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GENERALIZED LOGISTIC DISTRIBUTIONS BASED ON VARIOUS METHODS FOR CONSTRUCTING THE STANDARD LOGISTIC DISTRIBUTION

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Abstract

The objective of this paper is to identify generalizations of logistic distribution through various methods for constructing the standard logistic distribution.

Methods considered are based on transformations, Burr differential equation, mixtures and the difference of two distributed independent Gumbel random variables.

For each method, distributions used are generalized. The generalized logistic distributions identified by Johnson [6] of Types I, II, III and IV, have their extended versions obtained. A new generalization namely "extended standard logistic distribution" has been introduced.

1. Introduction

Several different forms of generalizations of logistic distribution have been proposed in literature.

The question is: Is there any pattern one can follow to identify these generalizations? The answer to this question, leads to the objective of this paper which is to identify generalized logistic distributions through various methods for constructing the standard logistic distribution.

Methods considered are based on transformations, discussed in section 2, Burr differential equation in section 3, mixtures in section 4 and the difference of two Gumbel random variables in section 5. For each method, distributions used are generalized. Concluding remarks are in section 6.

The notations used are GLI, GLII, GLIII and GLIV to represent generalized logistic distributions of type I, II, III and IV respectively.

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2. Generalized Logistic Distributions Based on Transformations

Let

$$X = \phi(Y), \tag{2.1}$$

where X is the new variable and Y is the old or parent variable.

$$\therefore\ Y = \phi^{-1}(X) \tag{2.2}$$

assuming the inverse exists.

If $f(x)$ and $g(y)$ are the probability density function (pdf) of X and Y respectively, then by the change of variable technique,

$$f(x) = g(\phi^{-1})(X) \left| \frac{dy}{dx} \right|. \tag{2.3}$$

We now apply the transformation formula (2.3) for the case when the old variable is uniform, Laplace Pareto I.

2.1. When the parent variable is uniform or exponential

Let

$$X = \ln\left(\frac{1-U}{U}\right), \tag{2.4}$$

where U is uniform [0, 1].

The pdf of X is

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2},\ -\infty < x < \infty, \tag{2.5}$$

which is standard logistic distribution. It can be generalized by letting

a.

$$X = \ln\left(\frac{U^{\frac{1}{p}}}{1-U^{\frac{1}{p}}}\right), \tag{2.6}$$

where U is uniform [0, 1] and $p > 0$.

Then

$$f(x) = \frac{pe^{-px}}{(1+e^{-x})^{p+1}}\ -\infty < x < \infty,\ p > 0 \tag{2.7}$$

which is generalized logistic type II (GLII) as obtained by Balakrishnan and Leung [1].

b.

$$X = -\ln\left(\frac{\lambda U^{\frac{1}{p}}}{1 - U^{\frac{1}{p}}}\right), \tag{2.8}$$

where U is uniform $[0, 1]$, $\lambda > 0;\ p > 0$.

Then the pdf of X becomes

$$f(x) = \frac{p\lambda e^{-px}}{(1 + e^{-x})^{p+1}}\ -\infty < x < \infty,\ \lambda,\ p > 0. \tag{2.9}$$

This is extended GLII, according to Olapade [10].

c.

$$X = \ln\left(\frac{1 - Y}{Y}\right), \tag{2.10}$$

where Y is beta 1 with parameter a and b.

Then

$$f(x) = \frac{1}{\beta(a,\ b)} \frac{e^{-ax}}{(1 + e^{-x})^{a+b}} \text{ for } -\infty < x < \infty,\ \lambda > 0,\ a > 0,\ b > 0 \tag{2.11}$$

which Y is GLIV distribution as obtained by Prentice [11].

Remark. Uniform distribution is a special case of a beta 1 distribution with parameters a and b.

2.2. When the parent variable is Laplacian

Let

$$X = \ln\left(\frac{1 - \frac{1}{2}e^{-Y}}{\frac{1}{2}e^{-Y}}\right),$$

$$= \ln(2e^{Y} - 1), \tag{2.12}$$

where Y is standard Laplace; i.e.

$$g(y) = \frac{1}{2}e^{-|y|},\ y > 0 \tag{2.13}$$

then X is standard logistic distribution, which can be extended by letting

$$X = \ln\left(\frac{1 - \frac{1}{2}e^{-Y}}{\frac{\lambda}{2}e^{-Y}}\right), \tag{2.14}$$

where Y is standard Laplace.

Then

$$f(x) = \frac{\lambda e^{-x}}{(\lambda + e^{-x})}, \; -\infty < x < \infty, \; \lambda > 0 \tag{2.15}$$

we introduce this as the, extended standard logistic distribution.

2.3. When the old variable is Pareto I

Let

$$X = -\ln\left(\frac{Y}{\beta} - 1\right), \tag{2.16}$$

where Y is Pareto I distributed with parameter α and β; i.e.,

$$g(y) = \frac{\alpha\beta^{\alpha}}{y^{\alpha+1}} \; y > \beta > 0; \; \alpha > 0. \tag{2.17}$$

Then

$$f(x) = \frac{\alpha e^{-x}}{(1 + e^{-x})^{\alpha+1}} \; -\infty < x < \infty, \; \alpha > 0 \tag{2.18}$$

which is GLI as obtained by Blakrishnan and Leung [1].

When $\alpha = 1$, we have the standard logistic.

3. Generalized Logistic Distribution Based on Burr Differential Equation

Burr [3] introduced a system of distributions based on a differential equation of the form.

$$y' = y(1 - y)\,g(x, y) \tag{3.1}$$

where $y = F(x)$, a cdf of a continuous random variable. $y' = f(x)$, a pdf

$g(x, y)$ is a non negative function of x and y.

When

$$g(x, y) = g(x) \tag{3.2}$$

then 3.1 becomes

$$y' = y(1 - y)g(x) \tag{3.3}$$

$$\therefore \int \frac{dy}{y(1 - y)} = \int g(x)dx \tag{3.4}$$

which boils down to

$$y = F(x) = [e^{-\int g(x)dx} + 1]^{-1} \tag{3.5}$$

which was obtained by Burr [3].

If

$$g(x) = 1. \tag{3.6}$$

Then

$$F(x) = \frac{1}{1 + e^{-x}}, \; -\infty < x < \infty \tag{3.7}$$

which is the cdf of a standard logistic distribution.

Furthermore, Burr [3] considered the power of the cdf $F(x)$, i.e.

$$G(x) = [F(x)]^{\alpha}$$

$$= (1 + e^{-x})^{-\alpha}, \; -\infty < x < \infty.$$

The corresponding pdf is

$$g(x) = \frac{\alpha e^{-x}}{(1 + e^{-x})^{\alpha+1}} \; -\infty < x < \infty, \; \alpha > 0 \tag{3.8}$$

which is GLI according to Johnson, Kotz and Balakrishana [6].

It is also called exponentiated logistic distribution. According to Burr system of distributions, it is Burr II distribution.

4. Generalized Logistic Distributions Based on Gumbel Mixture

A continuous mixture is defined as

$$f(x) = \int_{-\infty}^{\infty} f(x/\lambda)g(\lambda)d\lambda \tag{4.1}$$

where $f(x/\lambda)$ is a conditional pdf or pmf; $g(x)$ is a continuous mixing distribution and $f(x)$ is the mixed distribution or mixture

We are going to look at Gumbel mixtures.

4.1. Gumbel I Distribution and its Mixtures

Let

$$Y = e^{-X},$$

where Y is standard exponential.

Then the pdf of X is

$$f(x) = e^{-x} \exp(-e^{-x}), -\infty < x < \infty \tag{4.2}$$

which is standard Gumbel distribution or type I extreme value distribution. Since there is another form of Gumbel distribution discussed in subsection 4.3, we shall call this one standard Gumbel I distribution.

The standard Gumbel I distribution has no parameter.

To introduce a parameter, we suppose

$$X = -\ln Y$$

where Y is exponential with a varying parameter λ.

Then,

$$f(x/\lambda) = \lambda e^{-x} \exp(-\lambda e^{-x}), -\infty < x < \infty, \lambda > 0 \tag{4.3}$$

and Gumbel I mixture becomes.

$$f(x) = \int_0^\infty \lambda e^{-x}\{\exp(-\lambda e^{-x}) g(\lambda)\} d\lambda. \tag{4.4}$$

Let us consider the following cases of $g(\lambda)$

(i) When

$$g(\lambda) = e^{-\lambda}, \lambda > 0 \tag{4.5}$$

which is a standard exponential distribution. Then $f(\lambda)$ is a standard logistic distribution. Generalizing the standard logistic distribution to gamma distribution with we have;

(ii)

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda}, \ \lambda > 0; \ \alpha > 0 \tag{4.6}$$

which is a gamma mixing distribution with parameter α.

The mixture becomes,

$$f(x) = \frac{\alpha e^{-x}}{(1 + e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty, \ \alpha > 0 \tag{4.7}$$

which is GLI distribution.

(iii)

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1}, \ \lambda > 0; \ \alpha, \beta > 0 \tag{4.8}$$

which is a gamma mixing distribution with parameter α and β.

$$\therefore \ f(x) = \frac{\alpha}{\beta} \frac{e^{-x}}{\left(1 + \frac{1}{\beta} e^{-x}\right)^{\alpha+1}}$$

$$= \frac{\alpha p e^{-x}}{\left(1 + \frac{1}{\beta} e^{-x}\right)^{\alpha+1}}, \ -\infty < x < \infty, \ \alpha > 0, \ p = \frac{1}{\beta} > 0. \tag{4.9}$$

This is extended type I generalized logistic distribution as given by Olapade [10].

4.2. Generalized Gumbel I Distribution and its Mixtures.

Let

$$Y = e^{-X},$$

where Y is a gamma distribution with two parameters α fixed and λ varying;

$$\therefore \ f(x/\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\alpha x} \exp\ (-\lambda e^{-x}) x > 0; \ \alpha > 0; \ \lambda > 0 \tag{4.10}$$

which is generalized Gumbel I distribution. For the generalized conditional Gumbel I distribution, we have;

$$f(x/\lambda) = \int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\alpha x} \exp\ (-\lambda e^{-x}) g(\lambda) d\lambda. \tag{4.11}$$

Let us consider the following cases

(i) $g(\lambda) = e^{-\lambda}, \lambda > 0.$

Then

$$f(x) = \frac{\alpha e^{-\alpha x}}{(1 + e^{-x})^{\alpha+1}}, -\infty < x < \infty; \alpha > 0 \tag{4.12}$$

which is GLII distribution.

(ii)

$$g(\lambda) = \beta e^{-\beta\lambda}, \lambda > 0; \beta > 0 \tag{4.13}$$

$$\therefore f(x) = \frac{\alpha\beta e^{-\alpha x}}{(\beta + e^{-x})^{\alpha+1}} -\infty < x < \infty; \alpha > 0; \beta > 0. \tag{4.14}$$

This is extended GLII distribution.

(iii)

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda}, \lambda > 0; \alpha > 0. \tag{4.15}$$

Then

$$f(x) = \frac{e^{-\alpha x}}{\beta(\alpha, \alpha)(1 + e^{-x})^{2\alpha}}, -\infty < x < \infty; \alpha > 0. \tag{4.16}$$

This is GLIII distribution as introduced by Johnson et al [6].

(iv)

$$g(\lambda) = \frac{\lambda^{\beta-1}}{\Gamma(\beta)} e^{-\lambda}, \lambda > 0; \beta > 0$$

$$\therefore f(x) = \frac{e^{-\alpha x}}{B(\alpha, \beta)(1 + e^{-x})^{\alpha+\beta}} \tag{4.17}$$

which is GLIV

(v)

$$g(\lambda) = \frac{\phi^{\beta}}{\Gamma(\beta)} e^{-\phi\lambda} \lambda^{\beta-1}, \lambda > 0; \phi > 0, \beta > 0$$

$$\therefore f(x) = \frac{\phi^{\beta}}{\beta(\alpha, \beta)} \frac{e^{-\alpha x}}{(\phi + e^{-x})^{\alpha+\beta}} \text{ for } -\infty < x < \infty; \alpha, \beta, \phi < 0 \tag{4.18}$$

Which is extended GLIV according to Morais et al. [8].

4.3. Gumbel II distribution and its mixtures

Another form of Gumbel distribution is given by letting

$$Y = e^{X},$$

where Y is a standard exponential.

Then the pdf of X is

$$f(x) = e^{x} \exp(-e^{x}), \; -\infty < x < \infty \tag{4.19}$$

we shall call this pdf, standard Gumbel II. To introduce a parameter, we suppose

$$Y = e^{X},$$

where Y is exponential with a varying parameter λ.

$$\therefore \; f(x/\lambda) = \lambda e^{x} \exp(-\lambda e^{x}), \; -\infty < x < \infty, \; \lambda > 0 \tag{4.20}$$

and Gumbel II mixtures is

$$f(x) = \int_0^\infty \lambda e^{x} \exp(-\lambda e^{x}) g(\lambda) d\lambda. \tag{4.21}$$

Let us consider the following cases of $g(\lambda)$:

(i)

$$g(\lambda) = \frac{e^{-x}}{(1 + e^{-x})^2}, \; -\infty < x < \infty \tag{4.22}$$

a standard logistic distribution.

(ii)

$$g(\lambda) = \frac{\lambda^{\alpha - 1}}{\Gamma(\alpha)} e^{-\lambda}, \; \lambda > 0, \; \alpha < 0 \tag{4.23}$$

implies that

$$f(x) = \frac{\alpha e^{-\alpha x}}{(1 + e^{-x})^{\alpha + 1}}, \; -\infty < x < \infty; \; \alpha > 0 \tag{4.24}$$

which is GLII

(iii)

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\lambda}\lambda^{\alpha-1}, \lambda > 0, \alpha, \beta > 0 \tag{4.25}$$

implies that

$$f(x) = \frac{\alpha p e^{x}}{(1 + pe^{x})^{\alpha+1}}, -\infty < x < \infty; \alpha > 0, p = \frac{1}{\beta}. \tag{4.26}$$

This can be expressed further as.

$$f(x) = \frac{\alpha p e^{-\alpha x}}{(p + e^{-x})^{\alpha+1}}, -\infty < x < \infty; \alpha > 0, p > 0 \tag{4.27}$$

which is extended GLII distribution.

4.4. Generalized Gumbel II Distribution and its Mixtures

Let

$$Y = e^{X},$$

where Y is gamma distributed with parameters α and λ; i.e.

$$\therefore f(x/\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{\alpha x} \exp(-\lambda e^{x}) - \infty; x < \infty; \alpha > 0; \lambda > 0 \tag{4.28}$$

which is generalized Gumbel II mixture. For generalized Gumbel II mixture we have;

$$f(x) = \int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{\alpha x} \exp(-\lambda e^{x}) g(\lambda) d\lambda. \tag{4.29}$$

Let us consider the following cases of $g(\lambda)$

(i) When

$$g(\lambda) = e^{-\lambda}, \lambda > 0.$$

Then

$$f(x) = \frac{\alpha e^{\alpha x}}{(1 + e^{x})^{\alpha+1}}$$

$$= \frac{\alpha e^{-x}}{(1 + e^{-x})^{\alpha+1}}, -\infty < x < \infty; \alpha > 0 \tag{4.30}$$

which is GLI distribution.

(ii) For

$$g(\lambda) = \beta e^{-\beta\lambda}, \ \lambda > 0; \ \beta > 0. \tag{4.31}$$

$$f(x) = \frac{\alpha\beta e^{\alpha x}}{(\beta + e^{x})^{\alpha+1}}$$

$$= \frac{\alpha\beta\, e^{-x}}{(1 + \beta e^{-x})^{\alpha+1}} \tag{4.32}$$

which is extended GLI distribution.

(iii) For

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda}, \ \lambda > 0, \ \alpha > 0 \tag{4.33}$$

$$f(x) = \frac{1}{\beta(\alpha, \alpha)} \frac{e^{\alpha x}}{(1 + e^{x})^{2\alpha}} \tag{4.34}$$

$$= \frac{1}{\beta(\alpha, \alpha)} \frac{e^{-\alpha x}}{(1 + e^{-x})} \text{ for } -\infty < x < \infty; \ \alpha > 0; \ \beta > 0. \tag{4.35}$$

This is GLIII distribution as obtained by Johnson et al. [6].

(iv) For

$$g(\lambda) = \frac{\lambda^{\beta-1}}{\Gamma(\beta)} e^{-\lambda}, \ \lambda > 0, \ \beta > 0 \tag{4.36}$$

$$f(x) = \frac{1}{B(\alpha, \beta)} \frac{e^{\alpha x}}{(1 + e^{x})^{\alpha+\beta}}$$

$$= \frac{1}{B(\alpha, \beta)} \frac{e^{-\beta x}}{(1 + e^{-x})^{\alpha+\beta}} \text{ for}$$

$$-\infty < x < \infty; \ \alpha > 0; \ \beta > 0 \tag{4.37}$$

this is GLIV distribution as obtained by Prentice [11].

(v) For

$$g(\lambda) = \frac{\phi}{\Gamma(\beta)} e^{-\phi\lambda} \lambda^{\beta-1}, \ \lambda > 0; \ \phi, \ \beta > 0 \tag{4.38}$$

$$f(x) = \frac{\phi^{\beta}}{B(\alpha, \beta)} \frac{e^{\alpha x}}{(\phi + e^{x})^{\alpha+\beta}}$$

$$= \frac{\phi^{\beta}}{B(\alpha, \beta)} \frac{e^{-\beta x}}{(1 + \phi e^{-x})^{\alpha+\beta}} \tag{4.39}$$

which is extended GLIV.

4.5. Lev-Sev mixture

Villa and Escobar [12] call Gumbel I, largest extreme value (LEV) and Gumbel II to be smallest extreme value (SEV).

They introduced the location and scale parameters. Thus the LEV distribution is given by

$$f_{Lev}(x/\mu, \sigma) = \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} \exp\left(-e^{-\frac{(x-\mu)}{\sigma}}\right)$$

$$\text{for } -\infty < x < \infty;\ -\infty < \mu < \infty;\ \sigma > 0. \tag{4.40}$$

The Sev distribution is given by

$$f_{Sev}(x/\mu, \sigma) = \frac{1}{\sigma} e^{\frac{x-\mu}{\sigma}} \exp\left(-e^{\frac{x-\mu}{\sigma}}\right)$$

$$\text{for } -\infty < x < \infty;\ -\infty < \mu < \infty;\ \text{and } \sigma > 0. \tag{4.41}$$

Fixing σ and varying μ, we have Lev-Sev mixture given by

$$f_{Lev}(x) = \int_{-\infty}^{\infty} f_{Lev}(x/\mu) g_{Sev}(\mu) d\mu \tag{4.42}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} \exp\left(-e^{-\frac{x-\mu}{\sigma}}\right) g_{Sev}(\mu) d\mu. \tag{4.43}$$

Put $\sigma = 1$ in 4.43

$$f_{Lev}(x) = \int_{-\infty}^{\infty} e^{-(x-\mu)} \exp(-e^{-(x-\mu)}) g_{Sev}(\mu) d\mu. \tag{4.44}$$

Consider various cases of $g_{Sev}(\mu)$

(a) For $g_{Sev}(\mu) = e^{\mu} \exp(-e^{\mu})$, the standard Gumbel II distribution,

$$f_{Lev}(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \; -\infty < x < \infty \tag{4.45}$$

which is standard logistic.

(b) For $g_{Sev}(\mu) = e^{\mu-\xi} \exp(-e^{\mu-\xi})$, the Gumbel II with location mean ξ and variance 1,

$$f_{Lev}(x) = \frac{\lambda e^{-x}}{(1+e^{-x})^2}, \; -\infty < x < \infty; \; \lambda = e^{-\xi} > 0 \tag{4.46}$$

which is the extended standard logistic distribution.

(c) For $g_{Sev}(\mu) = \lambda e^{\mu} \exp(-e^{\lambda\mu})$, the standard Gumbel II distribution with parameter λ,

$$f_{Lev}(x) = \frac{\lambda e^{-x}}{(\lambda+e^{-x})^2}, \; -\infty < x < \infty \tag{4.47}$$

which is the extended standard logistic distribution.

(d) $g_{Sev}(\mu) = \frac{e^{\alpha\mu}}{\Gamma(\alpha)} \exp(-e^{\mu})$, the generalized Gumbel with one parameter α

$$f_{Lev}(x) = \frac{\alpha e^{-x}}{(1+e^{-x})^{\alpha+1}}, \; \alpha > 0; \; -\infty < x < \infty \tag{4.48}$$

which is GLI, when $\alpha = 1$ we obtain the standard logistic distribution.

(e) For $g_{Sev}(\mu) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{\alpha\mu} \exp(-\lambda e^{\mu})$, the Gumbel II distribution with parameters α and λ,

$$f_{Lev}(x) = \frac{\lambda^{\alpha} e^{-x}}{(\lambda+e^{-x})^{\alpha+1}}, \; \lambda > 0; \; \alpha > 0; \; -\infty < x < \infty \tag{4.49}$$

which is the extended standard logistic distribution.

4.6. Lev-Sev mixture

(a) $g_{Sev}(\mu) = e^{\mu} \exp(-e^{\mu})$, the standard Gumbel II distribution,

$$f_{Sev}(x) = \frac{e^{x}}{(1+e^{x})^2}, \; -\infty < x < \infty \tag{4.50}$$

which is standard logistic distribution.

(b) $g_{Sev}(\mu) = e^{\mu-\xi}\exp(-e^{\mu-\xi})$, the Gumbel II with location mean ξ and variance 1,

$$f_{Sev}(x) = \frac{\lambda e^{x}}{(\lambda + e^{x})^{2}}, \lambda > 0; \alpha > 0; -\infty < x < \infty \tag{4.51}$$

which is the extended standard logistic distribution.

(c) $g_{Sev}(\mu) = \lambda e^{\mu}\exp(-e^{\lambda\mu})$, the standard Gumbel II distribution with parameter λ,

$$f_{Sev}(x) = \frac{\lambda e^{x}}{(\lambda + e^{x})^{2}}, \lambda > 0; \alpha > 0; -\infty < x < \infty \tag{4.52}$$

which is the extended standard logistic distribution.

(d) $g_{Sev}(\mu) = \frac{e^{\alpha\mu}}{\Gamma(\alpha)}\exp(-e^{\mu})$, the generalized Gumbel with one parameter α

$$f_{Sev}(x) = \frac{\alpha e^{x}}{(1 + e^{x})^{\alpha+1}}, \alpha > 0; -\infty < x < \infty \tag{4.53}$$

which is GLI,

(e) $g_{Sev}(\mu) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}e^{\alpha\mu}\exp(-\lambda e^{\mu})$, the Gumbel II distribution with parameters α and λ,

$$f_{Sev}(x) = \frac{\alpha\lambda^{\alpha} e^{x}}{(1 + e^{x})^{\alpha+1}}, \lambda > 0; \alpha > 0; -\infty < x < \infty \tag{4.54}$$

when $\alpha = 1$ we obtain the standard logistic distribution and when $\alpha = \lambda = 1$ we obtain the standard logistic distribution.

5. Difference of two Gumbel Random Variables

Let $Z = X_1 - X_2$, where X_1 and X_2 are independent random variables.

Therefore the cdf of Z is

$$\begin{aligned} G(z) &= \text{Prob }\{Z \leq z\} \\ &= \text{Prob }\{X_1 - X_2 \leq z\} \\ &= \text{Prob }\{X_1 \leq z + x_2\} \end{aligned}$$

$$= \text{Prob}\ \{X_1 \le z + x_2\},\ -\infty < x_2 < \infty$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z+x_2} f_1(x) f_2(x) dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{z+x_2} f_1(x)\, dx_1 \right\} f_2(x) dx_2$$

$$= \int_{-\infty}^{\infty} F_1(z + x_2) f_2(x)\, dx_2 \tag{5.1}$$

$$\therefore\ g(z) = \frac{d}{dx} G(z) = \int_{-\infty}^{\infty} f_1(z + x_2) f_2(x) dx_2. \tag{5.2}$$

5.1. Suppose X_1 is Standard Gumbel I

Then

$$F_1(x) = \exp\ (-e^{-x_1})$$

and

$$f_1(x) = e^{-x} \exp\ (-e^{-x}).$$

(a) If X_2 is also standard Gumbel I distributed, then

$$G(x) = \int_{-\infty}^{\infty} F_1(z + x_2) f_2(x) dx_2$$

$$= \int_{-\infty}^{\infty} \exp\ (-e^{-(z+x_2)}) \cdot e^{-x_2} \exp\ (-e^{-x_2}) dx_2$$

$$= \int_{-\infty}^{\infty} \exp\ (-e^{-z} e^{-x_2} - e^{-x_2}) e^{-x_2}\, dx_2$$

$$= \int_{-\infty}^{\infty} \exp\ (-(1 + e^{-z}) e^{-x_2}) e^{-x_2}\, dx_2.$$

Let $y = e^{-x_2} \Rightarrow dy = -e^{-x_2} dx_2$

$$\therefore\ G(x) = \int_0^{\infty} e^{-(1+e^{-z})y}\, dy$$

$$= \frac{1}{1 + e^{-z}}$$

and

$$g(z) = \frac{e^{-z}}{(1 + e^{-z})^2}, \ -\infty < z < \infty \tag{5.3}$$

which is standard logistic.

Remark. We now wish to obtain the standard logistic distribution by generalizing the distribution of X_2. That is to determine various cases of $f_2 x$ in the formula $g(z) = \int_{-\infty}^{\infty} f_1(z + x_2) f_2(x) dx_2$.

(a) If X_2 is Gumbel with parameter λ, then

$$g(z) = \int_{-\infty}^{\infty} e^{-(z+x_2)} \exp\left(e^{-(z+x_2)}\right) \lambda e^{-x_2} \exp\left(-\lambda e^{-x_2}\right) dx_2$$

$$\therefore \ g(z) = e^{-z} \lambda \int_{-\infty}^{\infty} e^{-x_2} \exp\left(-(\lambda + e^{-z})\right) e^{-x_2} dx_2$$

$$= \lambda e^{-z} \int_{0}^{\infty} y e^{-(\lambda + e^{-z})y} dy$$

$$= \frac{\lambda e^{-z}}{(\lambda + e^{-z})^2}, \ -\infty < z < \infty, \ \lambda > 0 \tag{5.4}$$

which is an extended standard logistic distribution.

(b) If X_2 is generalized Gumbel with parameter α, then

$$g(z) = \int_{-\infty}^{\infty} e^{-(z+x_2)} \exp\left(-e^{-(z+x_2)}\right) \frac{e^{-\alpha x_2}}{\Gamma(\alpha)} \exp\left(-e^{-x_2}\right) dx_2$$

$$= \frac{e^{-z}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\alpha x_2} \exp\left(-(1 + e^{-z}) e^{-x_2}\right) e^{-x_2} dx_2$$

$$= \frac{e^{-z}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} y^{\alpha} e^{-(1+e^{-z})y} dy$$

$$= \frac{e^{-z}}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 1)}{(1 + e^{-z})^{\alpha+1}}$$

$$= \frac{\alpha e^{-z}}{(1 + e^{-z})^{\alpha+1}}, \ -\infty < z < \infty, \ \lambda > 0 \tag{5.5}$$

which is GLI.

(c) If X_2 is generalized Gumbel I with two parameters α and λ.

Then

$$g(z) = \int_{-\infty}^{\infty} e^{-(z+x_2)} \exp\left(-e^{-(z+x_2)}\right) \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\alpha x_2} \exp\left(-\lambda e^{-x_2}\right) dx_2$$

$$= \frac{\lambda\alpha}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\alpha x_2} \exp\left(-(\lambda + e^{-z})\right) e^{-x_2} dx_2$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} y^{\alpha} e^{-(\lambda + e^{-z})y} dy$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-z} \frac{\Gamma(\alpha + 1)}{(\lambda + e^{-z})^{\alpha+1}}$$

$$= \frac{\alpha}{\lambda} \frac{e^{-z}}{\left(1 + \frac{1}{\lambda} e^{-z}\right)^{\alpha+1}}$$

$$= \frac{\alpha p e^{-z}}{(1 + p e^{-z})}, \ -\infty < z < \infty, \ \alpha > 0, \ p = \frac{1}{\lambda} > 0 \tag{5.6}$$

this is an extended GLI distribution.

5.2. Suppose X_1 is Gumbel I with parameter λ

(a) If X_2 is standard Gumbel, then

$$\therefore \ g(z) = \int_{-\infty}^{\infty} \lambda e^{-(z+x_2)} \exp\left(-\lambda e^{-z+x_2}\right) e^{-x_2} e^{-e^{-x_2}} dx_2$$

$$= \lambda e^{-z} \int_{-\infty}^{\infty} e^{-x_2} \exp\left(-(\lambda e^{-z} + 1) e^{-x_2}\right) e^{-x_2} dx_2$$

$$= \lambda e^{-z} \int_{-\infty}^{\infty} y e^{-(1+\alpha e^{-z})y} dy$$

$$= \frac{\lambda e^{-z}}{(1 + \lambda e^{-z})^2}$$

$$= \frac{\lambda e^{-z}}{\left(\frac{\lambda}{\lambda} + \lambda e^{-z}\right)^2}$$

$$= \frac{\frac{1}{\lambda} e^{-z}}{\left(\frac{1}{\lambda} + e^{-z}\right)^2}$$

$$= \frac{\beta e^{-z}}{(\beta + e^{-z})}, \; -\infty < z < \infty, \; \beta = \frac{1}{\lambda} > 0 \tag{5.7}$$

which is an extended standard logistic.

(b) If X_2 is generalized Gumbel I with one parameter α, then.

$$g(z) = \int_{-\infty}^{\infty} \lambda e^{-(z+x_2)} \exp(-\lambda e^{-(z+x_2)}) \frac{e^{-\alpha x_2}}{\Gamma(\alpha)} \exp(-e^{-x_2}) dx_2$$

$$= \frac{\lambda e^{-z}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\alpha x_2} \exp(-(1 + \lambda e^{-z}) e^{-x_2}) dx_2$$

$$= \frac{\lambda}{\Gamma(\alpha)} e^{-z} \int_{-\infty}^{\infty} y^{\alpha} e^{-(1+\lambda e^{-z})y} dy$$

$$= \frac{\lambda}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 1)}{(1 + \lambda e^{-z})^{\alpha+1}}$$

$$= \frac{\alpha\lambda e^{-z}}{(1 + \lambda e^{-z})^{\alpha+1}} \tag{5.8}$$

which is extended GLI distribution.

5.3. Suppose X_1 is generalized Gumbel I with parameter α

Then

$$f_1(x) = \frac{e^{-\alpha x}}{\Gamma(\alpha)} \exp(-e^{-z}), \; -\infty < z < \infty, \; \alpha > 0 \tag{5.8}$$

(a) If X_2 is standard Gumbel, then

$$g(z) = \int_{-\infty}^{\infty} \frac{e^{\alpha(z+x_2)}}{\Gamma(\alpha)} \exp(-e^{-(z+x_2)}) e^{-x_2} \exp(-e^{-x_2}) dx_2$$

$$= \frac{e^{-\alpha z}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\alpha x_2} \exp(-(1 + e^{-z}) e^{-x_2}) dx_2$$

$$\therefore \; g(z) = \frac{e^{-\alpha z}}{\Gamma(\alpha)} \int_{0}^{\infty} y^{\alpha} e^{-(1+e^{-z})y} dy$$

$$= \frac{e^{-\alpha z}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(1+e^{-z})^{\alpha+1}}$$

$$= \frac{\alpha e^{-\alpha z}}{(1+e^{-z})^{\alpha+1}} \tag{5.10}$$

which is GLII distribution.

(b) X_2 is Gumbel with parameter λ

$$\therefore\ g(z) = \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)} \exp\ (-e^{-(z+x_2)})\lambda e^{-x_2} \exp\ (-\lambda e - x_2)dx$$

$$= \frac{\lambda}{\Gamma(\alpha)} e^{-\alpha z} \int_{-\infty}^{\infty} e^{-\alpha x_2} e^{-(\lambda+e^{-z})e^{-x_2}} e^{-x_2} dx_2$$

$$= \frac{\lambda}{\Gamma} e^{-\alpha z} \int_{-\infty}^{\infty} y^{\alpha} e^{-(\lambda+e^{-z})y} dy$$

$$= \frac{\lambda}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\lambda+e^{-z})}$$

$$= \frac{\alpha\lambda\ e^{-\alpha z}}{(\lambda+e^{-z})^{\alpha+1}},\ -\infty < z < \infty;\ \lambda,\ \alpha > 0 \tag{5.11}$$

which is extended GLII distribution.

(c) If X_2 is also generalized Gumbel with parameter α, then

$$g(z) = \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)} \exp\ (-e^{-(z+x_2)}) \cdot \frac{e^{-\alpha x_2}}{\Gamma(\alpha)} \exp\ (-e^{-x_2})dx_2$$

$$= \frac{e^{-\alpha z}}{\Gamma(\alpha)(\Gamma(\alpha))} \int_{-\infty}^{\infty} e^{-z\alpha x_2} e^{-(1+e^{-z})e^{-x_2}} sx_2.$$

Let $y = e^{-x_2} \Rightarrow dy = e^{-x} dx_2 \Rightarrow dx_2 = \frac{-dy}{y}$

$$\therefore\ g(z) = \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\alpha)} \int_{-\infty}^{\infty} y^{2\alpha} e^{-(1+e^{-z})y} \frac{dy}{y}$$

$$= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\alpha)} \frac{\Gamma(2\alpha)}{(1+e^{-z})^{2\alpha}}$$

$$= \frac{e^{-\alpha z}}{\beta(\alpha, \alpha)(1 + e^{-z})^{2\alpha}} \quad -\infty < z < \infty; \alpha > 0 \tag{5.12}$$

which is GLIII distribution as was obtained by Davidson [3].

(d) If X_2 is generalized Gumbel with one parameter β, then,

$$g(z) = \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)} \exp\left(-e^{-(z+x_2)}\right) \frac{e^{-\beta x_2}}{\Gamma(\alpha)} \exp\left(-e^{-x_2}\right) dx_2$$

$$\therefore\ g(z) = \frac{e^{e^{-\alpha z}}}{\Gamma(\alpha)\Gamma(\beta)} \int_{-\infty}^{\infty} e^{-(\alpha+\beta)x_2} \exp\left(-(1 + e^{-z})e^{-x_2}\right) dx_2$$

$$= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} y^{\alpha+\beta} e^{-(1+e^{-z})y} \frac{dy}{y}$$

$$= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+\beta)}{(1 + e^{-z})^{\alpha+\beta}}$$

$$= \frac{1}{\beta(\alpha, \beta)} \frac{e^{-\alpha z}}{(1 + e^{-z})^{\alpha+\beta}}, \quad -\infty < z < \infty, \alpha, \beta > 0 \tag{5.13}$$

this is GLIV distribution.

(e) If X_2 is generalized Gumbel with two parameter ϕ and m then

$$g(z) = \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)} \exp\left(-e^{-(z+x_2)}\right) \frac{\phi^m}{\Gamma(m)} e^{-m x_2} \exp\left(-\phi^{-x_2}\right) dx_2$$

$$= \frac{\phi^m e^{-x_2}}{\Gamma(\alpha)\Gamma(m)} \int_{-\infty}^{\infty} e^{-(\alpha+m)x_2} \exp\left(-(\phi + e^{-z})e^{-x_2}\right) dx_2$$

$$= \frac{\phi^m e^{-x_2}}{\Gamma(\alpha)\Gamma(m)} \int_{-\infty}^{\infty} y^{\alpha+m} e^{-(\phi+e^{-z})y} \frac{dy}{y}$$

$$= \frac{\phi^m e^{-\alpha z}}{\Gamma(\alpha)(\Gamma(m))} \frac{\Gamma(\alpha + m)}{(\phi + e^{-z})^{\alpha+m}}$$

$$\therefore\ g(z) = \frac{\phi^m}{\beta(\alpha, m)} \frac{e^{-\alpha z}}{(\phi + e^{-z})^{\alpha+m}}, \quad -\infty < z < \infty; \alpha, \phi, m > 0 \tag{5.14}$$

which is extended GLIV distribution as suggested by Morais et al. [8].

6. Conclusion

Using Gumbel mixture or the difference of two Gumbel random variables, we have obtained; the standard logistic, extended standard logistic, GL I, extended GL I, GL II, extended GL II, GL III, GL IV and extended GL IV distributions.

The Burr differential equation approach gives us only the standard logistic and GL I distributions.

The transformation technique requires appropriate choice of transformations for generalizing the standard logistic distribution.

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