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COMPUTATIONAL APPROACH FOR TRANSIENT BEHAVIOUR OF FINITE SOURCE RETRIAL QUEUEING MODEL WITH MULTIPLE VACATIONS AND CLOSED DOWN

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Abstract

A new computational approach is used to analyze time dependent transient behaviour of finite source retrial queuing model with multiple vacations and closed down in which the primary arrival rate λ which follows a Poisson distribution and the service time μ follows an exponential distribution. We have assumed that the finite calling population size *F*. The vacation time follows an exponential distribution with parameter *α*. All transitions are assembled by using an infinitesimal generator matrix. Cayley Hamilton method is used to obtain the Time dependent and Steady state solutions. Numerical studies have been done for time dependent average number of customers in the queue, transient probabilities of server free/busy/vacation for various values of *F*, α, γ, μ, λ.

1. Introduction

In this paper, we study about the transient behaviour of finite source Retrial queueing system with multiple vacations and closed down by new computational approach. An effective algorithm is used to obtain transient probabilities and time dependent system performance measures. All the

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numerical calculations are done by using SCILAB. The detailed discussions are given below:

Retrial queues and its studies have been found in Miller (1983), Falin (1990) and Artalejo (2009, 2010). Considerable attention on finite source queues studies from various researchers such as Dong-Yuh Yang and Ying-Yi Wu (2014) have studied Transient Behaviour Analysis of a Finite capacity queue with Working Breakdowns and server Vacations.

Shin, Y. W. (2015) has studied an Algorithmic approach to Markovian multi-server retrial queues with vacations. Vijayashree and K. V., Jananai B., (2016) have analysed Transient analysis of an $M/M/c$ queue subject to multiple exponential vacation, Sudhesh, R., Azhagappan, A., and Dharmaraja, S. (2017) have analysed Transient analysis of $M/M/1$ queue with working vacation, heterogeneous service and customers' impatience.

Ioannis Dimitriou and Christos Langaris (2009) has studied A Queueing Model with Start-Up/Close-Down Times and Retrial Customers. Zhisheng, Niu Tao Shu, Yoshitaka Takahashi (2003) have analyzed vacation queue with setup and close-down times and batch Markovian arrival processes. S. S. Mishra and Dinesh Kumar Yadav (2008) have worked on Cost and Profit Analysis of $M/Ek/1$ Queueing System with Removable Service Station. Also, Bhupender Kumar Som, Sunny Seth (2017) have analysed an $M/M/1/N$ Queuing system with Encouraged Arrivals.

2. The Mathematical Model and Its Solutions

Model Description

Consider a finite source retrial queuing system with multiple vacation and closed down in which the primary arrival rate λ follows a Poisson distribution and the service time follows an exponential distribution with parameter μ. Further, we have assumed that the calling population is finite of size F.

Retrial queues

If server not idle then arriving occupant goes to orbit becomes a group of recurrent customers. This group of recurrent customers viewed as sort of queue. It obeys a Poisson, recurrent customers with intensity σ. If incoming recurrent customers find the server free, it served and leaves system; these recurrent customers vanished after service.

Multiple vacations

The concept of multiple vacations is incorporated in this work in such a way that after completion of a service if server finds no one in orbit then he goes for a vacation. After finishing of the vacation period if server finds at least one customer in the orbit then he returns to the system and waiting for service otherwise he will go for vacation once again. It follows an exponential distribution with variable α.

Closed down

After completing a service, if there are no customers present in the system then the server will close down the system. Close down time is generally distributed with exponential distribution γ.

Retrial Policy

The retrial policy states that the probability of an customers from an orbit to try for service during the time interval $(t, t + \Delta t)$ given that there were n customers in orbit at time t is $n \ \sigma\Delta t + O(\Delta t)$. This regulation for access for the server from the retrial group is called a classical retrial rate policy.

3. Representation of Random Process

Let $C(t)$ be the number of customers in the orbit at time t and $S(t)$ be the status of the server (i.e.) idle/busy/vacation/closed down at time t. The random process is described as $\{\langle C(t), S(t)\rangle / C(t) = 0, 1, 2, 3, \ldots, F; S(t)\} \cup \{\langle C(t), S(t)\rangle / C(t) = 0, 1, 2, 3, \ldots, F - 1, S(t) = 1\} \cup \langle C(t), S(t)\rangle / C(t) = 0, 1, 2, 3, \ldots, F; S(t) = 2\} \cup \{\langle C(t), S(t)\rangle / C(t) = 0; 1, 3, \ldots, F, S(t) = 3\}$, where $S(t) = 0$ if the server is free/idle at time t, $S(t) = 1$ if the server is busy at time t, $S(t) = 2$ if the server is in vacation at time t and, $S(t) = 3$ if the server is closed down at time t.

Kolmogorov balanced equations of this model are given below the server is in idle:

$$\left.\begin{aligned}
&P'_{10}(t) = -[(F-1)\lambda + \sigma]P_{10}(t) + \mu P_{11}(t) + \alpha P_{12}(t)\\
&P'_{20}(t) = -[(F-2)\lambda + 2\sigma]P_{20}(t) + \mu P_{21}(t) + \alpha P_{22}(t)\\
&\ldots\\
&P'_{n0}(t) = -[(F-n)\lambda + n\sigma]P_{n0}(t) + \mu P_{n1}(t) + \alpha P_{n2}(t);\, n = 1, 2, \ldots, F-1\\
&P'_{F0}(t) = -[F\sigma]P_{F0}(t) + \alpha P_{F2}(t)
\end{aligned}\right\} \quad (1)$$

The server is in busy:

$$\left.\begin{aligned}
&P'_{01}(t) = -[(F-1)\lambda + \mu]P_{01}(t) + \sigma P_{10}(t)\\
&P'_{11}(t) = -[(F-2)\lambda + \mu]P_{11}(t) + (F-1)\lambda P_{10}(t) + (F-1)\lambda P_{01}(t) + 2\sigma P_{20}(t)\\
&P'_{21}(t) = -[(F-3)\lambda + \mu]P_{21}(t) + (F-2)\lambda P_{20}(t) + (F-2)\lambda P_{11}(t) + 3\sigma P_{30}(t)\\
&\ldots\\
&P'_{n1}(t) = -[(F-(n+1))\lambda + n\sigma]P_{n1}(t) + (F-n)\lambda P_{n0}(t) + (F_s - n)\lambda P_{n-11}(t)\\
&+ (n+1)\sigma P_{n+10}(t);\, n = 1, 2, \ldots, F-1
\end{aligned}\right\} \quad (2)$$

The server is in vacation:

$$\left.\begin{aligned}
&P'_{02}(t) = -[F\lambda]P_{02}(t) + \gamma P_{03}(t)\\
&P'_{12}(t) = -[(F-1)\lambda + \alpha]P_{12}(t) + F\lambda P_{02}(t)\\
&P'_{22}(t) = -[(F-2)\lambda + \alpha]P_{22}(t) + (F-1)\lambda P_{12}(t)\\
&\ldots\\
&P'_{n2}(t) = -[(F-n)\lambda + \alpha]P_{n2}(t) + (F-(n-))\lambda P_{n-12}(t);\, n = 1, 2, \ldots, F-1\\
&P'_{F2}(t) = -\alpha P_{F2}(t) + (F-(n-1))\lambda P_{F-12}(t)
\end{aligned}\right\} \quad (3)$$

The server is in closed down:

$$P'_{03}(t) = -\gamma P_{03}(t) + \mu P_{01}(t) \quad (4)$$

The infinitesimal generator matrix L for this model is given below

$$L = \begin{pmatrix}
L_{00} & L_{01} & L_{02} & L_{03} & L_{04} & \cdots & L_{0F}\\
L_{10} & L_{11} & L_{12} & L_{13} & L_{14} & \cdots & L_{1F}\\
L_{20} & L_{21} & L_{22} & L_{23} & L_{24} & \cdots & L_{2F}\\
L_{30} & L_{31} & L_{32} & L_{33} & L_{34} & \cdots & L_{3F}\\
. & . & . & . & . & \cdots & \cdots\\
L_{F-10} & L_{F-11} & L_{F-12} & L_{F-13} & L_{F-14} & \cdots & L_{F-1F}\\
L_{F0} & L_{F1} & L_{F2} & L_{F3} & L_{F4} & \cdots & L_{FF}
\end{pmatrix}$$

The matrices $L_{00}, L_{01}, L_{10}, L_{11}, L_{21}, L_{22}, L_{32}, L_{33}, \ldots, L_{F-1F-1}$ are described in the L.

The infinitesimal transition rates of process X as follows

$$L_{00} = \begin{bmatrix} -((F-1)\lambda + \mu) & 0 \\ 0 & -F\lambda \end{bmatrix}, L_{01} = \begin{bmatrix} 0 \\ \gamma \\ -\gamma \end{bmatrix}, L_{10} = [\mu \quad 0]$$

$$L_{ii} = \begin{bmatrix} -((F-i)\lambda + i\sigma) & \mu & \alpha \\ (F-i)\lambda & -((F-(i+1))\lambda + \mu) & 0 \\ 0 & 0 & -((F-i)\lambda + \alpha) \end{bmatrix} \text{ for}$$

$$i = 1, 2, \ldots, F$$

$$L_{i-1i} = \begin{bmatrix} 0 & 0 \\ (F-i)\lambda & 0 \\ 0 & (F-(i-1))\lambda \end{bmatrix} \text{ for } i = 1, 2, \ldots, F-1$$

$$L_{ii+1} = \begin{bmatrix} i\sigma & 0 \\ 0 & 0 \end{bmatrix} \text{ for } i = 1, 2, \ldots, F, \; L_{F0} = [-F\sigma \quad \alpha],$$

$$L_{F2} = [\lambda \quad 0 \quad -\alpha]$$

Remaining all other entries are zero.

The equations (1)-(4) can be combined and expressed as $X'(t) = LX(t)$, where $L = R^T$ and $[X(t)]^T = [P_{00}(t), P_{01}(t), P_{02}(t), P_{03}(t), P_{10}(t), P_{11}(t), P_{12}(t), \ldots,$ $P_{F-1\,0}(t), P_{F-11}(t), P_{F-1\,2}(t), P_{F0}(t), P_{F2}(t)]^T$

Solving the equations, we get, $X(t) = e^{Lt}X_0$

When $t = 0$, $X_0 = X(0) = [1\ 0\ 0\ 0\ 0 \ldots 0\ 0\ 0]^T$

We define,

$P_{n0}(t)$: Probability that the server is idle and there are n customers in the orbit at t.

$P_{n1}(t)$: Probability that the server is busy and there are n customers in the orbit at t.

$P_{n2}(t)$: Probability that the server is in vacation when there are n customers in the orbit at t.

$P_{03}(t)$: Probability that the server is in closed down when there are no customers in the orbit at t.

4. Description of Computational Method

The following effective computational procedure is used to find the time dependent probabilities of number of customers in the orbit at time t. The time dependent probabilities is denoted by $X(t) = [P_{00}(t), P_{01}(t), P_{02}(t), P_{03}(t), P_{10}(t), P_{11}(t), P_{12}(t), \ldots, P_{F-10}(t), P_{F-11}(t), P_{F-12}(t), P_{F0}(t), P_{F2}(t)]^T$

Step 1. Find the Eigen values of the matrix t_L.

Step 2. Let $d_1, d_2, d_3, \ldots, d_{3F+2}$ be $3F + 2$ Eigen values.

Step 3. Represent this Eigen values in the Vandermonde's matrix

$$V = \begin{pmatrix} 1 & d_1 & d_1^2 & . & \ldots & d_1^{3F+1} \\ 1 & d_2 & d_2^2 & . & \ldots & d_2^{3F+1} \\ 1 & d_3 & d_3^2 & . & \ldots & d_3^{3F+1} \\ . & . & . & . & \ldots & . \\ . & . & . & . & \ldots & . \\ 1 & d_{3F+2} & d_{3F+2}^2 & . & \ldots & d_{3F+2}^{3F+1} \end{pmatrix}$$

Step 4. Let $C = (e^{d_1}\ e^{d_2}\ \ldots\ e^{d_{3F+2}})^T$ and $C = (\alpha_0\ \alpha_1\ \ldots\ \alpha_{3F+1})^T$.

Step 5. Find $\alpha = V^{-1}C$ and we get $\alpha = (\alpha_0\ \alpha_1\ \alpha_2\ \ldots\ \alpha_{3F+1})^T$.

Step 6. Using α in $e^{tL} = \alpha_0 I + \alpha_1(tL) + \alpha_2(tL)^2 + \ldots + \alpha_{3F+1}(tL)^{3F+1}$.

Step 7. Extract the first column of e^{tL} and store in $X(t)$.

Step 8. This probability vector $X(t)$ provides time dependent probabilities of number of customers in the queue at time t.

5. System Performance Measures

The following system measures are used to bring out the Transient behaviour of the finite source retrial queueing model with exhaustive type vacation under computational study for various values of $F, \lambda, \mu, \sigma, \alpha, \gamma$ and t.

a. Probability that the server is idle at time $t = P_{idle}(t) = \sum_{n=0}^{F} P_{n0}(t)$

b. Probability that the server is busy at time $t = P_{busy}(t) = \sum_{n=0}^{F-1} P_{n1}(t)$

c. Probability that the server is in vacation at time $t = P_{vacation}(t) = \sum_{n=0}^{F} P_{n2}(t)$

d. Probability that the server is in closed down at time $t = P_{closeddown}(t) = P_{03}(t)$

e. $Lq(t) =$ Mean number of customers in the queue at time $t = \sum_{n=1}^{F} n(P_{n0}(t) + P_{n2}(t)) + \sum_{n=1}^{F-1} nP_{n1}(t).$

6. Numerical Computations

Transient Probabilities and System Performance Measures have been expressed in the form of Tables for Various Values $F, \lambda, \mu, \sigma, \alpha, \gamma$ and t.

Table 1. Transient probability distribution of no. of customers in the orbit when the server is idle in the system 7 for $\lambda = 4, \mu = 6, \alpha = 2, \sigma = 3, \gamma = 1, F = 5$ and various values of t.

t	$p_{10}(t)$	$p_{20}(t)$	$p_{30}(t)$	$p_{40}(t)$	$p_{50}(t)$
0.1	0.0744	0.0352	0.0083	0.0008	0.0000
0.2	0.0600	0.0658	0.0354	0.0077	0.0000
0.3	0.0369	0.0667	0.0579	0.0206	0.0001
0.4	0.0238	0.0593	0.0708	0.0348	0.0003
0.5	0.0167	0.0520	0.0776	0.0473	0.0007
0.6	0.0128	0.0465	0.0813	0.0575	0.0012

0.7	0.0105	0.0426	0.0837	0.0654	0.0018
0.8	0.0091	0.0400	0.0854	0.0715	0.0024
0.9	0.0082	0.0383	0.0869	0.0762	0.0028
1	0.0075	0.0372	0.0883	0.0800	0.0032

Table 2. Transient probability distribution of no. of customers in the orbit when the server is idle in the system for $\lambda = 5, \mu = 7, \alpha = 3, \sigma = 4, \gamma = 2, F = 7$ and various values of t.

t	$p_{10}(t)$	$p_{20}(t)$	$p_{30}(t)$	$p_{40}(t)$	$p_{50}(t)$	$p_{60}(t)$	$p_{70}(t)$
0.1	0.0429	0.0444	0.0272	0.0100	0.0021	0.0002	0.0000
0.2	0.0131	0.0314	0.0438	0.0372	0.0179	0.0038	0.0000
0.3	0.0045	0.0159	0.0350	0.0479	0.0379	0.0134	0.0001
0.4	0.0021	0.0085	0.0251	0.0471	0.0515	0.0255	0.0003
0.5	0.0013	0.0052	0.0185	0.0433	0.0593	0.0365	0.0007
0.6	0.0009	0.0036	0.0145	0.0399	0.0637	0.0452	0.0010
0.7	0.0007	0.0027	0.0120	0.0373	0.0664	0.0515	0.0013
0.8	0.0005	0.0022	0.0106	0.0357	0.0683	0.0561	0.0015
0.9	0.0004	0.0019	0.0097	0.0347	0.0696	0.0594	0.0016
1	0.0004	0.0016	0.0091	0.0341	0.0708	0.0617	0.0016

Table 3. Transient probability distribution of no. of customers in the orbit when the server is idle in the system for $\lambda = 6, \mu = 8, \alpha = 4, \sigma = 5, \gamma = 3, F = 9$ and various values of t.

t	$p_{10}(t)$	$p_{20}(t)$	$p_{30}(t)$	$p_{40}(t)$	$p_{50}(t)$	$p_{60}(t)$	$p_{7}(t)$	$p_{80}(t)$	$p_{90}(t)$
0.1	0.0151	0.0273	0.0311	0.0236	0.0121	0.0040	0.0008	0.0001	0.0000
0.2	0.0018	0.0063	0.0157	0.0271	0.0323	0.0255	0.0121	0.0026	0.0000
0.3	0.0006	0.0017	0.0055	0.0147	0.0284	0.0371	0.0297	0.0111	0.0001
0.4	0.0003	0.0007	0.0023	0.0077	0.0204	0.0372	0.0418	0.0219	0.0003

0.5	0.0002	0.0004	0.0012	0.0046	0.0149	0.0343	0.0482	0.0313	0.0005
0.6	0.0002	0.0003	0.0008	0.0031	0.0118	0.0316	0.0514	0.0381	0.0007
0.7	0.0001	0.0002	0.0006	0.0024	0.0100	0.0299	0.0533	0.0426	0.0008
0.8	0.0001	0.0002	0.0004	0.0020	0.0090	0.0289	0.0544	0.0455	0.0007
0.9	0.0001	0.0001	0.0003	0.0017	0.0084	0.0283	0.0552	0.0474	0.0007
1	0.0001	0.0001	0.0003	0.0016	0.0081	0.0280	0.0558	0.0485	0.0006

Table 4. Transient probability distribution of no. of customers in the orbit when the server is idle in the system for $\lambda = 7$, $\mu = 9$, $\alpha = 5$, $\sigma = 6$, $\gamma = 4$, $F = 10$ and various values of t.

t	$p_{10}(t)$	$p_{20}(t)$	$p_{30}(t)$	$p_{40}(t)$	$p_{50}(t)$	$p_{60}(t)$	$p_7(t)$	$p_{80}(t)$	$p_{90}(t)$	$p_{10}(t)$
0.1	0.0061	0.0149	0.0236	0.0258	0.0198	0.0106	0.0037	0.0008	0.0001	0.0000
0.2	0.0006	0.0017	0.0053	0.0126	0.0224	0.0286	0.0247	0.0131	0.0032	0.0000
0.3	0.0003	0.0005	0.0013	0.0040	0.0111	0.0230	0.0334	0.0302	0.0129	0.0001
0.4	0.0002	0.0003	0.0005	0.0016	0.0055	0.0157	0.0318	0.0400	0.0238	0.0003
0.5	0.0001	0.0002	0.0003	0.0009	0.0033	0.0115	0.0288	0.0445	0.0320	0.0005
0.6	0.0001	0.0001	0.0002	0.0005	0.0023	0.0093	0.0266	0.0466	0.0372	0.0005
0.7	0.0001	0.0001	0.0001	0.0004	0.0018	0.0081	0.0254	0.0477	0.0402	0.0005
0.8	0.0000	0.0001	0.0001	0.0003	0.0016	0.0075	0.0248	0.0483	0.0420	0.0004
0.9	0.0000	0.0000	0.0001	0.0002	0.0014	0.0072	0.0245	0.0488	0.0430	0.0004
1	0.0000	0.0000	0.0000	0.0002	0.0013	0.0071	0.0244	0.0491	0.0436	0.0003

Table 5. Transient probability distribution of no. of customers in the orbit when the server is busy in the system for $\lambda = 4$, $\mu = 6$, $\alpha = 2$, $\sigma = 3$, $\gamma = 1$, $F = 5$ and various values of t.

t	$p_{01}(t)$	$p_{11}(t)$	$p_{21}(t)$	$p_{31}(t)$	$p_{41}(t)$
0.1	0.1175	0.2655	0.1884	0.0592	0.0070
0.2	0.0212	0.1414	0.2137	0.1444	0.0379
0.3	0.0077	0.0832	0.1865	0.1898	0.0770
0.4	0.0042	0.0555	0.1603	0.2115	0.1121
0.5	0.0027	0.0407	0.1409	0.2229	0.1404

0.6	0.0020	0.0323	0.1276	0.2301	0.1624
0.7	0.0016	0.0272	0.1187	0.2355	0.1796
0.8	0.0013	0.0240	0.1130	0.2401	0.1933
0.9	0.0012	0.0220	0.1094	0.2444	0.2043
1	0.0011	0.0206	0.1074	0.2485	0.2133

Table 6. Transient probability distribution of no. of customers in the orbit when the server is busy in the system for $\lambda = 5, \mu = 7, \alpha = 3, \sigma = 4, \gamma = 2, F = 7$ and various values of t.

t	$p_{01}(t)$	$p_{11}(t)$	$p_{21}(t)$	$p_{31}(t)$	$p_{41}(t)$	$p_{51}(t)$	$p_{61}(t)$
0.1	0.0296	0.1493	0.2253	0.1799	0.0819	0.0202	0.0021
0.2	0.0027	0.0370	0.1152	0.1931	0.1908	0.1052	0.0251
0.3	0.0007	0.0125	0.0557	0.1399	0.2121	0.1834	0.0701
0.4	0.0003	0.0055	0.0302	0.0991	0.1997	0.2310	0.1188
0.5	0.0002	0.0031	0.0188	0.0740	0.1827	0.2579	0.1604
0.6	0.0001	0.0020	0.0131	0.0592	0.1690	0.2738	0.1925
0.7	0.0001	0.0015	0.0101	0.0504	0.1598	0.2843	0.2163
0.8	0.0001	0.0012	0.0083	0.0450	0.1541	0.2919	0.2336
0.9	0.0000	0.0010	0.0072	0.0417	0.1508	0.2979	0.2462
1	0.0000	0.0008	0.0065	0.0397	0.1491	0.3029	0.2554

Table 7. Transient probability distribution of no. of customers in the orbit when the server is busy in the system for $\lambda = 6, \mu = 8, \alpha = 4, \sigma = 5, \gamma = 3, F = 9$ and various values of t.

t	$p_{01}(t)$	$p_{11}(t)$	$p_{21}(t)$	$p_{31}(t)$	$p_{41}(t)$	$p_{51}(t)$	$p_{61}(t)$	$p_{71}(t)$	$p_{81}(t)$
0.1	0.0057	0.0541	0.1381	0.1996	0.1841	0.1109	0.0425	0.0094	0.0009
0.2	0.0003	0.0053	0.0259	0.0764	0.1504	0.2007	0.1756	0.0914	0.0216
0.3	0.0001	0.0012	0.0063	0.0257	0.0765	0.1601	0.2244	0.1906	0.0746

0.4	0.0000	0.0006	0.0024	0.0106	0.0404	0.1136	0.2173	0.2529	0.1358
0.5	0.0000	0.0004	0.0013	0.0056	0.0245	0.0843	0.1995	0.2854	0.1865
0.6	0.0000	0.0003	0.0009	0.0036	0.0171	0.0679	0.1853	0.3027	0.2226
0.7	0.0000	0.0002	0.0006	0.0026	0.0133	0.0587	0.1763	0.3128	0.2467
0.8	0.0000	0.0001	0.0005	0.0020	0.0112	0.0535	0.1712	0.3195	0.2623
0.9	0.0000	0.0001	0.0003	0.0016	0.0100	0.0505	0.1685	0.3242	0.2724
1	0.0000	0.0001	0.0003	0.0014	0.0093	0.0488	0.1672	0.3277	0.2790

Table 8. Transient probability distribution of no. of customers in the orbit when the server is busy in the system for $\lambda = 7, \mu = 9, \alpha = 5, \sigma = 6, \gamma = 4, F = 10$ and various values of t.

t	$p_{01}(t)$	$p_{11}(t)$	$p_{21}(t)$	$p_{31}(t)$	$p_{41}(t)$	$p_{51}(t)$	$p_{61}(t)$	$p_{71}(t)$	$p_{81}(t)$	$p_{91}(t)$
0.1	0.0016	0.0220	0.0750	0.1483	0.1939	0.1736	0.1060	0.0424	0.0100	0.0011
0.2	0.0001	0.0013	0.0071	0.0269	0.0736	0.1450	0.2021	0.1902	0.1088	0.0287
0.3	0.0000	0.0004	0.0015	0.0060	0.0226	0.0682	0.1516	0.2323	0.2198	0.0972
0.4	0.0000	0.0002	0.0007	0.0023	0.0091	0.0346	0.1036	0.2167	0.2800	0.1685
0.5	0.0000	0.0002	0.0004	0.0012	0.0048	0.0212	0.0770	0.1967	0.3075	0.2207
0.6	0.0000	0.0001	0.0003	0.0008	0.0031	0.0152	0.0634	0.1836	0.3208	0.2538
0.7	0.0000	0.0001	0.0002	0.0005	0.0023	0.0123	0.0564	0.1764	0.3282	0.2734
0.8	0.0000	0.0000	0.0001	0.0004	0.0018	0.0108	0.0528	0.1728	0.3329	0.2849
0.9	0.0000	0.0000	0.0001	0.0003	0.0015	0.0100	0.0508	0.1711	0.3360	0.2915
1	0.0000	0.0000	0.0001	0.0002	0.0013	0.0095	0.0498	0.1704	0.3381	0.2954

Table 9. Transient probability distribution of no. of customers in the orbit when the server is in vacation in the system for $\lambda = 4, \mu = 6, \alpha = 2, \sigma = 3, \gamma = 1, F = 5$ and various values of t.

t	$p_{02}(t)$	$p_{12}(t)$	$p_{22}(t)$	$p_{32}(t)$	$p_{42}(t)$	$p_{52}(t)$
0.1	0.0081	0.0047	0.0019	0.0005	0.0001	0.0000
0.2	0.0114	0.0106	0.0080	0.0042	0.0014	0.0002
0.3	0.0114	0.0124	0.0123	0.0097	0.0051	0.0013
0.4	0.0106	0.0122	0.0136	0.0136	0.0101	0.0039
0.5	0.0097	0.0113	0.0134	0.0153	0.0144	0.0077

0.6	0.0089	0.0104	0.0126	0.0155	0.0172	0.0121
0.7	0.0081	0.0095	0.0116	0.0148	0.0185	0.0164
0.8	0.0074	0.0086	0.0106	0.0138	0.0187	0.0202
0.9	0.0067	0.0079	0.0096	0.0127	0.0182	0.0233
1	0.0061	0.0072	0.0088	0.0116	0.0173	0.0255

Table 10. Transient probability distribution of no. of customers in the orbit when the server is vacation in the system for $\lambda = 5, \mu = 7, \alpha = 3, \sigma = 4, \gamma = 2, F = 7$ and various values of t.

t	$p_{02}(t)$	$p_{12}(t)$	$p_{22}(t)$	$p_{32}(t)$	$p_{42}(t)$	$p_{52}(t)$	$p_{62}(t)$	$p_{72}(t)$
0.1	0.0085	0.0072	0.0048	0.0025	0.0009	0.0002	0.0000	0.0000
0.2	0.0082	0.0090	0.0093	0.0085	0.0063	0.0034	0.0012	0.0002
0.3	0.0069	0.0077	0.0087	0.0097	0.0099	0.0083	0.0049	0.0014
0.4	0.0056	0.0064	0.0073	0.0086	0.0100	0.0108	0.0091	0.0041
0.5	0.0046	0.0052	0.0060	0.0071	0.0088	0.0108	0.0116	0.0076
0.6	0.0038	0.0043	0.0049	0.0059	0.0073	0.0095	0.0122	0.0109
0.7	0.0031	0.0035	0.0041	0.0048	0.0060	0.0081	0.0115	0.0132
0.8	0.0026	0.0029	0.0033	0.0040	0.0049	0.0067	0.0102	0.0144
0.9	0.0021	0.0024	0.0027	0.0032	0.0040	0.0055	0.0087	0.0147
1	0.0017	0.0019	0.0022	0.0027	0.0033	0.0045	0.0073	0.0144

Table 11. Transient probability distribution of no. of customers in the orbit when the server is vacation in the system for $\lambda = 6, \mu = 8, \alpha = 4, \sigma = 5, \gamma = 3, F = 9$ and various values of t.

t	$p_{02}(t)$	$p_{12}(t)$	$p_{22}(t)$	$p_{32}(t)$	$p_{42}(t)$	$p_{52}(t)$	$p_{62}(t)$	$p_{72}(t)$	$p_{82}(t)$	$p_{92}(t)$
0.1	0.0064	0.0064	0.0058	0.0045	0.0028	0.0013	0.0005	0.0001	0.0000	0.0000
0.2	0.0050	0.0055	0.0060	0.0065	0.0068	0.0062	0.0047	0.0027	0.0010	0.0002
0.3	0.0037	0.0041	0.0045	0.0051	0.0059	0.0067	0.0071	0.0065	0.0042	0.0014

0.4	0.0027	0.0030	0.0034	0.0038	0.0044	0.0053	0.0064	0.0074	0.0070	0.0038
0.5	0.0020	0.0022	0.0025	0.0028	0.0033	0.0039	0.0049	0.0064	0.0078	0.0063
0.6	0.0015	0.0017	0.0018	0.0021	0.0024	0.0029	0.0037	0.0050	0.0071	0.0080
0.7	0.0011	0.0012	0.0014	0.0016	0.0018	0.0022	0.0027	0.0038	0.0058	0.0085
0.8	0.0008	0.0009	0.0010	0.0012	0.0013	0.0016	0.0020	0.0028	0.0046	0.0083
0.9	0.0006	0.0007	0.0008	0.0009	0.0010	0.0012	0.0015	0.0021	0.0035	0.0075
1	0.0005	0.0005	0.0006	0.0006	0.0007	0.0009	0.0011	0.0015	0.0026	0.0065

Table 12. Transient probability distribution of no. of customers in the orbit when the server is vacation in the system for $\lambda = 7, \mu = 9, \alpha = 5, \sigma = 6, \gamma = 4, F = 10$ and various values of t.

s.no.	$p_{02}(t)$	$p_{12}(t)$	$p_{22}(t)$	$p_{32}(t)$	$p_{42}(t)$	$p_{52}(t)$	$p_{62}(t)$	$p_{72}(t)$	$p_{82}(t)$	$p_{92}(t)$
0.1	0.0054	0.0057	0.0057	0.0053	0.0042	0.0027	0.0014	0.0005	0.0001	0.0000
0.2	0.0037	0.0040	0.0044	0.0049	0.0054	0.0058	0.0058	0.0048	0.0031	0.0013
0.3	0.0025	0.0027	0.0030	0.0033	0.0038	0.0044	0.0051	0.0059	0.0060	0.0045
0.4	0.0016	0.0018	0.0020	0.0022	0.0025	0.0030	0.0036	0.0044	0.0056	0.0062
0.5	0.0011	0.0012	0.0013	0.0015	0.0017	0.0020	0.0024	0.0030	0.0041	0.0057
0.6	0.0007	0.0008	0.0009	0.0010	0.0011	0.0013	0.0016	0.0020	0.0028	0.0044
0.7	0.0005	0.0005	0.0006	0.0007	0.0008	0.0009	0.0011	0.0014	0.0019	0.0032
0.8	0.0003	0.0004	0.0004	0.0005	0.0005	0.0006	0.0007	0.0009	0.0013	0.0022
0.9	0.0002	0.0002	0.0003	0.0003	0.0003	0.0004	0.0005	0.0006	0.0009	0.0015
1	0.0001	0.0002	0.0002	0.0002	0.0002	0.0003	0.0003	0.0004	0.0006	0.0010

Table 13. Transient probability distribution of no. of customers in the orbit when the server is p_n closed down in the system for $\lambda = 4, \mu = 6, \alpha = 2, \sigma = 3, \gamma = 1, F = 5$ and various values of t.

t	$p_{03}(t)$
0.1	0.2284
0.2	0.2367
0.3	0.2214
0.4	0.2035
0.5	0.1861

0.6	0.1697
0.7	0.1545
0.8	0.1407
0.9	0.1280
1	0.1164

Table 14. Transient probability distribution of no. of customers in the orbit when the server is closed down in the system for $\lambda = 5, \mu = 7, \alpha = 3, \sigma = 4, \gamma = 2, F = 7$ and various values of t.

t	$p_{03}(t)$
0.1	0.1606
0.2	0.1377
0.3	0.1136
0.4	0.0932
0.5	0.0765
0.6	0.0627
0.7	0.0514
0.8	0.0421
0.9	0.0345
1	0.0283

Table 15. Transient probability distribution of no. of customers in the orbit when the server is closed down in the system for $\lambda = 6, \mu = 8, \alpha = 4, \sigma = 5, \gamma = 3, F = 9$ and various values of t.

t	$p_{03}(t)$
0.1	0.1127
0.2	0.0844

0.3	0.0626
0.4	0.0464
0.5	0.0344
0.6	0.0255
0.7	0.0189
0.8	0.0140
0.9	0.0104
1	0.0077

Table 16. Transient probability distribution of no. of customers in the orbit when the server is closed down in the system for $\lambda = 7$, $\mu = 9$, $\alpha = 5$, $\sigma = 6$, $\gamma = 4$, $F = 10$ and various values of t.

t	$p_{03}(t)$
0.1	0.0898
0.2	0.0604
0.3	0.0405
0.4	0.0272
0.5	0.0182
0.6	0.0122
0.7	0.0082
0.8	0.0055
0.9	0.0037
1	0.0025

Table 17. Time dependent system measures for $\lambda = 4$, $\mu = 6$, $\alpha = 2$, $\sigma = 3$, $\gamma = 1$, $F = 5$ and various values of t.

t	pidle	pbusy	pvac	pcld	Lq
0.1	0.1188	0.6376	0.0152	0.2284	1.0065
0.2	0.1688	0.5586	0.0358	0.2367	1.0968
0.3	0.1822	0.5442	0.0522	0.2214	1.3538
0.4	0.1890	0.5435	0.0640	0.2035	1.5647
0.5	0.1944	0.5477	0.0718	0.1861	1.7298
0.6	0.1994	0.5543	0.0766	0.1697	1.8599
0.7	0.2040	0.5626	0.0789	0.1545	1.9634
0.8	0.2083	0.5717	0.0793	0.1407	2.0463
0.9	0.2124	0.5812	0.0784	0.1280	2.1130
1	0.2163	0.5909	0.0764	0.1164	2.1670

Table 18. Time dependent system measures for $\lambda = 5$, $\mu = 7$, $\alpha = 3$, $\sigma = 4$, $\gamma = 2$, $F = 7$ and various values of t.

T	pidle	pbusy	pvac	pcld	Lq
0.1	0.1268	0.6884	0.0242	0.1606	1.4294
0.2	0.1473	0.6691	0.0460	0.1377	2.2201
0.3	0.1545	0.6743	0.0576	0.1136	2.8298
0.4	0.1601	0.6846	0.0620	0.0932	3.2490
0.5	0.1648	0.6970	0.0618	0.0765	3.5389
0.6	0.1687	0.7098	0.0588	0.0627	3.7412
0.7	0.1720	0.7224	0.0542	0.0514	3.8835
0.8	0.1748	0.7341	0.0489	0.0421	3.9841
0.9	0.1772	0.7449	0.0434	0.0345	4.0558
1	0.1792	0.7545	0.0380	0.0283	4.1071

Table 19. Time dependent system measures for $\lambda = 6, \mu = 8, \alpha = 4, \sigma = 5, \gamma = 3, F = 9$ and various values of t.

t	pidle	pbusy	pvac	pcld	Lq
0.1	0.1141	0.7453	0.0279	0.1127	2.2848
0.2	0.1234	0.7475	0.0446	0.0844	3.7549
0.3	0.1287	0.7595	0.0492	0.0626	4.6620
0.4	0.1327	0.7736	0.0472	0.0464	5.2168
0.5	0.1358	0.7876	0.0423	0.0344	5.5598
0.6	0.1381	0.8002	0.0362	0.0255	5.7740
0.7	0.1398	0.8112	0.0301	0.0189	5.9088
0.8	0.1412	0.8203	0.0245	0.0140	5.9943
0.9	0.1422	0.8278	0.0196	0.0104	6.0490
1	0.1430	0.8338	0.0155	0.0077	6.0843

Table 20. Time dependent system measures for $\lambda = 7, \mu = 9, \alpha = 5, \sigma = 6, \gamma = 4, F = 10$ and various values of t.

t	pidle	pbusy	pvac	pcld	Lq
0.1	0.1054	0.7739	0.0310	0.0898	3.0297
0.2	0.1123	0.7838	0.0435	0.0604	4.8456
0.3	0.1167	0.7998	0.0430	0.0405	5.8480
0.4	0.1198	0.8158	0.0372	0.0272	6.4047
0.5	0.1219	0.8298	0.0300	0.0182	6.7170
0.6	0.1234	0.8411	0.0232	0.0122	6.8934
0.7	0.1245	0.8499	0.0174	0.0082	6.9937
0.8	0.1252	0.8565	0.0128	0.0055	7.0512
0.9	0.1257	0.8613	0.0093	0.0037	7.0843
1	0.1260	0.8649	0.0066	0.0025	7.1036

7. Results

Table 1, Table 2, Table 3 and Table 4 show Transient probabilities of no. of customers in the orbit when the server is idle in the system for various values of $F, \lambda, \mu, \sigma, \alpha, \gamma$ and t.

Table 5, Table 6, Table 7 and Table 8 show Transient probabilities of no. of customers in the orbit when the server is in busy for various values of $F, \lambda, \mu, \sigma, \alpha, \gamma$ and t.

Table 9, Table 10, Table 11 and Table 12 show Transient probabilities of no. of customers in the orbit when the server is vacation in the system for various values of $F, \lambda, \mu, \sigma, \alpha, \gamma$ and t.

Table 13, Table 14, Table 15 and Table 16 show Transient probabilities of no. of customers in the orbit when the server is in closed down in the system for various values of $F, \lambda, \mu, \sigma, \alpha, \gamma$ and t.

Table 17, Table 18, Table 19 and Table 20 show Time dependent System performance measures for various values of $F, \lambda, \mu, \sigma, \alpha, \gamma$ and t.

We infer:

$P_{busy}(t)$ increases as arrival rate λ increases for all values of t.

$P_{vacation}(t)$ decreases as arrival rate λ increases for all values of t.

$P_{closeddown}(t)$ decreases as arrival rate λ increases for all values of t.

The value of t increases and for various values of $F, \lambda, \mu, \sigma, \alpha, \gamma$ and $P_{idle}(t) \to P_{idle}$, $P_{busy}(t) \to P_{busy}$, $P_{vacation}(t) \to P_{vacation}$, $P_{closeddown} \to P_{closeddown}(t)$, $L_q(t) \to L_q$.

8. Conclusion

In this paper, a new computational approach was used to evaluate the transient solution of finite source Retrial queuing model with multiple vacations and closed down using Cayley Hamilton method and infinitesimal generator matrix. System performance measures and time dependent probability distributions are determined.

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