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SOME FIXED-POINT RESULTS IN S_b-METRIC SPACES

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Abstract

In this paper, we establish some fixed point and common fixed-point theorems in S_b-metric spaces using implicit relation. The results presented in this paper extend and generalize several results from the existing literature.

1. Introduction

In 1906, Maurice Fréchet [4] introduced the concept of metric spaces. Later, in the year 1922, Stefan Banach [2] proved a very famous theorem called "Banach Fixed Point Theorem". In 2006, Z. Mustafa and B. Sims [5] introduced G-metric spaces. In 2012, Sedghi, Shobe and Aliouche [11] introduced S-metric spaces and they claimed that S-metric spaces are the generalization of G-metric spaces. But, later Dung, Hieu and Radojevic [3] have given examples that S-metric spaces are not the generalization of G-metric spaces or vice versa. Therefore, the collection of G-metric spaces and S-metric spaces are different. In 1989, I. A. Bakhtin [1] introduced b-metric spaces as a generalization of metric spaces. In 2016, N. Souayah, N. Mlaiki [12] introduced S_b-metric spaces as the generalizations of b-metric spaces and S-metric spaces. But, very recently Tas and Ozur [6] studied some relations between S_b-metric spaces and some other metric spaces. S. Sedghi and N. V. Dung [9] introduced an implicit relation to investigate some fixed-

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point theorems on S-metric spaces. In 2015, Prudhvi [7] proved some fixed-point theorems on S-metric spaces, which extends the results of Sedgi and Dung [9].

Inspired by G. S. Saluja [8], Prudhvi [7], S. Sedghi, N. V. Dung [9] and some others, we establish some fixed point and common fixed-point theorems in S_b-metric spaces satisfying an implicit relation.

2. Preliminaries

Definition 2.1[11]**.** Let Ω be a nonempty set. An S-metric on Ω is a function $S : \Omega^3 \to [0, \infty)$ that satisfies the following conditions, for each $\varsigma, \vartheta, w, a \in \Omega$,

(S1) $S(\varsigma, \vartheta, w) > 0$ for all $\varsigma, \vartheta, w \in \Omega$ with $\varsigma \neq \vartheta \neq w$.

(S2) $S(\varsigma, \vartheta, w) = 0$ if $\varsigma = \vartheta = w$.

(S3) $S(\varsigma, \vartheta, w) \leq [S(\varsigma, \varsigma, a) + S(\vartheta, \vartheta, a) + S(w, w, a)]$.

The pair (Ω, S) is called S-metric space.

Example 2.1[3]**.** Let $\Omega = R$, the set of all real numbers and let $S(\varsigma, \vartheta, w) = |\vartheta + w - 2\varsigma| + |\vartheta - w|$ $\forall \varsigma, \vartheta, w \in \Omega$. Then (Ω, S) is an S-metric space.

Definition 2.2[1]**.** Let Ω be a nonempty set. A b-metric on Ω is a function $d : \Omega^2 \to [0, \infty)$ if there exists a real number $s \geq 1$ such that the following conditions holds for all $\varsigma, \vartheta \in \Omega$

(i) $d(\varsigma, \vartheta) = 0 \Leftrightarrow \varsigma = \vartheta$.

(ii) $d(\varsigma, \vartheta) = d(\vartheta, \varsigma)$

(iii) $d(\varsigma, \vartheta) \leq s[d(\varsigma, w) + d(w, \vartheta)]$

The pair (Ω, d) is called a b-metric space.

Definition 2.3[12]**.** Let Ω be a nonempty set and let $s \geq 1$ be a given number.

A function $S_b : \Omega^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $\forall \varsigma, \vartheta, w, a \in \Omega,$ the following conditions hold:

(i) $S_b(\varsigma, \vartheta, w) = 0$ if $\varsigma = \vartheta = w$.

(ii) $S_b(\varsigma, \vartheta, w) \leq s[S_b(\varsigma, \varsigma, a) + S_b(\vartheta, \vartheta, a) + S_b(w, w, a)]$,

The pair (Ω, S_b) is called an S_b-metric space.

Remark 2.1. We note that every S-metric space is an S_b-metric space with $s = 1$, but the converse statement is not true.

Example 2.2[6]**.** Let $\Omega = R$, the set of all real numbers and let $S_b(\varsigma, \vartheta, w) = \frac{1}{16}(|\varsigma - \vartheta| + |\vartheta - w| + |\varsigma - w|)^2$, for all $\varsigma, \vartheta, w \in \Omega$.

Then (Ω, S_b) is an S_b-metric space with $s = 4$, but it is not an S-metric space. Indeed, for $\varsigma = 4$, $\vartheta = 6$, $w = 8$ and $a = 5$, we get

$$S_b(4, 6, 8) = 4 > S_b(4, 4, 5) + S_b(6, 6, 5) + S_b(8, 8, 5).$$

Thus, S_b-metric spaces are more general than S-metric spaces.

Definition 2.4[6]**.** A S_b-metric S_b is said to be symmetric if

$$S_b(\varsigma, \varsigma, \vartheta) = S_b(\vartheta, \vartheta, \varsigma)\ \forall \varsigma, \vartheta \in \Omega.$$

Lemma 2.1[10]**.** *In S_b-metric space, we have*

(i) $S_b(\varsigma, \varsigma, \vartheta) \leq sS_b(\vartheta, \vartheta, \varsigma)$ *and* $S_b(\vartheta, \vartheta, \varsigma) \leq sS_b(\varsigma, \varsigma, \vartheta)$

(ii) $S_b(\varsigma, \varsigma, w) \leq 2sS_b(\varsigma, \varsigma, \vartheta) + s^2 S_b(\vartheta, \vartheta, w)$.

Definition 2.5[12]**.** If (Ω, S_b) is an S_b-metric space and a sequence $\{\varsigma_n\}$ in Ω. Then

(i) $\{\varsigma_n\}$ is called a S_b-Cauchy sequence, if to each $\epsilon > 0, \exists n_0 \in N$ such that $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \leq \epsilon, \forall n, m > n_0$.

(ii) $\{\varsigma_n\} \to \varsigma \Leftrightarrow$ to each $\epsilon > 0, \exists n_0 \in N$ such that $S_b(\varsigma_n, \varsigma_n, \varsigma) < \epsilon$ and $S_b(\varsigma, \varsigma, \varsigma_n) < \epsilon\ \forall n \geq n_0$, and we write as $\lim_{n \to \infty} \varsigma_n = \varsigma$.

Definition 2.6[12]**.** We say that (Ω, S_b) is complete if every S_b-Cauchy sequence is S_b-Convergent in Ω.

Tas and Ozgur [6] proved the following theorems in S_b-metric spaces.

Theorem 2.1[6]**.** *If (Ω, S_b) is a complete S_b-metric space with $s \geq 1$ and T is a self map on Ω satisfying*

$$S_b(T\varsigma, T\varsigma, T\vartheta) \leq cS_b(\varsigma, \varsigma, \vartheta), \forall \varsigma, \vartheta \in \Omega, \text{ where } 0 < c < \frac{1}{s^2}.$$

Then T has a unique fixed point ς in Ω.

Example 2.3[10]**.** Let (Ω, S) be a S-metric space and $S_*(\varsigma, \vartheta, w) = [S(\varsigma, \vartheta, w)]^q$, where $q > 1$ is a real number.

Note that S_* is a S_b-metric with $s = 2^{2(q-1)}$. Obvisously, S_* satisfies conditions

(i) $0 < S_*(\varsigma, \vartheta, w)$, for all $\varsigma, \vartheta, w \in \Omega$ with $\varsigma \neq \vartheta \neq w$.

(ii) $S_*(\varsigma, \vartheta, w) = 0$ if $\varsigma = \vartheta = w$.

If $1 < q < \infty$, then the convexity of the function $f(\varsigma) = \varsigma^q$, $(\varsigma > 0)$ implies that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$.

Thus, for each $\varsigma, \vartheta, w, a \in \Omega$, we obtain,

$$\begin{aligned} S_*(\varsigma, \vartheta, w) &= S(\varsigma, \vartheta, w)^q \\ &\leq ([S(\varsigma, \varsigma, a) + S(\vartheta, \vartheta, a)] + S(w, w, a))^q \\ &\leq 2^{q-1}([S(\varsigma, \varsigma, a) + S(\vartheta, \vartheta, a)]^q + S(w, w, a)^q) \\ &\leq 2^{2-1}([2^{q-1}(S(\varsigma, \varsigma, a)^q + S(\vartheta, \vartheta, a)^q)] + 2^{q-1}S(w, w, a)^q) \\ &\leq 2^{2(q-1)}(S(\varsigma, \varsigma, a)^q + S(\vartheta, \vartheta, a)^q + S(w, w, a)^q). \\ &\leq 2^{2(q-1)}(S_*(\varsigma, \varsigma, a) + S_*(\vartheta, \vartheta, a) + S_*(w, w, a)). \end{aligned}$$

So, S_* is a S_b-metric with $s = 2^{2(q-1)}$.

Now, we introduce an implicit relation to prove some fixed point and common fixed-point theorems in S_b-metric spaces.

Definition 2.7 (Implicit Relation). Let Ψ be the family of all real valued continuous functions $\psi : R_+^5 \to R_+$ non-decreasing in the first argument for five variables. For some $q \in \left[0, \frac{1}{s^2}\right]$, where $s \geq 1$, we consider the following conditions.

(R1) For $\varsigma, \vartheta \in R_+$, if $\varsigma \leq \psi(\vartheta, s\varsigma, s\vartheta, s\varsigma, \varsigma + s\vartheta)$ then $\varsigma \leq q\vartheta$.

(R2) For $\varsigma, \vartheta \in R_+$, if $\varsigma \leq \psi(0, 0, \varsigma, 0, 0)$ then $\varsigma = 0$.

(R3) For $\varsigma \in R_+$, if $\varsigma \leq \psi\left(\varsigma, 0, 0, 0, \frac{\varsigma}{2}\right)$ then $\varsigma = 0$.

3. Main Results

In this section, we shall prove some fixed point and common fixed-point theorems satisfying an implicit relation in S_b-metric spaces.

Theorem 3.1. *Let T be a self map on a complete S_b-metric space (Ω, S_b) with $s \geq 1$ and*

$$S_b(T\varsigma, T\vartheta, Tw) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, T\varsigma), S_b(w, w, Tw), S_b(\varsigma, \varsigma, T\vartheta),$$
$$\frac{1}{2s}[S_b(\vartheta, \vartheta, T\vartheta) + S_b(w, w, T\varsigma)]) \quad (1)$$

for all $\varsigma, \vartheta, w \in \Omega$ and $\psi \in \Psi$. If ψ satisfies the conditions $(R1)$, $(R2)$ and (R3), *then T has a unique fixed point in Ω.*

Proof. Let $\varsigma_0 \in \Omega$ be arbitrary and define a sequence $\{\varsigma_n\}$ in Ω such that $\varsigma_{n+1} = T\varsigma_n$, for any $n \in N$. If for some $n \in N$, $\varsigma_{n+1} = \varsigma_n$. Then, $\varsigma_n = T\varsigma_n$. Hence, T has a fixed point. Now, we may assume that $\varsigma_{n+1} \neq \varsigma_n$, for all $n \in N$. It follows from inequality (1) and Lemma 2.1, we consider

$$S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) = S_b(T\varsigma_n, T\varsigma_n, T\varsigma_{n-1})$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), S_b(\varsigma_n, \varsigma_n, T\varsigma_n), S_b(\varsigma_{n-1}, \varsigma_{n-1}, T\varsigma_{n-1}),$$

$$S_b(\varsigma_n, \varsigma_n, T\varsigma_n), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, T\varsigma_n) + S_b(\varsigma_{n-1}, \varsigma_{n1}, T\varsigma_{n-1})])$$

$$= \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), S_b(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n),$$

$$S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1})S_b(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n)])$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}),$$

$$sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), \frac{1}{2s}[sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n)$$

$$+2sS_b(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n) + sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n)])$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}),$$

$$sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), \frac{1}{2s}[2sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) + 2s^2S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1})])$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}), sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}),$$

$$sS_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n), [S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) + sS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1})]) \tag{2}$$

Since $\psi \in \Psi$ satisfies the condition (R1), there exists $q \in \left[0, \frac{1}{s^2}\right)$ such that

$$S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_n) \leq qS_b(\varsigma_n, \varsigma_n, \varsigma_{n-1}) \leq q^n S_b(\varsigma_1, \varsigma_1, \varsigma_0) \tag{3}$$

For $n, m \in N$ with $n < m$, using Lemma 2.1 and equation (3), we have

$$S_b(\varsigma_n, \varsigma_n, \varsigma_m) \leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_m)$$

$$\leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2[2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_{n+2}) + s^2S_b(\varsigma_{n+2}, \varsigma_{n+2}, \varsigma_m)]$$

$$\leq 2aq^n[1 + s^2q + (s^2q)^2 + \ldots]S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

$$\leq \left(\frac{2sq^n}{1-s^2q}\right) S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

Since $q \in \left[0, \frac{1}{s^2}\right]$ and $s \geq 1$. Taking the limit as $n \to \infty$, we get $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \to 0$. This proves that the sequence $\{\varsigma_n\}$ is a cauchy sequence in the complete S_b-metric space (Ω, S_b). Then, there exists $\rho \in \Omega$ such that $\lim_{n\to\infty} \varsigma_n = \rho$. Now we prove that ρ is a fixed point of T. Again by using inequality (1), we obtain

$$S_b(\varsigma_n, \varsigma_n, T\rho) = S_b(T\varsigma_n, T\varsigma_n, T\rho)$$

$$\leq \psi(S_b(\varsigma_n, \varsigma_n, \rho), S_b(\varsigma_n, \varsigma_n, T\varsigma_n), S_b(\rho, \rho, T\rho),$$

$$S_b(\varsigma_n, \varsigma_n, T\varsigma_n), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, T\varsigma_n) + S_b(\rho, \rho, T\varsigma_n)])$$

$$= \psi(S_b(\varsigma_n, \varsigma_n, \rho), S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), S_b(\rho, \rho, T\rho),$$

$$S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}), \frac{1}{2s}[S_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + S_b(\rho, \rho, \varsigma_{n+1})])$$

Letting $n \to \infty$, we get

$$S_b(\rho, \rho, T\rho) \leq \psi(S_b(\rho, \rho, \rho), S_b(\rho, \rho, \rho), S_b(\rho, \rho, T\rho),$$

$$S_b(\rho, \rho, \rho), \frac{1}{2s}[S_b(\rho, \rho, \rho) + S_b(\rho, \rho, \rho)])$$

$$\text{that is, } S_b(\rho, \rho, T\rho) \leq \psi(0, 0, S_b(\rho, \rho, T\rho), 0, 0)$$

Since $\psi \in \Psi$ satisfies the condition (R2), then we get

$$S_b(\rho, \rho, T\rho) \leq q S_b(\rho, \rho, T\rho)$$

$$\text{that is, } (1-q)S_b(\rho, \rho, T\rho) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\rho, \rho, T\rho) = 0$. Hence $T\rho = \rho$.

Thus, ρ is a fixed point of T. Now, we show that fixed point of T is unique.

For this, let ρ^* be another fixed point of T. It follows from inequality (1) and Lemma 2.1, we get

$$S_b(\rho, \rho, \rho^*) = S_b(T\rho, T\rho, \rho^*)$$

$$\leq \psi(S_b(\rho, \rho, \rho^*), S_b(\rho, \rho, T\rho), S_b(\rho^*, \rho^*, T\rho^*),$$

$$S_b(\rho, \rho, T\rho), \frac{1}{2s}[S_b(\rho, \rho, T\rho) + S_b(\rho^*, \rho^*, T\rho)])$$

$$= \psi(S_b(\rho, \rho, \rho^*), S_b(\rho, \rho, \rho), S_b(\rho^*, \rho^*, \rho^*),$$

$$S_b(\rho, \rho, \rho), \frac{1}{2s}[S_b(\rho, \rho, \rho) + S_b(\rho^*, \rho^*, \rho)])$$

$$\leq \psi(S_b(\rho, \rho, \rho^*), 0, 0, 0, \frac{1}{2} S_b(\rho, \rho, \rho^*))$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\rho, \rho, \rho^*) \leq q S_b(\rho, \rho, \rho^*)$$

$$\text{that is, } (1 - q) S_b(\rho, \rho, \rho^*) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\rho, \rho, \rho^*) = 0$. Hence $\rho = \rho^*$. Thus the fixed point of T is unique.

Theorem 3.2. *Let T_1 and T_2 be two selfmaps on a complete S_b-metric space (Ω, S_b) with $s \geq 1$ and*

$$S_b(T_1\varsigma, T_1\vartheta, T_2 w) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, T_1\varsigma), S_b(w, w, T_2 w),$$

$$S_b(\varsigma, \varsigma, T_1\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1\vartheta) + S_b(w, w, T_1\varsigma)]) \tag{4}$$

for all $\varsigma, \vartheta, w \in \Omega$ and $\psi \in \Psi$. If ψ satisfies the conditions $(R1)$, $(R2)$ and (R3), *then T_1 and T_2 have a unique fixed point in Ω.*

Proof. Let $\varsigma_0 \in X$ be arbitrary and a sequence $\{\varsigma_n\}$ in X defined by $\varsigma_{2n+1} = T_1\varsigma_{2n}$ and $\varsigma_{2n+2} = T_2\varsigma_{2n+1}$, for $n = 0, 1, 2, 3, \ldots$.

It follows from inequality (4) and Lemma 2.1, we have

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) = S_b(T_1\varsigma_{2n}, T_1\varsigma_{2n}, T_2\varsigma_{2n-1})$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, T_2\varsigma_{2n-1}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, T_1\varsigma_{2n})])$$

$$= \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n+1})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), sS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}),$$

$$sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), \frac{1}{2s}[sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})$$

$$+2sS_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}) + sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}),$$

$$\frac{1}{2s}[2sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) + 2s^2S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1})]) \tag{5}$$

Since $\psi \in \Psi$ satisfies the condition ($R1$), there exists $q \in \left[0, \frac{1}{s^2}\right)$ such that

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) \leq qS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}) \leq q^{2n}S_b(\varsigma_1, \varsigma_1, \varsigma_0) \tag{6}$$

For $n, m \in N$ with $n < m$, by using Lemma 2.1 and equation (6), we have

$$S_b(\varsigma_n, \varsigma_n, \varsigma_m) \leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_m)$$

$$\leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2[2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_{n+2}) + s^2S_b(\varsigma_{n+2}, \varsigma_{n+2}, \varsigma_m)]$$

$$\leq 2sq^n[1 + s^2q + (s^2q)^2 + \ldots]S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

$$\leq \left(\frac{2sq^n}{1-s^2q}\right) S_b(\varsigma_0, \varsigma_0, \varsigma_1).$$

Since $q \in \left[0, \frac{1}{s^2}\right]$ and $s \geq 1$. Taking the limit as $n \to \infty$, we get $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \to 0$. This proves that the sequence $\{\varsigma_n\}$ is a cauchy sequence in the complete S_b-metric space (Ω, S_b). Then, there exists $\sigma \in \Omega$ such that $\lim_{n\to\infty} \varsigma_n = \sigma$. Now we prove that σ is a common fixed point of T_1 and T_2.

For this Consider,

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, T_1\sigma) = S_b(T_1\varsigma_{2n}, T_1\varsigma_{2n}, T_1\sigma)$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \sigma), S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), S_b(\sigma, \sigma, T_1\sigma),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, T_1\varsigma_{2n}) + S_b(\sigma, \sigma, T_1\varsigma_{2n})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \sigma), S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), S_b(\sigma, \sigma, T_1\sigma),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}) + S_b(\sigma, \sigma, \varsigma_{2n+1})]) \tag{7}$$

Letting $n \to \infty$, we get

$$S_b(\sigma, \sigma, T_1\sigma) \leq \psi(S_b(\sigma, \sigma, \sigma), S_b(\sigma, \sigma, \sigma), S_b(\sigma, \sigma, T_1\sigma),$$

$$S_b(\sigma, \sigma, \sigma), \frac{1}{2s}[S_b(\sigma, \sigma, \sigma) + S_b(\sigma, \sigma, \sigma)])$$

$$\text{that is, } S_b(\sigma, \sigma, T_1\sigma) \leq \psi(0, 0, S_b(\sigma, \sigma, T_1\sigma), 0, 0)$$

Since $\psi \in \Psi$ satisfies the condition (R2), then we get

$$S_b(\sigma, \sigma, T_1\sigma) \leq qS_b(\sigma, \sigma, T_1\sigma)$$

$$\text{that is, } (1-q)S_b(\sigma, \sigma, T_1\sigma) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\sigma, \sigma, T_1\sigma) = 0$. Hence $T_1\sigma = \sigma$.

Similarly, we can show that $T_2\sigma = \sigma$. This shows that σ is a common fixed point of T_1 and T_2. Now we prove that T_1 and T_2 have a unique common fixed point. For this, let σ^* be another common fixed point of T_1 and T_2. It follows from equation (4) and Lemma 2.1, we have

$$S_b(\sigma, \sigma, \sigma^*) = S_b(T_1\sigma, T_1\sigma, T_2\sigma^*)$$

$$\leq \psi(S_b(\sigma, \sigma, \sigma^*), S_b(\sigma, \sigma, T_1\sigma), S_b(\sigma^*, \sigma^*, T_2\sigma^*),$$

$$S_b(\sigma, \sigma, T_1\sigma), \frac{1}{2s}[S_b(\sigma, \sigma, T_1\sigma) + S_b(\sigma^*, \sigma^*, T_2\sigma)])$$

$$= \psi(S_b(\sigma, \sigma, \sigma^*), S_b(\sigma, \sigma, \sigma), S_b(\sigma^*, \sigma^*, \sigma^*)$$

$$S_b(\sigma, \sigma, \sigma), \frac{1}{2s}[S_b(\sigma, \sigma, \sigma) + S_b(\sigma^*, \sigma^*, \sigma)])$$

$$= \psi\left(S_b(\sigma, \sigma, \sigma^*), 0, 0, 0, \frac{1}{2}S_b(\sigma, \sigma, \sigma^*)\right).$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\sigma, \sigma, \sigma^*) \leq qS_b(\sigma, \sigma, \sigma^*)$$

$$\text{that is, } (1 - q)S_b(\sigma, \sigma, \sigma^*)$$

Since $1 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\sigma, \sigma, \sigma^*) = 0$. Hence $\sigma = \sigma^*$. This shows that σ is the unique common fixed point of T_1 and T_2.

Theorem 3.3. *Let T_1 and T_2 be two continuous selfmaps on a complete Sbmetric space (Ω, S_b) with $s \geq 1$ and*

$$S_b(T_1^p\varsigma, T_1^p\vartheta, T_2^pw) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, T_1^p\varsigma), S_b(w, w, T_2^pw),$$

$$S_b(\varsigma, \varsigma, T_1^p\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, T_1^p\vartheta) + S_b(w, w, T_1^p\varsigma)]) \tag{8}$$

for all $\varsigma, \vartheta, w \in \Omega$, where p and q are integers and $\psi \in \Psi$. If ψ satisfies the conditions $(R1)$, $(R2)$ and $(R3)$, then T_1 and T_2 have a unique fixed point in Ω.

Proof. Since T_1^p and T_2^p satisfies the conditions of Theorem 3.2. Let λ be the common fixed point.

Then, we have $T_1^p\lambda = \lambda \Rightarrow T_1(T_1^p\lambda) = T_1\lambda \Rightarrow T_1^p(T_1\lambda) = T_1\lambda.$

If $T_1\lambda = \lambda_0,$ then $T_1^p\lambda_0 = \lambda_0.$ So, $T_1\lambda$ is a fixed point of $T_1^p.$

Similarly, $T_2(T_2^q\lambda) = T_2^q(T_2\lambda) = T_2\lambda.$ Now, using equation (8) and Lemma 2.1, we obtain

$$S_b(\lambda, \lambda, T_1\lambda) = S_b(T_1^p\lambda, T_1^p\lambda, T_1^p(T_1\lambda))$$

$$\leq \psi(S_b(\lambda, \lambda, T_1\lambda), S_b(\lambda, \lambda, T_1^p\lambda), S_b(T_1\lambda, T_1\lambda, T_1^p(T_1\lambda)),$$

$$S_b(\lambda, \lambda, T_1^p\lambda), \frac{1}{2s}[S_b(\lambda, \lambda, T_1^p\lambda) + S_b(T_1\lambda, T_1\lambda, T_1^p\lambda)])$$

$$\leq \psi(S_b(\lambda, \lambda, T_1\lambda), S_b(\lambda, \lambda, \lambda), S_b(T_1\lambda, T_1\lambda, T_1\lambda),$$

$$S_b(\lambda, \lambda, \lambda), \frac{1}{2s}[S_b(\lambda, \lambda, \lambda) + S_b(T_1\lambda, T_1\lambda, \lambda)])$$

$$\leq \psi\left(S_b(\lambda, \lambda, T_1\lambda), 0, 0, 0, \frac{1}{2}[S_b(\lambda, \lambda, T_1\lambda)]\right).$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\lambda, \lambda, T_1\lambda) \leq kS_b(\lambda, \lambda, T_1\lambda)$$

$$\text{that is, } (1-k)S_b(\lambda, \lambda, T_1\lambda) \leq 0.$$

Since $0 \leq k \leq \frac{1}{s^2}$ and $s \geq 1.$ Therefore we get $S_b(\lambda, \lambda, T_1\lambda) = 0.$ Hence $T_1\lambda = \lambda.$ Similarly, we can show that $T_2\lambda = \lambda.$ This shows that λ is a common fixed point of T_1 and $T_2.$ For uniqueness of λ, Let $\lambda^* \neq \lambda$ be another common fixed point of T_1 and $T_2.$ Then clearly λ^* is also a common fixed point of T_1^p and T_2^q, which implies $\lambda = \lambda^*.$ Hence T_1 and T_2 have a unique common fixed point.

Theorem 3.4. *Let* $\{G_\alpha\}$ *be a family of continuous selfmaps on a complete* S_b*-metric space* (Ω, S_b) *with* $s \geq 1$ *and*

$$S_b(G_\alpha\varsigma, G_\alpha\vartheta, G_\beta w) \leq \psi(S_b(\varsigma, \vartheta, w), S_b(\vartheta, \vartheta, G_\alpha\varsigma), S_b(w, w, G_\beta w),$$

$$S_b(\varsigma, \varsigma, G_\alpha\vartheta), \frac{1}{2s}[S_b(\vartheta, \vartheta, G_\alpha\vartheta) + S_b(w, w, G_\alpha\varsigma)]) \tag{9}$$

for all $\varsigma, \vartheta, w \in \Omega$, *and* $\alpha, \beta \in R^+$ *with* $\alpha \neq \beta$. *Then there exists a unique* $\eta \in \Omega$ *satisfying* $G_\alpha\eta = \eta$, *for all* $\alpha \in \Psi$.

Proof. Let $\varsigma_0 \in \Omega$ be arbitrary and a sequence $\{\varsigma_n\}$ in Ω defined by $\varsigma_{2n+1} = G_\alpha\varsigma_{2n}$ and $\varsigma_{2n+2} = G_\beta\varsigma_{2n+1}$, for $n = 0, 1, 2, 3, \ldots$.

It follows from inequality (9) and Lemma 2.1, we have

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) = S_b(G_\alpha\varsigma_{2n}, G_\alpha\varsigma_{2n}, G_\beta\varsigma_{2n-1})$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, G_\alpha\varsigma_{2n}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, G_\beta\varsigma_{2n-1})$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, G_\alpha\varsigma_{2n}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, G_\alpha\varsigma_{2n}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, G_\alpha\varsigma_{2n})])$$

$$= \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}),$$

$$S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}), \frac{1}{2s}[S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n+1}) + S_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n+1})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), sS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}),$$

$$sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), \frac{1}{2s}[sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})$$

$$+2sS_b(\varsigma_{2n-1}, \varsigma_{2n-1}, \varsigma_{2n}) + sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n})])$$

$$\leq \psi(S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}), sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), sS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}),$$

$$sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}), \frac{1}{2s}[2sS_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) + 2s^2S_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1})]) \tag{10}$$

Since $\psi \in \Psi$ satisfies the condition (R1), there exists $q \in \left[0, \frac{1}{s^2}\right)$ such that

$$S_b(\varsigma_{2n+1}, \varsigma_{2n+1}, \varsigma_{2n}) \leq qS_b(\varsigma_{2n}, \varsigma_{2n}, \varsigma_{2n-1}) \leq q^{2n}S_b(\varsigma_1, \varsigma_1, \varsigma_0) \quad (11)$$

For $n, m \in N$ with $n < m$, by using Lemma 2.1 and equation (11), we have

$$S_b(\varsigma_n, \varsigma_n, \varsigma_m) \leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_m)$$

$$\leq 2sS_b(\varsigma_n, \varsigma_n, \varsigma_{n+1}) + s^2[2S_b(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_{n+2}) + s^2S_b(\varsigma_{n+2}, \varsigma_{n+2}, \varsigma_m)]$$

$$\leq 2sq^n[1 + s^2q + (s^2q)^2 + \ldots]S_b(\varsigma_0, \varsigma_0, \varsigma_1)$$

$$\leq \left(\frac{2sq^n}{1 - s^2q}\right)S_b(\varsigma_0, \varsigma_0, \varsigma_1).$$

Since $q \in \left[0, \frac{1}{s^2}\right]$ and $s \geq 1$. Taking the limit as $n \to \infty$, we get $S_b(\varsigma_n, \varsigma_n, \varsigma_m) \to 0$. This proves that the sequence $\{\varsigma_n\}$ is a cauchy sequence in the complete S_b-metric space (Ω, S_b). Then, there exists $\eta \in \Omega$ such that $\lim_{n\to\infty} \varsigma_n = \eta$. By the continuity of G_α and G_β, it is clear that $G_\alpha\eta = G_\beta\eta = \eta$. Therefore η is a common fixed point of G_α and G_β, for all $\alpha \in \Psi$. In order to prove the uniqueness, let us take another common fixed point η^* of G_α and G_β, where $\eta \neq \eta^*$. Then using equation (9) and Lemma 2.1, we obtain

$$S_b(\eta, \eta, \eta^*) = S_b(G_\alpha\eta, G_\alpha\eta, G_\beta\eta^*)$$

$$\leq \psi(S_b(\eta, \eta, \eta^*), S_b(\eta, \eta, G_\alpha\eta), S_b(\eta^*, \eta^*, G_\beta\eta^*),$$

$$S_b(\eta, \eta, G_\alpha\eta), \frac{1}{2s}[S_b(\eta, \eta, G_\alpha\eta) + S_b(\eta^*, \eta^*, G_\alpha\eta)])$$

$$\leq \psi(S_b(\eta, \eta, \eta^*), S_b(\eta, \eta, \eta), S_b(\eta^*, \eta^*, \eta^*),$$

$$S_b(\eta, \eta, \eta), \frac{1}{2s}[S_b(\eta, \eta, \eta) + S_b(\eta^*, \eta^*, \eta)])$$

$$\leq \psi\left(S_b(\eta, \eta, \eta^*), 0, 0, 0, \frac{1}{2} S_b(\eta, \eta, \eta^*)\right).$$

Since $\psi \in \Psi$ satisfies the condition (R3), then we get

$$S_b(\eta, \eta, \eta^*) \leq q S_b(\eta, \eta, \eta^*)$$

$$\text{that is, } (1 - q) S_b(\eta, \eta, \eta^*) \leq 0.$$

Since $0 \leq q \leq \frac{1}{s^2}$. Therefore we get $S_b(\eta, \eta, \eta^*) = 0$. Hence $\eta = \eta^*$. This shows that η is the unique common fixed point of G_α, for all $\alpha \in \Psi$.

Corollary 3.1. *Let* (Ω, S_b) *be a complete* S_b*-metric space. Suppose that the mapping* $T : \Omega \to \Omega$ *satisfies* $S_b(T\varsigma, T\vartheta, Tw) \leq \gamma S_b(\varsigma, \vartheta, w)$ *for all* $\varsigma, \vartheta, w \in \Omega$, *where* $\gamma \in [0, 1)$ *is a constant. Then T has a unique fixed point in* Ω. *Moreover, T is continuous at the fixed point.*

Proof. We can prove easily by using Theorem 3.1. with $\psi(a, b, c, d, e) = \gamma a$, for some $\gamma \in [0, 1)$ and $a, b, c, d, e \in R^+$.

Corollary 3.2. *Let* (Ω, S_b) *be a complete* S_b*-metric space. Suppose that the mappings* $T_1, T_2 : \Omega \to \Omega$ *satisfies* $S_b(T_1\varsigma, T_1\vartheta, T_2, w) \leq \delta S_b(\varsigma, \vartheta, w)$ *for all* $\varsigma, \vartheta, \omega \in \Omega$, *where* $\delta \in [0, 1)$ *is a constant. Then* T_1 *and* T_2 *have a unique fixed point in* Ω.

Proof. We can prove easiy by using Theorem 3.2. with $\psi(a, b, c, d, e) = \delta a$, for some $\delta \in [0, 1)$ and $a, b, c, d, e \in R^+$.

Example 3.1. Let (Ω, S_b) be a complete S_b-metric space with $s = 4$. Where $\Omega = [0, 1]$ and $S_b(\varsigma, \vartheta, w) = (|\varsigma - w| + |\vartheta - w|)^2$.

Now, we consider the mapping $T : \Omega \to \Omega$ defined by $T(\varsigma) = \frac{\varsigma}{5}$, for all $\varsigma \in [0, 1]$. Then $S_b(T\varsigma, T\vartheta, Tw) = (|T\varsigma - Tw| + |T\vartheta - Tw|)^2$

$$= \left(\left|\frac{\varsigma}{5} - \frac{w}{5}\right| + \left|\frac{\vartheta}{5} - \frac{w}{5}\right|\right)^2$$

$$= \frac{1}{25}(|\varsigma - w| + |\vartheta - w|)^2$$

$$\leq \frac{1}{25} S_b(\varsigma, \vartheta, w)$$

$$= \gamma S_b(\varsigma, \vartheta, w).$$

where $\gamma = \frac{1}{25} < 1$. Thus T satisfies all the conditions of corollary 3.1. and clearly $0 \in \Omega$ is the unique fixed point of T.

4. Conclusion

From this results, we can study the fixed-circle problem [13] using new contrations on different generalized metric spaces.

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