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SQUARE DIFFERENCE GEOMETRIC MEAN 3-EQUITABLE LABELING OF CERTAIN GRAPHS

V. ANNAMMA and JAWAHAR NISHA M I

L. N. Government College
Ponneri, 601204, Tamil Nadu, India

Department of Mathematics
Justice Basheer Ahmed Sayeed
College for Women (Autonomous)
Teynampet, Chennai, 600018
Tamil Nadu, India
E-mail: nishabdulbasith@gmail.com

Abstract

A Square Difference Geometric Mean (SDGM) 3-Equitable labeling of a graph $G = (V, E)$ is a surjective mapping $f : V(G) \to \{0, 1, 2\}$ such that the induced mapping $g : E(G) \to \{0, 1, 2\}$ is defined by $\left\lceil \sqrt{|(f(u))^2 - (f(v))^2|} \right\rceil, \forall uv \in E(G)$ with the condition $|v_f(i) - v_f(j)| \le 1$ and $|e_g(i) - e_g(j)| \le 1$ for all $0 \le i, j \le 2$. A graph is called a Square Difference Geometric Mean (SDGM) 3-Equitable graph if there exists a SDGM 3-Equitable labeling. In this paper we define the SDGM 3-Equitable labeling and we investigate the SDGM 3-Equitable labeling of certain graphs such as Path graph, Cycle graph, Star graph, Bistar graph and Comb graph.

1. Introduction

Here we are considering non trivial, simple, finite and undirected graphs. An assignment of integers to the vertices or edges, or both subject to certain conditions is called graph labeling [5]. The concept of cordial and 3-equitable labeling was introduced by Cahit [2]. Ponraj, Sivakumar and Sundaram introduced the concept of mean cordial labeling [7]. Geometric mean cordial

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labeling was introduced by K. Chitra Lakshmi, K. Nagarajan [3]. Meghpara Meera et al., points out that the name "Geometric Mean Cordial Labeling" should be "Geometric Mean 3-Equitable Labeling" as K. Chitra Lakshmi, K. Nagarajan are using $e_f(0)$, $e_f(1)$, $e_f(2)$ [6].

Motivated by these definitions, in this paper we define the new notion called Square Difference Geometric Mean (SDGM) 3-Equitable labeling. We investigate the SDGM 3-Equitable labeling of certain graphs such as Path graph, Cycle graph, Star graph, Bistar graph and Comb graph. Terms and definitions not defined here are used in the sense of Harary [4].

2. Preliminaries

Definition 2.1 [8]**.** The bistar graph $B[n, n]$ is the graph obtained by joining the center (apex) vertices of two copies of $K_{(1, n)}$ by an edge.

Definition 2.2 [1]**.** Let P_n be a path graph with n vertices and $n-1$ edges. The comb graph is defined as $P_n \cup K_1$. The comb is a graph formed by joining a pendant edge to each vertices of the path. The comb graph has $2n$ vertices and $2n-1$ edges.

3. Main Results

Definition 3.1. A Square Difference Geometric Mean (SDGM) 3-Equitable labeling of a graph $G = (V, E)$ is a surjective mapping $f : V(G) \to \{0, 1, 2\}$ such that the induced mapping $g : E(G) \to \{0, 1, 2\}$ is defined by $\left\lceil \sqrt{|(f(u))^2 - (f(v))^2|} \right\rceil, \forall\, uv \in E(G)$ with the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_g(i) - e_g(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph is called a Square Difference Geometric Mean (SDGM) 3-Equitable graph if there exists a SDGM 3-Equitable labeling.

Remarks 3.1. If we consider $f : V(G) \to \{0, 1\}$ the definition 3.1 coincides with that of cordial labeling. Hence we consider $f : V(G) \to \{0, 1, 2\}$.

Theorem 3.1. *Any Path graph P_n is a SDGM 3-Equitable graph.*

Proof. Let $G = (V, E)$ be a Path Graph P_n.

Let $V(G) = \{v_i / 1 \leq i \leq n\}$.

Let $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\}$.

Let $|V(G)| = l$ and $|E(G)| = k$. Then $l = n$ and $k = n - 1$.

The Path Graph P_n is shown below in Figure 1

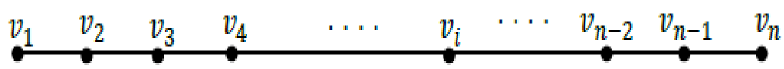

Figure 1. Path Graph P_n.

Define $f : V(G) \to \{0, 1, 2\}$. as follows:

$$f(v_i) = \begin{cases} 0, & i \equiv 0, 1 (\text{mod } 6) \\ 1, & i \equiv 2, 5 (\text{mod } 6) \\ 2, & i \equiv 3, 4 (\text{mod } 6) \end{cases} \text{ for all } 1 \leq i \leq n$$

Case (i). $n \equiv 0 (\text{mod } 3)$

Let $n = 3t, t \geq 1$

Here $l = n = 3t$ and $k = n - 1 = 3t - 1$.

Then $v_f(0) = v_f(1) = v_f(2) = t$ and $e_g(0) = t - 1, e_g(1) = e_g(2) = t$.

Case (ii). $n \equiv 1 (\text{mod } 3)$

Let $n = 3t + 1, t \geq 0$

Here $l = n = 3t + 1$ and $k = n - 1 = 3t$.

Subcase (i). t is odd

$v_f(0) = v_f(1) = t, v_f(2) = t + 1$ and $e_g(0) = e_g(1) = e_g(2) = t$.

Subcase (ii). t is even

$v_f(0) = t + 1, v_f(1) = v_f(2) = t$ and $e_g(0) = e_g(1) = e_g(2) = t$.

Case (iii). $n \equiv 2(\text{mod}\, 3)$

Let $n = 3t + 2$, $t \geq 0$

Here $l = n = 3t + 2$ and $k = n - 1 = 3t + 1$.

Subcase (i). t is odd

$v_f(0) = t$, $v_f(1) = v_f(2) = t + 1$ and $e_g(0) = e_g(1) = t$, $e_g(2) = t + 1$.

Subcase (ii). t is even

$v_f(0) = v_f(1) = t + 1$, $v_f(2) = t$ and $e_g(0) = e_g(2) = t$, $e_g(1) = t + 1$.

In all the above three cases, we see that $|v_f(i) - v_f(j)| \leq 1$ and $|e_g(i) - e_g(j)| \leq 1$ for all $0 \leq i, j \leq 2$.

Hence Path graph P_n is a SDGM 3-Equitable graph.

Illustration 3.1. SDGM 3-Equitable Labeling of Path graph P_{14} is shown in Figure 2

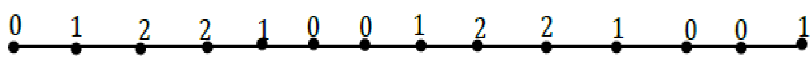

Figure 2. Path Graph P_{14}.

Here $v_f(0) = v_f(1) = 5$, $v_f(2) = 4$ and $e_g(0) = e_g(2) = 4$, $e_g(1) = 5$.

Therefore $|v_f(i) - v_f(j)| \leq 1$ and $|e_g(i) - e_g(j)| \leq 1$ for all $0 \leq i, j \leq 2$.

Theorem 3.2. *The Cycle graph C_n is a SDGM 3-Equitable graph except for $n \equiv 3(\text{mod}\, 6)$.*

Proof. Let $G = (V, E)$ be a Cycle Graph C_n.

Let $V(G) = \{v_i / 1 \leq i \leq n\}$.

Let $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n - 1\} \cup \{v_n v_1\}$

Let $|V(G)| = l$ and $|E(G)| = k$. Then $l = n$ and $k = n$.

The Cycle Graph C_n is shown below in Figure 3

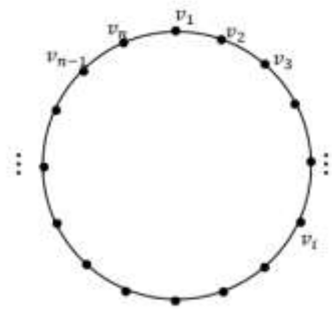

Figure 3. Cycle Graph C_n.

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

Case (i). $n \equiv 2 (\bmod 6)$

Let $n = 6t + 2$, $t > 0$

$$f(v_i) = \begin{cases} 0, & i \equiv 1,\ 4 (\bmod 6) \\ 1, & i \equiv 2,\ 3 (\bmod 6) \text{ for all } 1 \leq i \leq n-2 \\ 2, & i \equiv 0,\ 5 (\bmod 6) \end{cases}$$

$f(v_{n-1}) = 1$

$f(v_n) = 0$

Here $l = n = 6t + 2$ and $k = n = 6t + 2$.

Then $v_f(0) = v_f(1) = 2t + 1$, $v_f(2) = 2t$ and $e_g(0) = e_g(1) = 2t + 1$, $e_g(2) = 2t$.

Case (ii). $n \not\equiv 2 (\bmod 6)$

$$f(v_i) = \begin{cases} 0, & i \equiv 0,\ 1 (\bmod 6) \\ 1, & i \equiv 2,\ 5 (\bmod 6) \text{ for all } 1 \leq i \leq n \\ 2, & i \equiv 3,\ 4 (\bmod 6) \end{cases}$$

Subcase (i). $n \equiv 0 (\bmod 6)$

Let $n = 6t$, $t > 0$

Here $l = n = 6t$ and $k = n = 6t$.

Then $v_f(0) = v_f(1) = v_f(2) = 2t$ and $e_g(0) = e_g(1) = e_g(2) = 2t$.

Subcase (ii). $n \equiv 1 (\text{mod}\, 6)$

Let $n = 6t + 1,\ t > 0$

Here $l = n = 6t + 1$ and $k = n = 6t + 1$.

Then $v_f(0) = 2t + 1, v_f(1) = v_f(2) = 2t$ and $e_g(0) = 2t + 1, e_g(1) = e_g(2) = 2t$.

Subcase (iii). $n \equiv 4 (\text{mod}\, 6)$

Let $n = 6t + 4,\ t \geq 0$

Here $l = n = 6t + 4$ and $k = n = 6t + 4$.

Then $v_f(0) = v_f(1) = 2t + 1,\ v_f(2) = 2t + 2$ and $e_g(0) = e_g(1) = 2t + 1$, $e_g(2) = 2t + 2$.

Subcase (iv). $n \equiv 5 (\text{mod}\, 6)$

Let $n = 6t + 5,\ t \geq 0$

Here $l = n = 6t + 5$ and $k = n = 6t + 5$.

Then $v_f(0) = 2t + 1,\ v_f(1) = v_f(2) = 2t + 2$ and $e_g(0) = 2t + 1,\ e_g(1) = e_g(2) = 2t + 2$.

In all the above cases, we see that $| v_f(i) - v_f(j) | \leq 1$ and $| e_g(i) - e_g(j) | \leq 1$ for all $0 \leq i,\ j \leq 2$.

Hence Cycle graph C_n is a SDGM 3-Equitable graph.

Illustration 3.2. SDGM 3-Equitable Labeling of Cycle graph C_{16} is shown in Figure 4

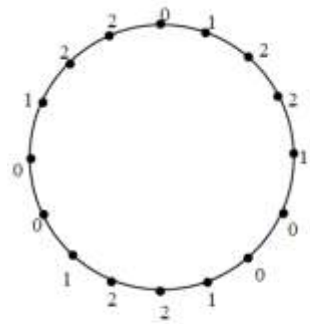

Figure 4. Cycle Graph C_{16}.

Here $v_f(0) = v_f(1) = 5$, $v_f(2) = 6$ and $e_g(0) = e_g(1) = 5$, $e_g(2) = 6$.

Therefore $|v_f(i) - v_f(j)| \leq 1$ and $|e_g(i) - e_g(j)| \leq 1$ for all $0 \leq i, j \leq 2$.

Note. For $n \equiv 3 (\text{mod } 6)$, the cycle graph C_n is not a SDGM 3-Equitable graph.

Theorem 3.3. *The Star graph $K_{1,n}$ is a SDGM 3-Equitable graph for all $n \geq 1$.*

Proof. Let $G = (V, E)$ be a Star Graph $K_{1,n}$.

Let $V(G) = \{v, v_i / 1 \leq i \leq n\}$.

Let $E(G) = \{vv_i / 1 \leq i \leq n\}$.

Let $|V(G)| = l$ and $|E(G)| = k$. Then $l = n + 1$ and $k = n$.

The Star Graph $K_{1,n}$ is shown below in Figure 5

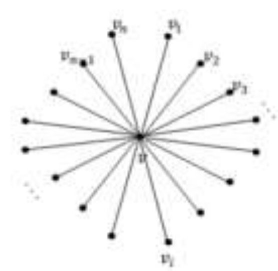

Figure 5. Star Graph $K_{1,n}$.

Define $f : V(G) \to \{0, 1, 2\}$ as follows.

$f(v) = 0$

$$f(v_i) = \begin{cases} 0, & i \equiv 0 (\bmod 3) \\ 1, & i \equiv 2 (\bmod 3) \\ 2, & i \equiv 1 (\bmod 3) \end{cases} \text{ for all } 1 \leq i \leq n$$

Case (i). $n \equiv 0 (\bmod 3)$

Let $n = 3t,\ t > 0$

Here $l = n + 1 = 3t + 1$ and $k = n = 3t$.

Then $v_f(0) = t + 1,\ v_f(1) = v_f(2) = t$ and $e_g(0) = e_g(1) = e_g(2) = t$.

Case (ii). $n \equiv 1 (\bmod 3)$

Let $n = 3t + 1,\ t \geq 0$

Here $l = n + 1 = 3t + 2$ and $k = n = 3t + 1$.

Then $v_f(0) = v_f(2) = t + 1,\ v_f(1) = t$ and $e_g(0) = e_g(1) = t,\ e_g(2) = t + 1$.

Case (iii). $n \equiv 2 (\bmod 3)$

Let $n = 3t + 2,\ t \geq 0$

Here $l = n + 1 = 3t + 3$ and $k = n = 3t + 2$.

Then $v_f(0) = v_f(1) = v_f(2) = t + 1$ and $e_g(0) = t,\ e_g(1) = e_g(2) = t + 1$.

In all the above three cases, we see that $| v_f(i) - v_f(j) | \leq 1$ and $| e_g(i) - e_g(j) | \leq 1$ for all $0 \leq i,\ j \leq 2$.

Hence Star graph $K_{1,n}$ is a SDGM 3-Equitable graph.

Illustration 3.3. SDGM 3-Equitable Labeling of Star graph $K_{1,16}$ is shown in Figure 6

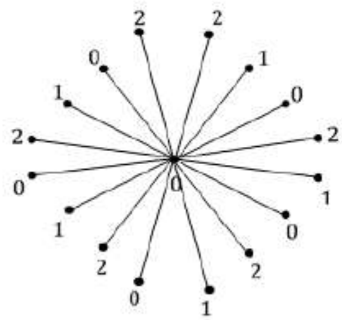

Figure 6. Star Graph $K_{1,16}$.

Here $v_f(0) = v_f(2) = 6$, $v_f(1) = 5$ and $e_g(0) = e_g(1) = 5$, $e_g(2) = 5$.

Therefore $| v_f(i) - v_f(j) | \leq 1$ and $| e_g(i) - e_g(j) | \leq 1$ for all $0 \leq i, j \leq 2$.

Theorem 3.4. *The Bistar graph* $B(n, n)$ *is a SDGM* 3-*Equitable graph for all* $n \geq 1$.

Proof. Let $G = (V, E)$ be a Bistar Graph $B(n, n)$.

Let $V(G) = \{u, v, u_i, v_i / 1 \leq i \leq n\}$.

Let $E(G) = \{uu_i, vv_i / 1 \leq i \leq n\}$.

Let $| V(G) | = l$ and $| E(G) | = k$. Then $l = 2n + 2$ and $k = 2n + 1$.

The Bistar Graph $B(n, n)$ is shown below in Figure 7

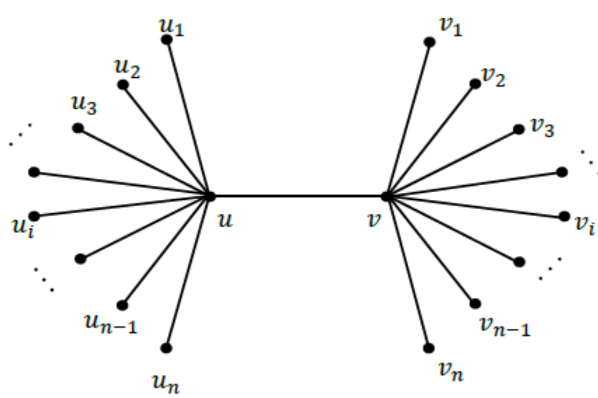

Figure 7. Bistar Graph $B[n, n]$.

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

Case (i). $n \equiv 0 (\text{mod } 3)$

Let $n = 3t, t \geq 1$

$f(u) = f(v) = 0$

$f(u_i) = \begin{cases} 0, & 1 \leq i \leq t \\ 1, & t+1 \leq i \leq 3t \end{cases}$ and $f(v_i) = \begin{cases} 0, & 1 \leq i \leq t-1 \\ 2, & t \leq i \leq 3t \end{cases}$

Here $l = 2n + 2 = 6t + 2$ and $k = 2n + 1 = 6t + 1$.

Then $v_f(0) = v_f(2) = 2t + 1, v_f(1) = 2t$ and $e_g(0) = e_g(1) = 2t, e_g(2) = 2t + 1$.

Case (ii). $n \equiv 1 (\text{mod } 3)$

Let $n = 3t + 1, t \geq 0$

$f(u) = f(v) = 0$

$f(u_i) = \begin{cases} 0, & 1 \leq i \leq t \\ 1, & t+1 \leq i \leq 3t+1 \end{cases}$ and $f(v_i) = \begin{cases} 0, & 1 \leq i \leq t \\ 2, & t+1 \leq i \leq 3t+1 \end{cases}$

Here $l = 2n + 2 = 6t + 4$ and $k = 2n + 1 = 6t + 3$.

Then $v_f(0) = 2t + 2, v_f(1) = v_f(2) = 2t + 1$ and $e_g(0) = e_g(1) = e_g(2) = 2t + 1$.

Case (iii). $n \equiv 2 (\text{mod } 3)$

Let $n = 3t + 2, t \geq 0$

$f(u) = f(v) = 0$

$f(u_i) = \begin{cases} 0, & 1 \leq i \leq t \\ 1, & t+1 \leq i \leq 3t+2 \end{cases}$ and $f(v_i) = \begin{cases} 0, & 1 \leq i \leq t \\ 2, & t+1 \leq i \leq 3t+2 \end{cases}$

Here $l = 2n + 2 = 6t + 6$ and $k = 2n + 1 = 6t + 5$.

Then $v_f(0) = v_f(1) = v_f(2) = 2t + 2$ and $e_g(0) = 2t + 1, e_g(1) = e_g(2) = 2t + 2$.

In all the above three cases, we see that $|v_f(i) - v_f(j)| \leq 1$ and $|e_g(i) - e_g(j)| \leq 1$ for all $0 \leq i, j \leq 2$.

Hence Bistar graph $B(n, n)$ is a SDGM 3-Equitable graph.

Illustration 3.4. SDGM 3-Equitable Labeling of Bistar graph $B(8, 8)$ is shown in Figure 8

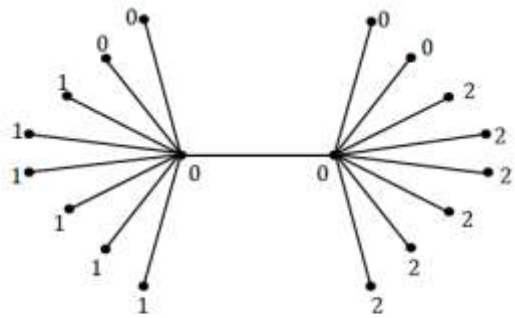

Figure 8. Bistar Graph $B[8, 8]$.

Here $v_f(0) = v_f(1) = v_f(2) = 6$ and $e_g(0) = 5$, $e_g(1) = e_g(2) = 6$.

Therefore $|v_f(i) - v_f(j)| \leq 1$ and $|e_g(i) - e_g(j)| \leq 1$ for all $0 \leq i, j \leq 2$.

Theorem 3.5. *The Comb graph* $[P_n \odot K_1]$ *is a SDGM* 3-*Equitable graph for all* n.

Proof. Let $G = (V, E)$ be a Comb Graph $[P_n \odot K_1]$.

Let $V(G) = \{v_i, u_i / 1 \leq i \leq n\}$, where v_i represents the vertices of the path and u_i represents the pendent vertices corresponding to each v_i respectively.

Let $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i u_i / 1 \leq i \leq n\}$.

Let $|V(G)| = l$ and $|E(G)| = k$. Then $l = 2n$ and $k = 2n - 1$

The Comb Graph $[P_n \odot K_1]$ is shown below in Figure 9

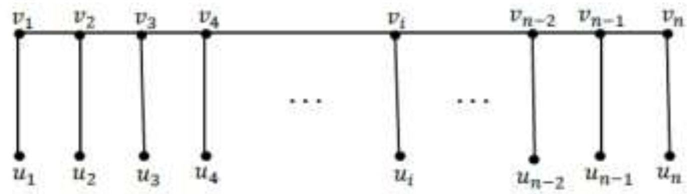

Figure 9. Comb Graph $[P_n \odot K_1]$.

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

Case (i). $n \equiv 0 (\bmod 3)$

Let $n = 3t, t \geq 1$

$\forall 1 \leq i \leq n,$

$$f(v_i) = \begin{cases} 0, & i \equiv 1, 2 (\bmod 3) \\ 2, & i \equiv 0 (\bmod 3) \end{cases} \text{ and } f(u_i) = \begin{cases} 1, & i \equiv 1, 2 (\bmod 3) \\ 2, & i \equiv 0 (\bmod 3) \end{cases}$$

Here $l = 2n = 6t$ and $k = 2n - 1 = 6t - 1$.

Then $v_f(0) = v_f(1) = v_f(2) = 2t$ and $e_g(0) = e_g(1) = 2t, e_g(2) = 2t - 1$.

Case (ii). $n \equiv 1 (\bmod 3)$

Let $n = 3t + 1, t \geq 0$

$\forall 1 \leq i \leq n,$

$$f(v_i) = \begin{cases} 0, & i \equiv 1, 2 (\bmod 3) \\ 2, & i \equiv 0 (\bmod 3) \end{cases} \text{ and } f(u_i) = \begin{cases} 1, & i \equiv 1, 2 (\bmod 3) \\ 2, & i \equiv 0 (\bmod 3) \end{cases}$$

Here $l = 2n = 6t + 2$ and $k = 2n - 1 = 6t + 1$.

Then $v_f(0) = v_f(1) = 2t + 1, v_f(2) = 2t$ and $e_g(0) = e_g(1) = 2t, e_g(1) = 2t + 1$.

Case (iii). $n \equiv 2 (\bmod 3)$

Let $n = 3t + 2, t \geq 0$

$f(v_n) = f(u_n) = 2$

$\forall 1 \leq i \leq n-1$,

$$f(v_i) = \begin{cases} 0, & i \equiv 1, 2 (\bmod 3) \\ 2, & i \equiv 0 (\bmod 3) \end{cases} \text{ and } f(u_i) = \begin{cases} 1, & i \equiv 1, 2 (\bmod 3) \\ 2, & i \equiv 0 (\bmod 3) \end{cases}$$

Here $l = 2n = 6t + 4$ and $k = 2n - 1 = 6t + 3$.

Then $v_f(0) = v_f(1) = 2t + 1$, $v_f(2) = 2t + 2$ and $e_g(0) = e_g(1) = e_g(2) = 2t + 1$.

In all the above three cases, we see that $| v_f(i) - v_f(j) | \leq 1$ and $| e_g(i) - e_g(j) | \leq 1$ for all $0 \leq i, j \leq 2$.

Hence Comb graph $[P_n \odot K_1]$ is a SDGM 3-Equitable graph.

Illustration 3.5. SDGM 3-Equitable Labeling of Comb graph $[P_{12} \odot K_1]$ is shown in Figure 10.

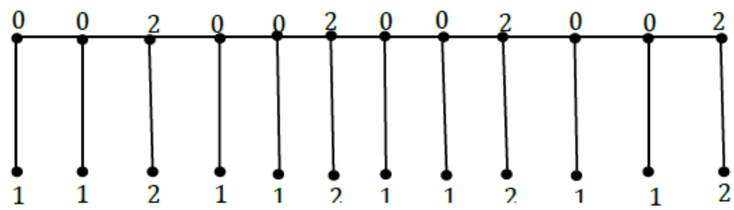

Figure 10. Comb Graph $P_{12} \odot K_1$.

Here $v_f(0) = v_f(1) = v_f(2) = 8$ and $e_g(0) = e_g(1) = 8$, $e_g(2) = 7$.

Therefore $| v_f(i) - v_f(j) | \leq 1$ and $| e_g(i) - e_g(j) | \leq 1$ for all $0 \leq i, j \leq 2$.

Conclusion

In this paper we introduced the concept of SDGM 3-Equitable labeling and we investigated the SDGM 3-Equitable labeling of certain graphs such as Path graph, Cycle graph, Star graph, Bistar graph and Comb graph. The future work includes SDGM 3-Equitable labeling of cycle and wheel related graphs.

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