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THE INTEGRAL EQUATION WITH THE DELAY ARGUMENTS IN SEMI-MARKOV RANDOM WALK PROCESSES

K. K. OMAROVA[2], U. C. IDRISOVA[1], SH. B. BAKHSHIYEV[2] and K. M. CABIYEV[2]

[1]Azerbaijan State Oil and Industry University
Baku, Azerbaijan
E-mail: ulviyyeidrisova@yahoo.com

[2]Institute of Control Systems of Azerbaijan
National Academy of Sciences
Baku, Azerbaijan
E-mail: omarovakonulk@gmail.com
shaxbazia@yahoo.com
Kamilce75@gmail.com

Abstract

In this work using the sequence of independent random variables a semi-Markov random walk process is constructed. Laplace transformation distributions of duration of time of stay of differential process of semi-Markov random walk is found.

1. Introduction

It is very well known that the semi-Markov processes are a generalization of Markov processes since the exponential distribution of time intervals is replaced with an arbitrary distribution. The semi-Markov processes have been introduced by Levy [2], Smith [7] and Takacs [8] in order to reduce the limitation induced by the exponential distribution of the corresponding time intervals. This is the immediate generalization of Markov chains since the Markov property is the typical consequence of the lack of

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memory of the exponential distribution. In [3], the process of the semi-Markov random walk is studied the asymptotic of distribution in two-sided boundary problems for random walks. In [4] and [6] various problems associated with the boundary functional of the random walk investigated. These studies found asymptotic expressions of various characteristics of the processes. In [5], an explicit form of the semi-Markov random walk with negative drift, nonnegative jumps and delaying screen. Recently, mathematical modeling of the semi-Markovian random walk processes with jumps and delaying screen by means of a fractional order differential equation was studied in [1].

In this paper, we study the class of the distribution of the walk and founded the Laplace-Stieltjes transform of complex and summary process of the semi-Markov random walk.

2. Problem Statement

Let on the probability space $(\Omega, F, P(\cdot))$ is given the sequence $\{\xi_k^+, \eta_k^+, \xi_k^-, \eta_k^-\}_{k=\overline{1, \infty}}$ of independent equal distributed positive and between themselves independent random variables $\xi_k^+, \eta_k^+, \xi_k^-, \eta_k^-, k = \overline{1, \infty}$.

Using these random variables, we can construct the following processes (see [4] and [6])

$$X^{\pm}(t) = \sum_{i=1}^{m-1} \eta_i^+, \text{ if } \sum_{i=1}^{m-1} \xi_i^{\pm} \leq t < \sum_{i=1}^{m} \xi_i^{\pm}, \; k = \overline{1, \infty}. \tag{1}$$

The process $X_1(t) = X^+(t) - X^-(t)$ we will call the process with differentiated random walk process.

One of realization of the processes will be the following

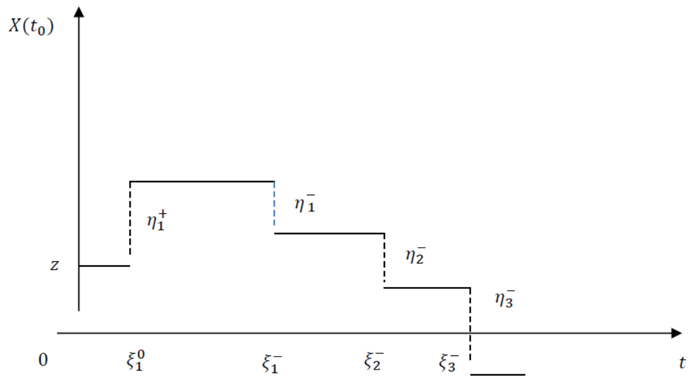

Figure 1. Complex semi-Markov random walk process.

Our aim is to find the Laplace-Stielties transform of the residence time of semi-Markov process.

We assume that the random variable $\xi_1^{\pm}$ is distributed according to an exponential law. Then $X(t)$ becomes a complex summary Markov process.

Let it given $X(t), t > 0$ process. We delay this process in the screen "a" and reflect in the screen "a" (see, [3] and [5])

We denote $R(t, x \mid X(0) = z) = P\{X(t) < x \mid X(0) = z\}, x > 0, z > 0.$

Event $\{X(t) < x\}$ occurs at the following five hypotheses:

$$H_1 = \{\xi_1^+ > t; \xi_1^- > t\},$$

$$H_2 = \{\xi_1^+ > t; \xi_1^- < t\},$$

$$H_3 = \{\xi_1^+ < t < \xi_1^-\},$$

$$H_4 = \{\xi_1^- < \xi_1^+ < t\},$$

$$H_5 = \{\xi_1^- < t < \xi_1^+\}.$$

From total probability formula, we have

$$R(t, x|z) = P\left(A \bigcup_{i=1}^{5} H_i\right) = P(AH_1) + P(AH_2) + P(AH_3) + P(AH_4) + P(AH_5)$$

$$= P\{\xi_1^+ > t; \xi_1^- > t; 0 < z < a | X(0) = z\}$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} P\{\xi_1^+ < s;\ \xi_1^- \in ds;\ X(s) \in dy | X(0) = z\} R(t-s,\ x | y)$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} P\{\xi_1^+ \in ds;\ \xi_1^- < t;\ X(s) \in dy | X(0) = z\} R(t-s,\ x | y)$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} P\{\xi_1^+ \in ds;\ \xi_1^- < s;\ X(s) \in dy | X(0) = z\} R(t-s,\ x | y)$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} P\{\xi_1^- \in ds;\ \xi_1^+ > t;\ X(s) \in dy | X(0) = z\} R(t-s,\ x | y).$$

Then that equation will become

$$R(t, x|z) = P\left(A \bigcup_{i=1}^{5} H_i\right) = P(AH_1) + P(AH_2) + P(AH_3) + P(AH_4) + P(AH_5)$$

$$= P\{\xi_1^+ > t;\ \xi_1^- > t;\ 0 < z < a | X(0) = z\}$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} d_y P\{\xi_1^+ < s; \xi_1^- \in ds; \min(a, | \min(a, z + X^+(s)) - \eta_1^- |)\} R(t-s, x|y)$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} d_y P\{\xi_1^+ \in ds;\ \xi_1^- > t;\ \min(a,\ z + \eta_1^+) < y\} R(t-s,\ x | y)$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} d_y P\{\xi_1^- < s; \xi_1^+ \in ds; \min[a, \min[z - x^-(s)] + \eta_1^+] < y\} R(t-s, x|y)$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} d_y P\{\xi_1^- \in ds\} \xi_1^+ > t; \min(a, | z - \eta_1^- | < y) R(t-s, x|y).$$

Then we have

$$R(t, x|z) = P\{\xi_+^1 > t\} P\{\xi_-^1 > t\}$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} R(t-s, x|y) P\{\xi_1^- \in ds\} d_y P\{\xi_1^+ < s; \min(a, | \min(a, z + X^+(s)) - \eta_1^- |) < y\}$$

$$+ P\{\zeta_1^+ > t\} + \int_{y=0}^{a} \int_{s=0}^{t} R(t-s,\ x \mid y) d_y P\{\xi_1^+ \in ds\} P\{\min(a,\ z + \eta_1^+) < y\}$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} P\{\xi_1^+ \in ds\} d_y P\{\xi_1^- < d; \min[a, \min[a, | z - x^-(s)|] + \eta_i^+] < y\}$$

$$R(t-s, x \mid y)$$

$$+ P\{\xi_1^+ > t\} + \int_{y=0}^{a} \int_{s=0}^{t} R(t-s, x \mid y); P\{\xi_1^- \in ds\} d_y P\{\min(a, | z + \eta_1^- | < y)\}.$$

As casual processes $X^+(t)$ and $X^-(t)$ are also independent random variables ξ_1^+ and ξ_1^- respectively. We will receive following form of a formula of total probability

$$P\{\xi_1^+ < s\} = P\{\xi_1^+ < s;\ \min(a, |\ \min(a,\ z + X^+(s)) - \eta_1^-\ |) < y\}$$

$$+ P\{\xi_1^+ < s;\ \min(a, |\ \min(a,\ z + X^+(s)) - \eta_1^-\ |) < y\}.$$

Then we have the following form

$$R(t,\ x \mid z) = P\{\xi_+^1 > t\} P\{\xi_-^1 > t\}$$

$$+ \int_{y=0}^{\alpha} \int_{s=0}^{t} R(t-s,\ x \mid y) P\{\xi_1^- \in dx\} d_y \sum_{k=1}^{\infty} \int_{u=0}^{s} P\{\xi_1^+ \in du\}$$

$$\times \left[P\left\{ \min\left(a,\ z + \sum_{i=1}^{k} \eta_i^+ \right) - \eta_1^- < y \right\} - P\left\{ \min\left(a,\ z + \sum_{i=1}^{k} \eta_i^+ \right) - \eta_1^- < y \right\} \right]$$

$$\times P\{v^+(s-u) = k-1\}] + P\{\xi_1^+ > t\} \int_{y=0}^{\alpha} \int_{s=0}^{t} R(t-s,\ x \mid y) P\{\xi_1^+ \in ds\} d_y$$

$$\times P\{\min(a,\ z + \eta_1^+) > y\} + \int_{y=0}^{\alpha} \int_{s=0}^{t} P\{\xi_1^+ > ds\} R(t-s,\ x \mid y) d_y$$

$$\times P\{\xi_1^- < s; \min[a, | z - x^-(s)|] + \eta_i^+ > y\}$$

$$\times P\{\xi_1^+ > t\} \int_{y=0}^{\alpha} \int_{s=0}^{t} R(t-s, x \mid y); P\{\xi_1^- \in ds\} d_y P\{(a, | z - \eta_1^- | > y)\}.$$

After differentiating both side of the received integral equation on z, we have

$$R(t,\ x \mid z) = P\{\xi_{+}^{1} > t\} P\{\xi_{-}^{1} > t\} +$$

$$+ \int_{y=0}^{a} \int_{s=01}^{t} R(t-s,\ x \mid y) P\{\xi_1^{-} \in ds\} d_y \sum_{k=1}^{\infty} \int_{k=1}^{s} P\{\xi_1^{+} \in du\}$$

$$\times \left[p\left\{ \min\left(a,\ z + \sum_{i=1}^{k} \eta_i^{+} \right) - \eta_1^{-} < y \right\} p\{v^{+}(s-u) = k-1\} \right.$$

$$- \int_{y=0}^{a} \int_{s=0}^{t} R(t-s, x \mid y) P\{\xi_1^{-} \in ds\} d_y \sum_{k=1}^{\infty} \int_{u=0}^{s} P\{\xi_1^{+} \in du\}$$

$$\times \left[p\left\{ \min\left(a, z + \sum_{i=1}^{k} \eta_i^{+} \right) - \eta_1^{-} < y \right\} p\{v^{+}(s-u) = k-1\} \right.$$

$$-P\{\xi_1^{+} > t\} \int_{y=0}^{a} \int_{s=0}^{t} R(t-s, x \mid y) P\{\xi_1^{+} \in ds\} d_y P\{z + \eta_1^{+}\} > y$$

$$+ \int_{y=0}^{a} \int_{s=0}^{t} P\{\xi_1^{+} \in ds\} R(t-s, x \mid y) d_y$$

$$\int_{\beta=0}^{\infty} P\{\xi_1^{-} < s; \min[a, |z - x^{-}(s)] > y - \beta\} d$$

$$\times P\{\eta_i^{+} < \beta\} + P\{\xi_1^{+} > t\} \int_{y=0}^{a} \int_{s=0}^{t} R(t-s, x \mid y);$$

$$P\{\xi_1^{-} \in ds\} d_y P\{z - \eta_1^{-} < y\}.$$

Both side of this integral equation having multiplied on $e^{\theta t}$ and integrated on t. We have

$$R(t, x \mid y) = \int_{t=0}^{\infty} e^{\theta t} P\{\xi_{+}^{1} > t\} P\{\xi_{-}^{1} > t\} dt$$

$$+\int_{y=0}^{a}\int_{s=0}^{t}R(t-s,x|y)P\{\xi_1^-\in ds\}\int_{u=0}^{s}P\{\xi_1^+\in du\}\sum_{k=1}^{\infty}P\{v^+(s-u)=k-1\}dy$$

$$\times\int_{\alpha=0}^{a-y}p\left\{\sum_{i=1}^{k}\eta_i^+>\alpha+y-z\right\}dP\{\eta_i^-<\alpha\}+\int_{y=0}^{a}\int_{s=0}^{t}R(t-s,x|y)P\{\xi_1^-\in ds\}$$

$$\times\int_{u=0}^{s}P\{\xi_1^+\in du\}\sum_{k=1}^{\infty}P\{v^+(s-u)=k-1\}dy$$

$$\int_{\alpha=0}^{a+y}P\left\{\sum_{i=1}^{k}\eta_i^+>\alpha-y-z\right\}dP\{\eta_i^-<\alpha\}$$

$$+P\{\xi_1^+>t\}\int_{y=0}^{a}\int_{s=0}^{t}R(t-s,x|y)P\{\xi_1^+\in ds\}dy\,P\{\eta_1^+>y-z\}$$

$$+\int_{y=0}^{a}\int_{s=0}^{t}P\{\xi_1^+\in du\}R(t-s,x|y)\sum_{k=1}^{\infty}\int_{u=0}^{s}P\{\xi_1^-\in du\}P\{v^-(s-u)=k-1\}dy$$

$$\times\int_{\beta=\max\{0,\,y-z\}}P\left\{\sum_{i=1}^{k}\eta_i^-<y+\beta+z\right\}dP\{\eta_1^-<\beta\}$$

$$+\int_{y=0}^{a}\int_{s=0}^{t}P\{\xi_1^+\in du\}R(t-s,x|y)$$

$$\times\sum_{k=1}^{\infty}\int_{u=0}^{s}P\{\xi_1^-\in du\}P\{v^-(s-u)=k-1\}dy\int_{\beta=0}^{y}P\left\{\sum_{i=1}^{k}\eta_1^-<y-\beta-z\right\}dP\{\eta_i^+<\beta\}$$

$$+P\{\xi_1^+>t\}\int_{y=0}^{a}\int_{s=0}^{t}R(t-s,x|y);\,P\{\xi_1^-\in ds\}dy\,P\{\|z-\eta_1^-|<-y\}.$$

So we received the integral equation for $\widetilde{K}(\theta/y)$ in the case (1). Thus

$$K(\theta/y)=\frac{1}{\lambda_++\lambda_-+\theta}+\frac{\lambda_+\lambda_-\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)\theta}e^{-\mu_-a}\int_{y=0}^{a}e^{-\mu_-y}\widetilde{R}(\theta,x/y)dy$$

$$+\frac{\lambda_+\lambda_-\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)\theta}e^{-\mu_-a}e^{\frac{\lambda_+\mu_+(a-z)}{\lambda_++\theta}}\int_{y=0}^{a}e^{-\mu_-y}\widetilde{R}(\theta, x/y)dy$$

$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)[\mu_+\theta+\mu_-(\lambda_++\theta)]}e^{\frac{\mu_+\theta}{\lambda_++\theta}}$$

$$\int_{y=0}^{a}e^{\mu_-y}e^{\frac{\mu_+\theta+\mu_-(\lambda_++\theta)}{\lambda_++\theta}}\widetilde{R}(\theta, x/y)dy$$

$$-\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)[\mu_+\theta+\mu_-(\lambda_++\theta)]}e^{-\mu_-z}\int_{y=0}^{a}e^{\mu_-y}\widetilde{R}(\theta, x/y)dy$$

$$-\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)[\mu_+\theta+\mu_-(\lambda_++\theta)]}e^{\frac{\mu_+\theta}{\lambda_++\theta}z}\int_{y=0}^{a}e^{-\frac{\mu_+\theta}{\lambda_++\theta}y}\widetilde{R}(\theta, x/y)dy$$

$$-\frac{\lambda_+\lambda_-\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)\theta}e^{-\mu_-a}\int_{y=0}^{a}e^{-\mu_-y}\widetilde{R}(\theta,\ x/y)dy$$

$$+\frac{\lambda_+\lambda_-\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)\theta}e^{-\mu_-z}\int_{y=0}^{a}e^{-\mu_-y}e^{\frac{\lambda_+\mu_+(a-z)}{\lambda_++\theta}}\widetilde{R}(\theta, x/y)dy$$

$$+\frac{\lambda_+\mu_+\lambda_-\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)[\mu_+\theta+\mu_-(\lambda_++\theta)]}e^{\frac{\mu_+\theta}{\lambda_++\theta}z}$$

$$\int_{y=0}^{a}e^{\mu_-y}e^{-\frac{\mu_+\theta\mu_-(\lambda_++\theta)}{\lambda_++\theta}}\widetilde{R}(\theta, x/y)$$

$$-e^{\frac{\mu_+\theta}{\lambda_++\theta}z}\int_{y=0}^{a}e^{\frac{\mu_+\theta}{\lambda_++\theta}y}\widetilde{R}(\theta,\ x/y)dy$$

$$+\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}e^{\mu_-z}\int_{y=0}^{a}e^{-\mu_+y}\widetilde{R}(\lambda_-+\theta, x/y)dy$$

$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\theta)(\lambda_-+\theta)[\mu_-\theta+\mu_+(\lambda_-+\theta)]}e^{\frac{-\mu_-\theta}{\lambda_-+\theta}z}$$

$$\int_{y=0}^{z}\left[e^{-\mu_+ y} - e^{-\frac{-\mu_-\theta}{\lambda_-+\theta}y}\right]\widetilde{R}(\theta,\ x/y)dy$$

$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \theta)(\lambda_- + \theta)[\mu_-\theta + \mu_+(\lambda_- + \theta)]}e^{\frac{-\mu_-\theta}{\lambda_-+\theta}z}$$

$$\int_{y=0}^{z}\left[e^{-\mu_+ y} - e^{-\mu_+ y}e^{\frac{\mu_-\theta+\mu_-(\lambda_-+\theta)}{\lambda_-+\theta}y}\right]\widetilde{R}(\theta,\ x/y)dy$$

$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \theta)(\lambda_- + \theta)[\mu_-\theta + \mu_+(\lambda_- + \theta)]}e^{\frac{-\mu_-\theta}{\lambda_-+\theta}z}$$

$$\int_{y=0}^{z}\left[e^{-\mu_+ y} - e^{\frac{-\mu_-\theta}{\lambda_-+\theta}y}\right]\widetilde{R}(\theta,\ x/y)dy$$

$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}e^{-\mu_- z}\int_{y=0}^{z}e^{\mu_- y}\widetilde{R}(\lambda_+ + \theta,\ x/y)dy.$$

Remark 1. From last integral equation it is possible to receive the ordinary differential equation.

3. Conclusion

Using the sequence of independent random variables, it is constructed process of semi-Markov random walk. Laplace transformation distributions of duration of time of stay of differential process of semi-Markov random walk is found.

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