of the mappings in less number of iteration. They have also produced various theorems and lemmas for the fast convergence of iterative sequences towards their fixed points of these fixed point iterations in by considering different spaces for example the study of convergence problem of iterative procedures for non expansive mappings in Banach spaces under various control conditions on parameters. Recall the a mapping $T : B \to B$ on a nonvoid closed convex subset B of a Banach space E is said to be non expansive if $\| Tb_1 - Tb_2 \| \leq \| b_1 - b_2 \|$ for all $b_1, b_2 \in B$.

The concept of three-step iterative method is very wide and has been extensively studied by many researchers. It has been shown Bnouhachem [15] that the three-step procedures are more efficient than the one-step and the two-step iterations to solve variational inequalities and these methods perform better to solve the problems of pure and applied mathematics.

Our aim is to study the convergence behaviour of Itr. P_1 and Itr. P_2 and to prove that these iterative algorithms have the same rate of convergence for non-expansive mappings.

2. Preliminaries

Definition 2.1 [9] (also see [12]). Let (M, d) is a complete metric space and let $Q : M \to M$ is a self-map of M. We define a set containing all fixed points of Q as $F_Q = \{m \in M | Qm = m\}$. Then $d(Qm, Qn) \leq ad(m, n)$ for all $m, n \in M$ and $a \in [0, 1)$ is known as the Banach contraction's condition.

Definition 2.2 [4] (also see [14]). Let $\{u_m\}(m = 0, 1, 2, \ldots, \infty)$ and $\{v_m\}(m = 0, 1, 2, \ldots, \infty)$ are two real sequences of converging to 'u' and 'v' respectively. If there is a real number l such that $\lim_{n \to \infty} | \frac{u_n - u}{v_n - v} | = l$', then (1) If $l = 0$, then $\{u_n\}(m = 0, 1, 2, \ldots, \infty)$ is said to fast converge to 'u' than $\{v_m\}(m = 0, 1, 2, \ldots, \infty)$ to 'v' and (2) $0 < l < 1$, then $\{u_n\}$ and $\{v_m\}$ have the same rate of convergence.

Definition 2.3 [6]**.** Abbas et al. in 2014 defined the iterative processes as given below

$$\begin{cases}\sigma_{n+1} = (1-\omega_n)H\vartheta_n + \omega_n H\rho_n \\ \vartheta_n = (1-\delta_n)H\sigma_n + \delta_n H\rho_n \\ \rho_n = (1-\varepsilon_n)\sigma_n + \varepsilon_n H\sigma_n\end{cases}$$

where the sequences $\{\omega_n\}, \{\delta_n\}$ and $\{\varepsilon_n\} \in (0, 1)$.

Definition 2.4 [7]**.** Thakur et al. in 2016 defined the iterative processes as

$$\begin{cases}\sigma_{n+1} = (1-\omega_n)H\rho_n + \omega_n H\vartheta_n \\ \vartheta_n = (1-\delta_n)\rho_n + \delta_n H\rho_n \\ \rho_n = (1-\varepsilon_n)\sigma_n + \varepsilon_n H\sigma_n\end{cases}$$

where the sequences $\{\omega_n\}, \{\delta_n\}$ and $\{\varepsilon_n\} \in (0, 1)$.

Let us now prove the main theorem.

3. Main Results

Theorem 3.1. *Let H be a self-map from a norm linear space E with a fixed point* σ^* *satisfying the inequality* $\| \sigma^* - H\sigma \| \le \mu \| \sigma^* - \sigma \|$ *for each* $\sigma \in E$ *and* $\mu \in [0, 1)$. *Let the sequences* $\{c_n\}$ *and* $\{d_n\}$ *be defined by Itr.* P_1 *and Itr.* P_2 *respectively with the sequences* $\{\omega_n\}, \{\delta_n\}$ *and* $\{\varepsilon_n\} \in (0, 1)$ *and* $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, *then Itr.* P_1 *and Itr.* P_2 *defined in the equations* (5) *and* (6) *have the same rate of convergence to* σ^* *of H.*

Proof. Using the inequality $\| \sigma^* - H\sigma \| \le \mu \| \sigma^* - \sigma \|$ and Itr. P_1 and Itr. P_2, we have the estimates for all $n \le 0$:

$$\| c_{n+1} - \sigma^* \| \le \mu^{2n} \| c_0 - \sigma^* \| \prod_{i=0}^{\infty} (1 - \varepsilon_i(1-\mu))$$

$$\| d_{n+1} - \sigma^* \| \le \mu^{2n} \| d_0 - \sigma^* \| \prod_{i=0}^{\infty} (1 - \varepsilon_i(1-\mu)).$$

Define

$$\alpha_n = \mu^{2n} \| c_0 - \sigma^* \| \prod_{i=0}^{\infty} (1 - \varepsilon_i (1 - \mu))$$

$$\beta_n = \mu^{2n} \| d_0 - \sigma^* \| \prod_{i=0}^{\infty} (1 - \varepsilon_i (1 - \mu)).$$

Since $c_0 = d_0$ and $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, it is evident that $\alpha_n \to 0$ and $\beta_n \to 0$ for all $n \geq 0$.

Hence

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = \lim_{n\to\infty} \frac{\mu^{2n} \| c_0 - \sigma * \| \prod_{i=0}^{\infty} (1 - \varepsilon_i (1 - \mu))}{\mu^{2n} \| d_0 - \sigma * \| \prod_{i=0}^{\infty} (1 - \varepsilon_i (1 - \mu))} = 1$$

yields that $\{c_n\}$ and $\{d_n\}$ iterative processes converge with same acceleration. Consequently, Itr. P_1 and Itr. P_2 have the same rate of convergence as desired. □

Following examples show that the iterative process (5) is equivalent to convergence of the iteration scheme (6) given in the introduction section of the manuscript. The above analytical proof is supported by considering the following numerical illustrations.

4. Numerical Illustrations

Example 4.1. Let $H : [0, 1] \to [0, 1] := \frac{\sigma}{2}, \omega_n = \delta_n = \varepsilon_n = 0,$ for $1 \leq n \leq 15$ and $\omega_n = \delta_n = \varepsilon_n = \frac{4}{\sqrt{n}}$, for $n \geq 16$. Clearly, H is non expansive operator with a unique fixed point 0.

Since $\omega_n = \delta_n = \varepsilon_n = 0$ for $1 \leq n \leq 15$, so $Itr.P_1 = \sigma_0 = Itr.P_2$ for $1 \leq n \leq 15$. Let $\sigma_0 \neq 0$, therefore for $Itr.P_1$, we have

$$\rho_n = (1 - \varepsilon_n)\sigma_n + \varepsilon_n H\sigma_n$$

$$= \left(1 - \frac{4}{\sqrt{n}}\right)\sigma_n + \frac{4}{\sqrt{n}} \frac{\sigma_n}{2}$$

$$= \left(1 - \frac{2}{\sqrt{n}}\right)\sigma_n$$

$$\vartheta_n = (1 - \delta_n)H\sigma_n + \delta_n H\rho_n$$

$$= \left(1 - \frac{4}{\sqrt{n}}\right)\frac{1}{2}\sigma_n + \frac{4}{\sqrt{n}}\frac{1}{2}\sigma_n\left(1 - \frac{2}{\sqrt{n}}\right)$$

$$= \left(\frac{1}{2} - \frac{4}{n}\right)\sigma_n$$

$$\sigma_{n+1} = (1 - \omega_n)H\vartheta_n + \omega_n H\rho_n$$

$$= \left(1 - \frac{4}{\sqrt{n}}\right)\left(\frac{1}{2} - \frac{4}{n}\right)\frac{1}{2}\sigma_n + \frac{4}{\sqrt{n}}\frac{1}{2}\left(1 - \frac{2}{\sqrt{n}}\right)\sigma_n$$

$$= \left(\frac{1}{4} - \frac{6}{n} + \frac{1}{\sqrt{n}} + \frac{8}{n\sqrt{n}}\right)\sigma_n$$

$$\ldots$$

$$\therefore\ Itr.P_1 = \prod_{k=1}^{n}\left(\frac{1}{4} - \frac{6}{k} + \frac{1}{\sqrt{k}} + \frac{8}{k\sqrt{k}}\right)\sigma_0.$$

For $Itr.\ P_2$ we derive:

$$\rho_n = (1 - \varepsilon_n)\sigma_n + \varepsilon_n H\sigma_n$$

$$= \left(1 - \frac{4}{\sqrt{n}}\right)\sigma_n \frac{4}{\sqrt{n}}\frac{\sigma_n}{2}$$

$$= \left(1 - \frac{2}{\sqrt{n}}\right)\sigma_n$$

$$\vartheta_n = (1 - \delta_n)\rho_n + \delta_n H\rho_n$$

$$= \left(1 - \frac{4}{\sqrt{n}}\right)\left(1 - \frac{2}{\sqrt{n}}\right)\sigma_n + \frac{4}{\sqrt{n}}H\left(\left(1 - \frac{2}{\sqrt{n}}\right)\sigma_n\right)$$

$$= \left(1 - \frac{6}{\sqrt{n}} + \frac{8}{n} + \frac{2}{\sqrt{n}} - \frac{4}{n}\right)\sigma_n$$

$$= \left(1 - \frac{4}{\sqrt{n}} + \frac{4}{n}\right)\sigma_n$$

$$\sigma_{n+1} = (1-\omega_n)H\rho_n + \omega_n H\vartheta_n$$

$$= \left(1-\frac{4}{\sqrt{n}}\right)\left(1-\frac{2}{\sqrt{n}}\right)\frac{1}{2}\sigma_n + \frac{4}{\sqrt{n}}\frac{1}{2}\left(1-\frac{4}{\sqrt{n}}+\frac{4}{n}\right)\sigma_n$$

$$= \left(\frac{1}{2}-\frac{1}{\sqrt{n}}-\frac{12}{n}+\frac{8}{n\sqrt{n}}\right)\sigma_n$$

$$\cdots$$

$$\therefore\ Itr.P_2 = \prod_{k=1}^{n}\left(\frac{1}{2}-\frac{1}{\sqrt{k}}-\frac{12}{k}+\frac{8}{k\sqrt{k}}\right)\sigma_0.$$

Therefore,

$$\left\|\frac{Itr.P_1-0}{Itr.P_2-0}\right\| = \left|\frac{\prod_{k=1}^{n}\left(\frac{1}{4}-\frac{6}{k}+\frac{1}{\sqrt{k}}+\frac{8}{k\sqrt{k}}\right)\sigma_0}{\prod_{k=1}^{n}\left(\frac{1}{2}-\frac{6}{\sqrt{k}}+\frac{12}{k}+\frac{8}{k\sqrt{k}}\right)\sigma_0}\right| < 1 \text{ yields that } Itr.\mathrm{P}_1 \approx Itr.P_2.$$

Hence, it is evident that $Itr.P_1$ and $Itr.P_2$ have the same rate of convergence.

Example 4.2. Define a map $H:[0,1]\to[0,1]$ by $H(\sigma)=\frac{\sigma}{2}$. Let $\delta_n = \varepsilon_n = \frac{3}{4} \in (0,1)$ with the initial value $\sigma_1 = 1$. It is easy to observe that H is non expansive map.

Following table and figure 1 represent the equivalence of convergence of (5) and (6):

Table 1. Comparison of rate of convergence of $Itr.P_1$ and $Itr.P_2$.

Step	$Itr.P_1$ (5)			$Itr.P_2$ (6)		
	σ_{n+1}	ϑ_n	ρ_n	σ_{n+1}	ϑ_n	ρ_n
1	0.2792968	0.3593750	0.6250000	0.2246093	0.3906250	0.625000
2	0.0780067	0.1003722	0.1745605	0.0504493	0.0877380	0.1403808
3	0.0217869	0.0280336	0.0487541	0.0113323	0.0197067	0.0315308
4	0.0060854	0.0078296	0.0136168	0.0025453	0.0044266	0.0070826
5	0.0016995	0.0021868	0.0038032	0.0005716	0.0009942	0.0015908
6	0.0004746	0.0006107	0.0010621	0.0001283	0.0002232	0.0003572
7	0.0001325	0.0001705	0.0002966	0.0000287	0.0000500	0.0000801
8	0.0000370	0.0000476	0.0000828	0.0000064	0.0000111	0.0000179
9	0.0000103	0.0000132	0.0000231	0.0000014	0.0000025	0.0000040
10	0.0000020	0.0000036	0.0000064	0.0000002	0.0000005	0.0000008
11	0.0000000	0.0000000	0.0000012	0.0000000	0.0000000	0.0000001

From Table 1, we see that both the iterative schemes converge to the fixed point $\tau = 0$ (corrected to seven decimal places) with the same rate of convergence i.e. at the 11^{th} step. Consequently, the iterative procedures (5) and (6) converges equivalently with the same rate of convergence. The convergence behaviour of these schemes is shown below:

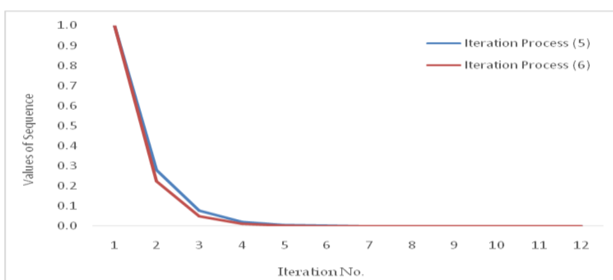


Figure 1. Graphical presentation of $Itr.P_1$ and $Itr.P_2$.

Example 4.3. Define a map $H : [0, 1] \to [0, 1]$ by $H(\sigma) = \frac{e^{\sigma}}{10}$. Let $\delta_n = \varepsilon_n = \frac{3}{4} \in (0, 1)$ with the initial value $\sigma_1 = 0.99$. We show that both the sequences converge to the fixed point $\tau = 0$ with the equal no. of iterations.

Following table and figure 2 show the equivalence of convergence rate of sequences $Itr.P_1$ and $Itr.P_2$ defined in (5) and (6) respectively:

Table 2. Comparison of convergence rate of $Itr.P_1$ and $Itr.P_2$.

Step	$Itr.P_1(5)$			$Itr.P_2(5)$		
	σ_{n+1}	ϑ_n	ρ_n	σ_{n+1}	ϑ_n	ρ_n
1	0.1476213	0.1848269	0.4493425	0.1335658	0.2298817	0.4493425
2	0.1129022	0.1138641	0.1238358	0.1122409	0.1142641	0.1191087
3	0.1118641	0.1118924	0.1121897	0.1118401	0.1118780	0.1119689
4	0.1118334	0.1118343	0.1118431	0.1118327	0.1118334	0.1118351
5	0.1118325	0.1118326	0.1118328	0.1118325	0.1118325	0.1118326

From Table 2, we see that both the iterative schemes reach to their common fixed point $\tau = 0.1118325$ correct up to the seven decimal places with the same convergence rate at the 5th step. Consequently, the iterative procedures (5) and (6) converges equivalently with the same rate of convergence.

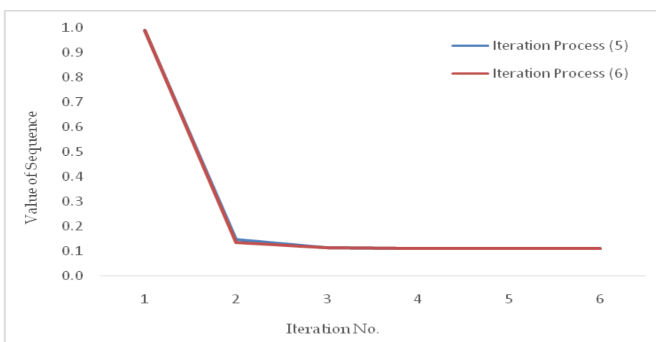
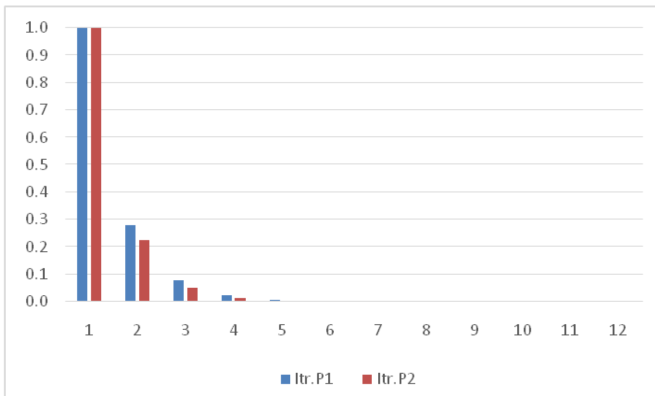


Figure 2. Graphical presentation of $Itr.P_1$ and $Itr.P_2$.

Hence both the comparative tables and figures indicate that $Itr.P_1$ and $Itr.P_2$ defined in (5) and (6) have the same rate of convergence.

5. Conclusion

In this study, two iterative algorithms $Itr.P_1$ and $Itr.P_2$ were compared and it is evident that these iterative processes have the same rate of convergence as they are completely independent of each other. It is concluded that $Itr.P_1$ is equivalent to the convergence of $Itr.P_2$. The graphical representation of $Itr.P_1$ and $Itr.P_2$ clearly depicts the convergence of $Itr.P_1$ and $Itr.P_2$ to the same fixed point. This study has brought a new perspective to define the iteration processes.



Convergence Equivalence of $Itr.P_1$ and $Itr.P_2$.

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